Solutions to

# Unit 3D Specialist Mathematics

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# Preface

The answers in the Sadler text book sometimes are not enough. For those times when your really need to see a fully worked solution, look here.

**It is essential that you use this sparingly!**

You should not look here until you have given your best effort to a problem. Understand the problem here, then go away and do it on your own.

## Errors

If you encounter any discrepancies between the work here and the solutions given in the Sadler text book, it is very likely that the error is mine. I have yet to find any errors in Sadler's solutions. Mine, however, have not been proofread as thoroughly and it is likely that there are errors in this work. Caveat discipulus!

# Contents

# Chapter 1

## Exercise 1A

1. The initial case, where $n = 1$,

$$1 = \frac{1}{2}(1)(1+1)$$

is true.

Assume the statement is true for $n = k$, i.e.

$$1 + 2 + 3 + 4 + \ldots + k = \frac{1}{2}k(k+1)$$

Then for $n = k + 1$

$$\begin{aligned} 1 + 2 + 3 + 4 + &\ldots + k + (k+1) \\ &= \frac{1}{2}k(k+1) + (k+1) \\ &= (\frac{1}{2}k + 1)(k+1) \\ &= \frac{1}{2}(k+2)(k+1) \\ &= \frac{1}{2}(k+1)\left((k+1)+1\right) \end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k + 1$.

Since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$. □

2. The initial case, where $n = 1$:

$$\begin{aligned} \text{L.H.S.} &= 1(1+1) \\ &= 2 \\ \text{R.H.S.} &= \frac{1}{3}(1+1)(1+2) \\ &= 2 \\ &= \text{L.H.S.} \end{aligned}$$

The statement is true for the initial case.

Assume the statement is true for $n = k$, i.e.

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + k(k+1) = \frac{k}{3}(k+1)(k+2)$$

Then for $n = k + 1$:

$$\begin{aligned} 1 \times 2 + 2 \times 3 + 3 \times 4 + &\ldots + k(k+1) + (k+1)(k+2) \\ &= \frac{k}{3}(k+1)(k+2) + (k+1)(k+2) \\ &= (\frac{k}{3} + 1)(k+1)(k+2) \\ &= \frac{1}{3}(k+3)(k+1)(k+2) \\ &= \frac{k+1}{3}(k+2)(k+3) \\ &= \frac{k+1}{3}\left((k+1)+1\right)\left((k+1)+2\right) \end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k + 1$.

Hence since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$. □

3. The initial case, where $n = 1$ is given:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^1) = 1$$

The statement is true for the initial case.

Assume the statement is true for $n = k$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^k) = kx^{k-1}$$

Then for $n = k + 1$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(x^{k+1}) &= \frac{\mathrm{d}}{\mathrm{d}x}(xx^k) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(x)(x^k) + (x)\left(\frac{\mathrm{d}}{\mathrm{d}x}(x^k)\right) \\ &= x^k + x\left(kx^{k-1}\right) \\ &= x^k + kx^k \\ &= (k+1)x^k \\ &= (k+1)x^{(k+1)-1} \end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k + 1$.

Hence since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$. □

4. The initial case, where $n = 1$:

$$2 = 2^2 - 2$$

The statement is true for the initial case.

Assume the statement is true for $n = k$, i.e.

$$2 + 4 + 8 + \ldots + 2^k = 2^{k+1} - 2$$

Then for $n = k + 1$

$$\begin{aligned} 2 + 4 + 8 + \ldots + 2^k + 2^{k+1} &= 2^{k+1} - 2 + 2^{k+1} \\ &= 2(2^{k+1}) - 2 \\ &= 2^{(k+1)+1} - 2 \end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k + 1$.

Hence since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$. □

5. The initial case, where $n = 1$:

$$\begin{aligned} \text{L.H.S.} &= 1(1+1)^3 \\ &= 1 \\ \text{R.H.S.} &= \frac{1^2}{4}(1+1)(1+2)^2 \\ &= 1 \\ &= \text{L.H.S.} \end{aligned}$$

The statement is true for the initial case.

Assume the statement is true for $n = k$, i.e.

$$1^3 + 2^3 + 3^3 + 4^3 + \ldots + k^3 = \frac{k^2}{4}(k+1)^2$$

Then for $n = k + 1$

$$\begin{aligned}
1^3 + 2^3 + 3^3 + 4^3 + \ldots + k^3 + (k+1)^3 \\
&= \frac{k^2}{4}(k+1)^2 + (k+1)^3 \\
&= \frac{k^2}{4}(k+1)^2 + (k+1)(k+1)^2 \\
&= \frac{k^2 + 4(k+1)}{4}(k+1)^2 \\
&= \frac{k^2 + 4k + 4}{4}(k+1)^2 \\
&= \frac{(k+2)^2}{4}(k+1)^2 \\
&= \frac{(k+1)^2}{4}(k+2)^2 \\
&= \frac{(k+1)^2}{4}((k+1)+1)^2
\end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k + 1$.

Hence since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$. □

6. (a) For $n = 2$, $(2n - 1) = 4 - 1 = 3$ and $n^2 = 4$ hence

$$1 + 3 = 4$$

is consistent with the rule.

For $n = 3$, $(2n - 1) = 6 - 1 = 5$ and $n^2 = 9$ hence

$$1 + 3 + 5 = 9$$

is consistent with the rule.

Verify the other statements similarly.

(b) The initial case, where $n = 1$: $2n - 1 = 1$ and

$$1 = 1^2$$

The statement is true for the initial case.

Assume the statement is true for $n = k$, i.e.

$$1 + 3 + 5 + \ldots + (2k-1) = k^2$$

Then for $n = k + 1$

$$\begin{aligned}
1 + 3 + 5 + \ldots + (2k-1) + (2(k+1) - 1) \\
&= k^2 + (2(k+1) - 1) \\
&= k^2 + 2k + 2 - 1 \\
&= k^2 + 2k + 1 \\
&= (k+1)^2
\end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k + 1$.

Hence since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$. □

7. The initial case, where $n = 1$:

$$\frac{1}{2} = \frac{2-1}{2}$$

The statement is true for the initial case.

Assume the statement is true for $n = k$, i.e.

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^k} = \frac{2^k - 1}{2^k}$$

Then for $n = k + 1$

$$\begin{aligned}
\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^k} + \frac{1}{2^{k+1}} \\
&= \frac{2^k - 1}{2^k} + \frac{1}{2^{k+1}} \\
&= \frac{2(2^k - 1)}{2^{k+1}} + \frac{1}{2^{k+1}} \\
&= \frac{2(2^k - 1) + 1}{2^{k+1}} \\
&= \frac{2^{k+1} - 2 + 1}{2^{k+1}} \\
&= \frac{2^{k+1} - 1}{2^{k+1}}
\end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k + 1$.

Hence since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$. □

8. The initial case, where $n = 1$:

$$\frac{1}{1(1+1)} = \frac{1}{1+1}$$

The statement is true for the initial case.

Assume the statement is true for $n = k$, i.e.

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

Then for $n = k + 1$

$$\begin{aligned}
\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2) + 1}{(k+1)(k+2)} \\
&= \frac{k^2 + 2k + 1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2} \\
&= \frac{k+1}{(k+1)+1}
\end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k + 1$.

Hence since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$. $\square$

9. The initial case, where $n = 1$:

$$\begin{aligned}\text{L.H.S.} &= 1(1+2)(1+4)\\ &= 10\\ \text{R.H.S.} &= \frac{1}{4}(1+1)(1+4)(1+5)\\ &= 10\\ &= \text{L.H.S.}\end{aligned}$$

The statement is true for the initial case.

Assume the statement is true for $n = k$, i.e.

$$\begin{aligned}1 \times 3 \times 5 + 2 \times 4 \times 6 + \ldots + k(k+2)(k+4)&\\ = \frac{k}{4}(k+1)(k+4)(k+5)&\end{aligned}$$

Then for $n = k + 1$

$$\begin{aligned}&1 \times 3 \times 5 + 2 \times 4 \times 6 + \ldots\\ &\quad + k(k+2)(k+4) + (k+1)(k+3)(k+5)\\ &= \frac{k}{4}(k+1)(k+4)(k+5) + (k+1)(k+3)(k+5)\\ &= (k+1)(k+5)\left(\frac{k}{4}(k+4) + (k+3)\right)\\ &= \frac{k+1}{4}(k+5)\left(k(k+4) + 4(k+3)\right)\\ &= \frac{k+1}{4}\left((k+1)+4\right)\left(k^2 + 4k + 4k + 12\right)\\ &= \frac{k+1}{4}\left((k+1)+4\right)\left(k^2 + 8k + 12\right)\\ &= \frac{k+1}{4}\left((k+1)+4\right)(k+2)(k+6)\\ &= \frac{k+1}{4}\left((k+1)+4\right)\left((k+1)+1\right)\left((k+1)+5\right)\\ &= \frac{k+1}{4}\left((k+1)+1\right)\left((k+1)+4\right)\left((k+1)+5\right)\end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k + 1$.

Hence since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$. $\square$

10. The initial case, where $n = 1$: $(x - 1)$ is a factor of $x^1 - 1$ since $x - 1 = x^1 - 1$.

The statement is true for the initial case.

Assume the statement is true for $n = k$, i.e.

$$x^k - 1 = a(x-1)$$

Then for $n = k + 1$

$$\begin{aligned}x^{k+1} - 1 &= x(x^k) - 1\\ &= x(x^k - 1 + 1) - 1\\ &= x(x^k - 1) + x - 1\\ &= ax(x-1) + (x-1)\\ &= (ax+1)(x-1)\end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k + 1$.

Hence since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$. $\square$

11. The initial case here is where $n = 7$, the first integer value satisfying $n > 6$:

$$\begin{aligned}\text{L.H.S.} &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7\\ &= 5040\\ \text{R.H.S.} &= 3^7\\ &= 2187\\ 5040 &> 2187\end{aligned}$$

The statement is true for the initial case.

Assume the statement is true for $n = k;\ k > 6$, i.e.

$$1 \times 2 \times 3 \times 4 \times \ldots \times k \geq 3^k$$

Then for $n = k + 1$

$$\begin{aligned}1 \times 2 \times 3 \times 4 \times \ldots \times k(k+1) &\geq 3^k(k+1)\\ 3^k(k+1) &= 3^{k+1}\frac{k+1}{3}\\ \text{Now}\qquad k &> 6\\ k+1 &> 7\\ \frac{k+1}{3} &> 1\\ \therefore\qquad 3^k(k+1) &> 3^{k+1}\\ \therefore\qquad 1 \times 2 \times 3 \times 4 \times \ldots \times k(k+1) &> 3^{k+1}\end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k + 1$.

Hence since the statement is true for $n = 7$ it follows by induction that it is true for all integer $n > 6$. $\square$

12. The initial case, where $n = 1$:

$$\begin{aligned}7^1 + 2 \times 13^1 &= 7 + 26\\ &= 33\\ &= 3 \times 11\end{aligned}$$

The statement is true for the initial case.

Assume the statement is true for $n = k$, i.e.

$$7^k + 2 \times 13^k = 3a,\ a \in \mathrm{I}$$

Then for $n = k + 1$

$$\begin{aligned}7^{k+1} + 2 \times 13^{k+1} &= 7 \times 7^k + 13 \times 2 \times 13^k\\ &= 7 \times 7^k + (7+6) \times 2 \times 13^k\\ &= 7 \times 7^k + 7 \times 2 \times 13^k + 12 \times 13^k\\ &= 7(7^k + 2 \times 13^k) + 3(4 \times 13^k)\\ &= 7(3a) + 3(4 \times 13^k)\\ &= 3(7a + 4 \times 13^k)\end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k + 1$.

Hence since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$. $\square$

13. The initial case, where $n = 1$:

$$\begin{aligned}\text{L.H.S.} &= 2\\ \text{R.H.S.} &= \frac{2}{3}(1 + (-1)^{1+1}2^1\\ &= \frac{2}{3}(1+2)\\ &= 2\\ &= \text{L.H.S.}\end{aligned}$$

The statement is true for the initial case.

Assume the statement is true for $n = k$, i.e.

$$2-4+8-16+\ldots+(-1)^{k+1}2^k = \frac{2}{3}(1+(-1)^{k+1}2^k)$$

Then for $n = k + 1$

$$\begin{aligned}&2 - 4+8 - 16 + \ldots + (-1)^{k+1}2^k + (-1)^{k+2}2^{k+1}\\ &= \frac{2}{3}(1 + (-1)^{k+1}2^k) + (-1)^{k+2}2^{k+1}\\ &= \frac{2}{3}(1 + (-1)^{k+1}2^k) + (-1)(-1)^{k+1}(2)2^k\\ &= \frac{2}{3}(1 + (-1)^{k+1}2^k) - 2(-1)^{k+1}2^k\\ &= 2\left(\frac{1 + (-1)^{k+1}2^k}{3} - (-1)^{k+1}2^k\right)\\ &= 2\left(\frac{1 + (-1)^{k+1}2^k}{3} - \frac{3(-1)^{k+1}2^k}{3}\right)\\ &= 2\left(\frac{1 + (-1)^{k+1}2^k - 3(-1)^{k+1}2^k}{3}\right)\\ &= 2\left(\frac{1 - 2(-1)^{k+1}2^k}{3}\right)\\ &= 2\left(\frac{1 - (-1)^{k+1}2^{k+1}}{3}\right)\\ &= 2\left(\frac{1 + (-1)(-1)^{k+1}2^{k+1}}{3}\right)\\ &= 2\left(\frac{1 + (-1)^{(k+1)+1}2^{k+1}}{3}\right)\\ &= \frac{2}{3}(1 + (-1)^{(k+1)+1}2^{k+1})\end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k + 1$.

Hence since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$. □

## Miscellaneous Exercise 1

1. (a) $$\begin{aligned}(7 + 3\text{i})(7 - 3\text{i}) &= 7^2 - (3\text{i})^2\\ &= 49 + 9\\ &= 58\end{aligned}$$

   (b) $$\begin{aligned}(5 + \text{i})(5 - 1\text{i}) &= 5^2 - (\text{i})^2\\ &= 25 + 1\\ &= 26\end{aligned}$$

   (c) $$\begin{aligned}(3 + 2\text{i})(2 - 3\text{i}) &= 6 - 9\text{i} + 4\text{i} - 6\text{i}^2\\ &= 6 - 5\text{i} + 6\\ &= 12 - 5\text{i}\end{aligned}$$

   (d) $$\begin{aligned}(1 - 5\text{i})^2 &= 1 - 10\text{i} + 25\text{i}^2\\ &= 1 - 10\text{i} - 25\\ &= -24 - 10\text{i}\end{aligned}$$

   (e) $$\begin{aligned}\frac{3 - 2\text{i}}{2 + \text{i}} &= \frac{(3 - 2\text{i})(2 - \text{i})}{(2 + \text{i})(2 - \text{i})}\\ &= \frac{6 - 3\text{i} - 4\text{i} + 2\text{i}^2}{4 - \text{i}^2}\\ &= \frac{6 - 7\text{i} - 2}{4 + 1}\\ &= \frac{4 - 7\text{i}}{5}\\ &= 0.8 - 1.4\text{i}\end{aligned}$$

   (f) $$\begin{aligned}\frac{1 + 2\text{i}}{3 - 4\text{i}} &= \frac{(1 + 2\text{i})(3 + 4\text{i})}{(3 - 4\text{i})(3 + 4\text{i})}\\ &= \frac{3 + 4\text{i} + 6\text{i} + 8\text{i}^2}{9 - 16\text{i}^2}\\ &= \frac{3 + 10\text{i} - 8}{9 + 16}\\ &= \frac{-5 + 10\text{i}}{25}\\ &= \frac{-1 + 2\text{i}}{5}\\ &= -0.2 + 0.4\text{i}\end{aligned}$$

2. (a) $$\begin{aligned}z + w &= 3 - 4\text{i} - 4 + 5\text{i}\\ &= -1 + \text{i}\end{aligned}$$

   (b) $$\begin{aligned}zw &= (3 - 4\text{i})(-4 + 5\text{i})\\ &= -12 + 15\text{i} + 16\text{i} - 20\text{i}^2\\ &= -12 + 31\text{i} + 20\\ &= 8 + 31\text{i}\end{aligned}$$

   (c) $\bar{z} = 3 + 4\text{i}$

   (d) $$\begin{aligned}z^2 &= (3 - 4\text{i})^2\\ &= 9 - 24\text{i} + 16\text{i}^2\\ &= 9 - 24\text{i} - 16\\ &= -7 - 24\text{i}\end{aligned}$$

(e) $\overline{zw} = \overline{(8+31\mathrm{i})}$
$= 8 - 31\mathrm{i}$

(f) $\bar{z}\bar{w} = (3+4\mathrm{i})(-4-5\mathrm{i})$
$= -12 - 15\mathrm{i} - 16\mathrm{i} - 20\mathrm{i}^2$
$= -12 - 31\mathrm{i} + 20$
$= 8 - 31\mathrm{i}$

(g) $q = \mathrm{Re}(\bar{w}) + \mathrm{Im}(\bar{z})\mathrm{i}$
$= \mathrm{Re}(-4-5\mathrm{i}) + \mathrm{Im}(3+4\mathrm{i})\mathrm{i}$
$= -4 + 4\mathrm{i}$

3. $(1+\mathrm{i})^5 = 1 + 5(\mathrm{i}) + 10(\mathrm{i}^2) + 10(\mathrm{i}^3) + 5(\mathrm{i}^4) + \mathrm{i}^5$
$= 1 + 5\mathrm{i} - 10 - 10\mathrm{i} + 5 + \mathrm{i}$
$= -4 - 4\mathrm{i}$

4. $(1-3\mathrm{i})^3 = 1^3 + 3(1^2)(-3\mathrm{i}) + 3(1)(-3\mathrm{i})^2 + (-3\mathrm{i})^3$
$= 1 - 9\mathrm{i} + 27\mathrm{i}^2 - 27\mathrm{i}^3$
$= 1 - 9\mathrm{i} - 27 + 27\mathrm{i}$
$= -26 + 18\mathrm{i}$
$\therefore\ \mathrm{Im}\left((1-3\mathrm{i})^3\right) = 18$

5. (a) $3 \times 2 = 6$

(b) $\mathrm{Re}\left((3-2\mathrm{i})(2+\mathrm{i})\right) = \mathrm{Re}(6 + 3\mathrm{i} - 4\mathrm{i} - 2\mathrm{i}^2)$
$= \mathrm{Re}(6 + -\mathrm{i} + 2)$
$= 8$

6. No working required.

7. (a) No working required.

(b) $6\,\mathrm{cis}\,\dfrac{5\pi}{3} = 6\,\mathrm{cis}\left(\dfrac{5\pi}{3} - 2\pi\right)$
$= 6\,\mathrm{cis}\left(\dfrac{5\pi}{3} - \dfrac{6\pi}{3}\right)$
$= 6\,\mathrm{cis}\left(-\dfrac{\pi}{3}\right)$

8. (a) No working required

(b) No working required

(c) $zw = (8 \times 2)\,\mathrm{cis}\left(\dfrac{3\pi}{4} + \dfrac{\pi}{3}\right)$
$= 16\,\mathrm{cis}\,\dfrac{13\pi}{12}$
$= 16\,\mathrm{cis}\left(\dfrac{13\pi}{12} - 2\pi\right)$
$= 16\,\mathrm{cis}\left(-\dfrac{11\pi}{12}\right)$

(d) Use the commutative property of multiplication and no working is needed.

(e) $\mathrm{i}w = \left(\mathrm{cis}\,\dfrac{\pi}{2}\right)\left(2\,\mathrm{cis}\,\dfrac{\pi}{3}\right)$
$= 2\,\mathrm{cis}\left(\dfrac{\pi}{3} + \dfrac{\pi}{2}\right)$
$= 2\,\mathrm{cis}\,\dfrac{5\pi}{6}$

(f) $\mathrm{i}z = 8\,\mathrm{cis}\left(\dfrac{3\pi}{4} + \dfrac{\pi}{2}\right)$
$= 8\,\mathrm{cis}\,\dfrac{5\pi}{4}$
$= 8\,\mathrm{cis}\left(\dfrac{5\pi}{4} - 2\pi\right)$
$= 8\,\mathrm{cis}\left(-\dfrac{3\pi}{4}\right)$

(g) $\dfrac{z}{w} = \dfrac{8}{2}\,\mathrm{cis}\left(\dfrac{3\pi}{4} - \dfrac{\pi}{3}\right)$
$= 4\,\mathrm{cis}\,\dfrac{5\pi}{1}2$

(h) No working required.

9. The initial case, where $n = 1$,

$$\begin{aligned}\text{L.H.S.} &= 5(1+2^0) + 2\\ &= 12\\ \text{R.H.S.} &= 1(1+6) + 5(2^1 - 1)\\ &= 7 + 5\\ &= 12\\ &= \text{L.H.S.}\end{aligned}$$

is true.

Assume the statement is true for $n = k$, i.e.

$$12 + 19 + 31 + 53 + \ldots + \left(5(1 + 2^{k-1}) + 2k\right) = k(k+6) + 5(2^k - 1)$$

Then for $n = k+1$

$$\begin{aligned}&12 + 19 + 31 + 53 + \ldots + \left(5(1+2^{k-1}) + 2k\right)\\ &\quad + \left(5(1+2^k) + 2(k+1)\right)\\ &= k(k+6) + 5(2^k - 1) + 5(1+2^k) + 2(k+1)\\ &= k(k+6) + 5 \times 2^k - 5 + 5 + 5 \times 2^k + 2k + 2\\ &= k(k+6) + 5 \times 2^k + 5 \times 2^k + 2k + 2\\ &= k(k+6) + 5 \times 2 \times 2^k + 2k + 2\\ &= k(k+6) + 5 \times 2^{k+1} + 2k + 2\\ &= k(k+6) + 5 \times 2^{k+1} - 5 + 5 + 2k + 2\\ &= k(k+6) + 5(2^{k+1} - 1) + 2k + 7\\ &= k(k+6) + 5(2^{k+1} - 1) + (k+6) + (k+1)\\ &= (k+1)(k+6) + 5(2^{k+1} - 1) + (k+1)\\ &= (k+1)(k+7) + 5(2^{k+1} - 1)\\ &= (k+1)\left((k+1) + 6\right) + 5(2^{k+1} - 1)\end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k+1$.

Since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$. □

# Chapter 2

## Exercise 2A

1. One solution is $z = 1 = \operatorname{cis} 0$ and the other five solutions divide the complex plane into six equal-sized regions:

$$z \in \left\{ \operatorname{cis} 0, \operatorname{cis} \frac{\pi}{3}, \operatorname{cis} \frac{2\pi}{3}, \operatorname{cis} \pi, \operatorname{cis} -\frac{\pi}{3}, \operatorname{cis} \frac{2\pi}{3} \right\}$$

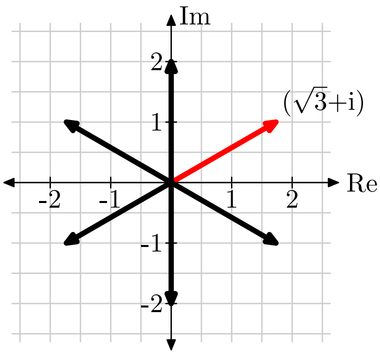
2. One solution is $z = 1 = \operatorname{cis} 0$ and the other five solutions divide the complex plane into eight equal-sized regions:

$$z \in \{\operatorname{cis} 0, \operatorname{cis} \pm 45°, \operatorname{cis} \pm 90°, \operatorname{cis} \pm 135°, \operatorname{cis} 180°\}$$

3. One solution is $z = 1 = \operatorname{cis} 0$ and the other five solutions divide the complex plane into seven equal-sized regions:

$$z \in \left\{ \operatorname{cis} 0, \operatorname{cis} \pm \frac{2\pi}{7}, \operatorname{cis} \pm \frac{4\pi}{7}, \pm \operatorname{cis} \frac{6\pi}{7} \right\}$$

4. Start with the one known root and mark in the other five, each rotated $\frac{\pi}{3}$ from the previous.

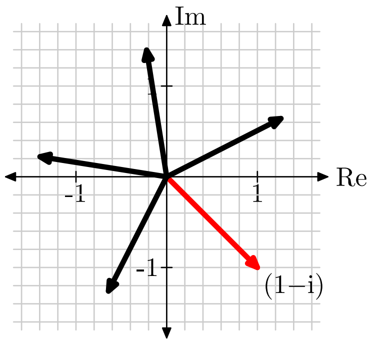



$$\sqrt{3} + \text{i} = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} \operatorname{cis} \left( \tan^{-1} \frac{1}{\sqrt{3}} \right)$$
$$= 2 \operatorname{cis} \frac{\pi}{6}$$

$\therefore z^6 = -64$ has roots

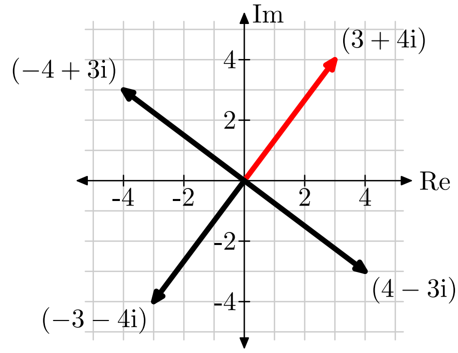
$$z \in \left\{ 2 \operatorname{cis} \pm \frac{\pi}{6}, 2 \operatorname{cis} \pm \frac{\pi}{2}, 2 \operatorname{cis} \pm \frac{5\pi}{6} \right\}$$

5. Start with the one known root and mark in the other four, each rotated $\frac{360}{5}^{\circ} = 72°$ from the previous.




$$1 - \text{i} = \sqrt{2} \operatorname{cis} -45°$$
$$\therefore \quad z \in \left\{ \sqrt{2} \operatorname{cis} -45°, \sqrt{2} \operatorname{cis} -117°, \sqrt{2} \operatorname{cis} 27°, \sqrt{2} \operatorname{cis} 99°, \sqrt{2} \operatorname{cis} 171° \right\}$$

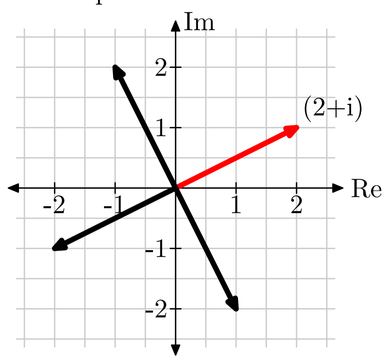
6. Start with the one known root and mark in the other four, each rotated 90° from the previous.




7. (a) $(2 + \text{i})^2 = 4 + 4\text{i} + \text{i} \quad 2$
$= 3 + 4\text{i}$

(b) $(2 + \text{i})^4 = \left((2 + \text{i})^2\right)^2$
$= (3 + 4\text{i})^2$
$= 9 + 24\text{i} + 16\text{i}^2$
$= -7 + 24\text{i}$

(c) We know that $z = 2 + \text{i}$ is a solution from the preceding work. Start with this solution and find the other three, each rotated 90° from the previous.




(d) Solutions are

$$z \in \{(2 + \text{i}), (-1 + 2\text{i}), (1 - 2\text{i}), (-2 - \text{i})\}$$

8. $k = (2 \operatorname{cis} 20°)^5$
$= (2 \operatorname{cis} 20°)(2 \operatorname{cis} 20°)(2 \operatorname{cis} 20°)^3$
$= (4 \operatorname{cis} 40°)(2 \operatorname{cis} 20°)^2(2 \operatorname{cis} 20°)$
$= (4 \operatorname{cis} 40°)(4 \operatorname{cis} 40°)(2 \operatorname{cis} 20°)$
$= (16 \operatorname{cis} 80°)(2 \operatorname{cis} 20°)$
$= 32 \operatorname{cis}(100)°$

(This is unnecessarily long-winded, but I haven't wanted to anticipate the next section. Students should be able to see how to do this in a single step.)

Now the other four solutions are simply 72° rotations of the first. (The first is also included for completeness.)

$$z \in \{2\operatorname{cis}20°, 2\operatorname{cis}92°, 2\operatorname{cis}164°, 2\operatorname{cis}(-52°), 2\operatorname{cis}(-124)°\}$$

9. The four solutions will be separated by 90°. From previous work it should be clear how this relates to the complex numbers in $a + b$i form so that no working is required. (See, for example, question 7.)

## Exercise 2B

It is useful for much of the work in this section to be familiar with Pascal's triangle and its role in binomial expansions:

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0: | | | | | | | 1 | | | | | | |
| 1: | | | | | | 1 | | 1 | | | | | |
| 2: | | | | | 1 | | 2 | | 1 | | | | |
| 3: | | | | 1 | | 3 | | 3 | | 1 | | | |
| 4: | | | 1 | | 4 | | 6 | | 4 | | 1 | | |
| 5: | | 1 | | 5 | | 10 | | 10 | | 5 | | 1 | |
| 6: | 1 | | 6 | | 15 | | 20 | | 15 | | 6 | | 1 |

etc.

1. To prove:

$$(\cos\theta + \mathrm{i}\sin\theta)^{-1} = \cos -\theta + \mathrm{i}\sin -\theta$$

Proof:

$$\begin{aligned}\text{L.H.S.} &= (\cos\theta + \mathrm{i}\sin\theta)^{-1}\\ &= \frac{1}{\cos\theta + \mathrm{i}\sin\theta}\\ &= \frac{\cos\theta - \mathrm{i}\sin\theta}{(\cos\theta + \mathrm{i}\sin\theta)(\cos\theta - \mathrm{i}\sin\theta)}\\ &= \frac{\cos\theta - \mathrm{i}\sin\theta}{\cos^2\theta - \mathrm{i}^2\sin^2\theta}\\ &= \frac{\cos\theta - \mathrm{i}\sin\theta}{\cos^2\theta + \sin^2\theta}\\ &= \cos\theta - \mathrm{i}\sin\theta\\ &= \cos -\theta + \mathrm{i}\sin -\theta\\ &= \text{R.H.S.}\end{aligned}$$

□

2. $$\begin{aligned}z^4 &= \cos\frac{4\pi}{6} + \mathrm{i}\sin\frac{4\pi}{6}\\ &= \cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\end{aligned}$$

3. $$\begin{aligned}z^5 &= 2^5\operatorname{cis}\frac{5\pi}{6}\\ &= 32\operatorname{cis}\frac{5\pi}{6}\end{aligned}$$

4. $$\begin{aligned}z &= 3^5\operatorname{cis}\frac{5\pi}{3}\\ &= 243\operatorname{cis}\left(-\frac{\pi}{3}\right)\\ &= 243\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)\end{aligned}$$

(matching the style of our answer to that of the question.)

5. $$\begin{aligned}\cos 2\theta + \mathrm{i}\sin 2\theta &= (\cos\theta + \mathrm{i}\sin\theta)^2\\ &= \cos^2\theta + 2\mathrm{i}\sin\theta\cos\theta + \mathrm{i}^2\sin^2\theta\\ &= \cos^2\theta - \sin^2\theta + 2\mathrm{i}\sin\theta\cos\theta\end{aligned}$$

Now equating real and imaginary parts we obtain

$$\begin{aligned}\cos 2\theta &= \sin^2\theta - \cos^2\theta\\ \text{and}\ \sin 2\theta &= 2\sin\theta\cos\theta\end{aligned}$$

6. $$\begin{aligned}\cos 3\theta + \mathrm{i}\sin 3\theta &= (\cos\theta + \mathrm{i}\sin\theta)^3\\ &= \cos^3\theta + 3\mathrm{i}\sin\theta\cos^2\theta\\ &\quad + 3\mathrm{i}^2\sin^2\theta\cos\theta + \mathrm{i}^3\sin^3\theta\\ &= \cos^3\theta + 3\mathrm{i}\sin\theta\cos^2\theta\\ &\quad - 3\sin^2\theta\cos\theta - \mathrm{i}\sin^3\theta\\ &= \cos^3\theta - 3\sin^2\theta\cos\theta\\ &\quad + \mathrm{i}(3\sin\theta\cos^2\theta - \sin^3\theta)\end{aligned}$$

Now equating real and imaginary parts we obtain

$$\begin{aligned}\sin 3\theta &= 3\sin\theta\cos^2\theta - \sin^3\theta\\ \text{and}\ \cos 3\theta &= \cos^3\theta - 3\sin^2\theta\\ &= \cos^3\theta - 3(1 - \cos^2\theta)\\ &= \cos^3\theta + 3\cos^2\theta - 3\end{aligned}$$

7. $$\begin{aligned}\cos 5\theta + \mathrm{i}\sin 5\theta &= (\cos\theta + \mathrm{i}\sin\theta)^5 \\ &= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta \\ &\quad - 10\mathrm{i}\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + \mathrm{i}\sin^5\theta \\ &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\ &\quad + \mathrm{i}(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)\end{aligned}$$
$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$
$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

8. $$\begin{aligned}(1+\mathrm{i})^6 &= \left(\sqrt{2}\,\mathrm{cis}\,\frac{\pi}{4}\right)^6 \\ &= \left(\sqrt{2}\right)^6\mathrm{cis}\,\frac{6\pi}{4} \\ &= 8\,\mathrm{cis}\,\frac{3\pi}{2} \\ &= 8\,\mathrm{cis}\,\frac{-\pi}{2}\end{aligned}$$

9. $$\begin{aligned}(\sqrt{3}+\mathrm{i})^5 &= \left(\sqrt{\left(\sqrt{3}\right)^2 + 1^2}\,\mathrm{cis}\left(\tan^{-1}\frac{1}{\sqrt{3}}\right)\right)^5 \\ &= \left(2\,\mathrm{cis}\,\frac{\pi}{6}\right)^5 \\ &= 2^5\,\mathrm{cis}\,\frac{5\pi}{6} \\ &= 32\,\mathrm{cis}\,\frac{5\pi}{6}\end{aligned}$$

10. $$\begin{aligned}(-3+3\sqrt{3}\mathrm{i})^4 &= \left(3(-1+\sqrt{3})\right)^4 \\ &= 3^4(-1+\sqrt{3})^4 \\ &= 81\left(2\,\mathrm{cis}\left(\tan^{-1}\frac{\sqrt{3}}{-1}\right)\right)^4 \\ &= 81\left(2\,\mathrm{cis}\,\frac{2\pi}{3}\right)^4 \\ &= 81\times 2^4\,\mathrm{cis}\,\frac{8\pi}{3} \\ &= 1296\,\mathrm{cis}\,\frac{2\pi}{3}\end{aligned}$$

11. $$\begin{aligned}\left(4-4\sqrt{3}\right)^{\frac{1}{3}} &= \left(\sqrt{16+48}\,\mathrm{cis}\left(\tan^{-1}\frac{-4\sqrt{3}}{4}\right)\right)^{\frac{1}{3}} \\ &= \left(8\,\mathrm{cis}\left(-\frac{\pi}{3}\right)\right)^{\frac{1}{3}} \\ &= 2\,\mathrm{cis}\left(-\frac{\pi}{9}\right)\end{aligned}$$

The other two cube roots are rotated by $\frac{2\pi}{3}$, i.e. $2\,\mathrm{cis}\left(-\frac{7\pi}{9}\right)$ and $2\,\mathrm{cis}\,\frac{5\pi}{9}$.

12. $$\begin{aligned}z^4 &= 16\mathrm{i} \\ &= 16\,\mathrm{cis}\,\frac{\pi}{2} \\ z &= \left(16\,\mathrm{cis}\,\frac{\pi}{2}\right)^{\frac{1}{4}} \\ &= 2\,\mathrm{cis}\,\frac{\pi}{8}\end{aligned}$$

The other three roots are rotated by $\frac{\pi}{2}$, i.e. $2\,\mathrm{cis}\left(-\frac{3\pi}{8}\right)$, $2\,\mathrm{cis}\left(-\frac{7\pi}{8}\right)$ and $2\,\mathrm{cis}\,\frac{5\pi}{8}$.

13. $$\sqrt{\left(8\sqrt{2}\right)^2 + \left(8\sqrt{2}\right)^2} = 16$$
$$\begin{aligned}z^4 &= 16\,\mathrm{cis}\,\frac{3\pi}{4} \\ z &= 2\,\mathrm{cis}\,\frac{3\pi}{16}\end{aligned}$$

The other three roots are rotated by $\frac{\pi}{2}$, i.e. $2\,\mathrm{cis}\left(-\frac{5\pi}{16}\right)$, $2\,\mathrm{cis}\left(-\frac{13\pi}{16}\right)$ and $2\,\mathrm{cis}\,\frac{11\pi}{16}$.

14. $$\begin{aligned}z^4 + 4 &= 0 \\ z^4 &= -4 \\ &= 4\,\mathrm{cis}\,\pi \\ z &= \sqrt{2}\,\mathrm{cis}\,\frac{\pi}{4}\end{aligned}$$

The other three roots are rotated by $\frac{\pi}{2}$, i.e. $\sqrt{2}\,\mathrm{cis}\left(-\frac{\pi}{4}\right)$, $\sqrt{2}\,\mathrm{cis}\left(-\frac{3\pi}{4}\right)$ and $\sqrt{2}\,\mathrm{cis}\,\frac{3\pi}{4}$.

15. $$\begin{aligned}|z_1| &= \frac{1}{2}\sqrt{\left(\sqrt{2}\right)^2 + \left(\sqrt{6}\right)^2} \\ &= \frac{1}{2}\sqrt{8} \\ &= \sqrt{2} \\ \arg(z_1) &= \tan^{-1}\frac{\sqrt{6}}{\sqrt{2}} \\ &= \tan^{-1}\sqrt{3} \\ &= \frac{\pi}{3} \\ \therefore\quad z_1 &= \sqrt{2}\,\mathrm{cis}\,\frac{\pi}{3} \\ |z_2| &= \frac{1}{2}\sqrt{\left(\sqrt{6}\right)^2 + \left(\sqrt{2}\right)^2} \\ &= \sqrt{2} \\ \arg(z_2) &= \tan^{-1}\frac{\sqrt{2}}{\sqrt{6}} \\ &= \tan^{-1}\frac{1}{\sqrt{3}} \\ &= \frac{\pi}{6} \\ \therefore\quad z_2 &= \sqrt{2}\,\mathrm{cis}\,\frac{\pi}{6} \\ z_1^6 &= \left(\sqrt{2}\right)^6\mathrm{cis}\,\frac{6\pi}{3} \\ &= 8\,\mathrm{cis}\,0 \\ z_2^3 &= \left(\sqrt{2}\right)^3\mathrm{cis}\,\frac{3\pi}{6} \\ &= 2\sqrt{2}\,\mathrm{cis}\,\frac{\pi}{2} \\ z_3^4 &= 2^4\,\mathrm{cis}\,\frac{4\pi}{8} \\ &= 16\,\mathrm{cis}\,\frac{\pi}{2} \\ \frac{z_1^6 z_2^3}{z_3^4} &= \frac{(8\,\mathrm{cis}\,0)\left(2\sqrt{2}\,\mathrm{cis}\,\frac{\pi}{2}\right)}{16\,\mathrm{cis}\,\frac{\pi}{2}} \\ &= \frac{16\sqrt{2}\,\mathrm{cis}\,\frac{\pi}{2}}{16\,\mathrm{cis}\,\frac{\pi}{2}} \\ &= \sqrt{2}\end{aligned}$$

## Exercise 2C

1–6 No working required.

7. $$3e^{\frac{4\pi i}{3}} = 3\operatorname{cis}\frac{4\pi}{3} = 3\operatorname{cis}\left(-\frac{2\pi}{3}\right)$$

8. $$2e^{2+\frac{\pi i}{3}} = 2\left(e^2\right)\left(e^{\frac{\pi i}{3}}\right) = 2e^2\operatorname{cis}\frac{\pi}{3}$$

9. $$\begin{aligned}9\sqrt{2}e^{\frac{-\pi i}{4}} &= 9\sqrt{2}\operatorname{cis}\frac{-\pi}{4}\\ &= 9\sqrt{2}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)\\ &= 9\sqrt{2}\left(\cos\frac{\pi}{4} i\sin\frac{\pi}{4}\right)\\ &= 9\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)\\ &= 9 - 9i\end{aligned}$$

10. $$\begin{aligned}2e^{\frac{-5\pi i}{6}} &= 2\operatorname{cis}\frac{-5\pi}{6}\\ &= 2\left(\cos\frac{-5\pi}{6} + i\sin\frac{-5\pi}{6}\right)\\ &= 2\left(-\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)\\ &= 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)\\ &= -\sqrt{3} - i\end{aligned}$$

11. $$\begin{aligned}10e^{\frac{2\pi i}{3}} &= 10\operatorname{cis}\frac{2\pi}{3}\\ &= 10\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\\ &= 10\left(-\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\\ &= 10\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\\ &= (-5 + 5\sqrt{3}i)\end{aligned}$$

12. $$\begin{aligned}10e^{\frac{3\pi i}{4}} &= 10\operatorname{cis}\frac{3\pi}{4}\\ &= 10\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\\ &= 10\left(-\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\\ &= 10\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\\ &= -5\sqrt{2} + 5\sqrt{2}i\end{aligned}$$

13. $$\begin{aligned}|6\sqrt{3} + 6i| &= \sqrt{6^2 \times 3 + 6^2}\\ &= 12\\ \arg(6\sqrt{3} + 6i) &= \tan^{-1}\frac{6}{6\sqrt{3}}\\ &= \frac{\pi}{6}\\ 6\sqrt{3} + 6i &= 12e^{\frac{i\pi}{6}}\end{aligned}$$

14. $$\begin{aligned}|-1 - \sqrt{3}i| &= \sqrt{1+3}\\ &= 2\\ \arg(-1 - \sqrt{3}i) &= \tan^{-1} -\sqrt{3}-1\\ &= -\frac{2\pi}{3} \qquad \text{(3rd quadrant)}\\ -1 - \sqrt{3}i &= 2e^{-\frac{2i\pi}{3}}\end{aligned}$$

15. $$3i = 3\operatorname{cis}\frac{\pi}{2} = 3e^{\frac{i\pi}{2}}$$

16. $$-2 = 2(-1) = 2e^{i\pi}$$

17. $$\begin{aligned}r &= \sqrt{5^2 + 12^2}\\ &= 13\\ \theta &= \tan^{-1}\frac{12}{5}\\ &= 1.18 \quad \text{(2d.p.)}\\ 5 + 12i &= 13e^{1.18i}\end{aligned}$$

18. $$\begin{aligned}r &= \sqrt{2^2 + 7^2}\\ &= \sqrt{53}\\ \theta &= \tan^{-1}\frac{-7}{2}\\ &= -1.29 \quad \text{(2d.p.)}\\ 2 - 7i &= \sqrt{53}e^{-1.29i}\end{aligned}$$

19. To find a complex conjugate, keep the modulus unchanged, and take the opposite of the argument. Thus if $z = re^{i\theta}$ then $\bar{z} = re^{-i\theta}$.

20. (a) $$\begin{aligned}\bar{z} &= 4e^{\frac{-\pi i}{3}}\\ &= 4\cos\frac{\pi}{3} - 4i\sin\frac{\pi}{3}\\ &= 2 - 2\sqrt{3}i\end{aligned}$$

(b) $$\begin{aligned}z^2 &= \left(4e^{\frac{\pi i}{3}}\right)^2\\ &= 16e^{\frac{2\pi i}{3}}\\ &= -16\cos\frac{\pi}{3} + 16i\sin\frac{\pi}{3}\\ &= -8 + 8\sqrt{3}i\end{aligned}$$

(c) $$\begin{aligned}\frac{1}{z} &= \left(4e^{\frac{\pi i}{3}}\right)^{-1}\\ &= \frac{1}{4}e^{\frac{-\pi i}{3}}\\ &= \frac{1}{4}\cos\frac{\pi}{3} - \frac{1}{4}i\sin\frac{\pi}{3}\\ &= \frac{1}{8} - \frac{\sqrt{3}}{8}i\end{aligned}$$

21. (a) $$\begin{aligned}\text{LHS} &= \frac{1}{\operatorname{cis} n\theta}\\ &= (\operatorname{cis} n\theta)^{-1}\\ &= \operatorname{cis}(-n\theta)\\ &= \text{RHS}\end{aligned}$$

□

(b) $$\begin{aligned}\text{LHS} &= \frac{1}{\operatorname{cis} n\theta}\\ &= \frac{1}{e^{n\theta i}}\\ &= e^{-n\theta i}\\ &= \operatorname{cis}(-n\theta)\\ &= \text{RHS}\end{aligned}$$

□

(c) $$\begin{aligned}\text{LHS} &= \frac{1}{\operatorname{cis} n\theta}\\ &= \frac{1}{\cos n\theta + i\sin n\theta}\\ &= \frac{\cos n\theta - i\sin n\theta}{(\cos n\theta + i\sin n\theta)(\cos n\theta - i\sin n\theta)}\\ &= \frac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta - i^2\sin^2 n\theta}\\ &= \frac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta + \sin^2 n\theta}\\ &= \cos n\theta - i\sin n\theta\\ &= \cos(-n\theta) + i\sin(-n\theta)\\ &= \operatorname{cis}(-n\theta)\\ &= \text{RHS}\end{aligned}$$

□

22. (a) First cosine:

$$\begin{aligned}\text{RHS} &= \frac{e^{i\theta} + e^{-i\theta}}{2}\\ &= \frac{\cos\theta + i\sin\theta + \cos -\theta + i\sin -\theta}{2}\\ &= \frac{\cos\theta + i\sin\theta + \cos\theta - i\sin\theta}{2}\\ &= \frac{2\cos\theta}{2}\\ &= \cos\theta\\ &= \text{LHS}\end{aligned}$$

□

then sine:

$$\begin{aligned}\text{RHS} &= \frac{e^{i\theta} - e^{-i\theta}}{2i}\\ &= \frac{\cos\theta + i\sin\theta - \cos -\theta - i\sin -\theta}{2i}\\ &= \frac{\cos\theta + i\sin\theta - \cos\theta + i\sin\theta}{2i}\\ &= \frac{2i\sin\theta}{2i}\\ &= \sin\theta\\ &= \text{LHS}\end{aligned}$$

□

23. (a) No working required: simple application of the chain rule.

(b) No working required: simple application of the chain rule.

(c) $$\begin{aligned}\frac{d}{d\theta}\left(e^{i\theta}e^2\right) &= \frac{d}{d\theta}\left(e^{i\theta+2}\right)\\ &= ie^{2+i\theta}\end{aligned}$$

Alternatively, bearing in mind that $e^2$ is a constant, we can do

$$\frac{d}{d\theta}\left(e^{i\theta}e^2\right) = ie^2e^{i\theta}$$

in a single step.

24. (a) $$\begin{aligned}\int e^{2ix}\,dx &= \frac{e^{2ix}}{2i} + c\\ &= \frac{ie^{2ix}}{-2} + c\\ &= -\frac{1}{2}ie^{2ix} + c\end{aligned}$$

(b) $$\begin{aligned}\int e^{3ix}\,dx &= \frac{e^{3ix}}{3i} + c\\ &= \frac{ie^{3ix}}{-3} + c\\ &= -\frac{1}{3}ie^{3ix} + c\end{aligned}$$

(c) $$\begin{aligned}\int e^{3+ix}\,dx &= \frac{e^{3+ix}}{i} + c\\ &= \frac{ie^{3+ix}}{-1} + c\\ &= -ie^{3+ix} + c\end{aligned}$$

25.
$$\begin{aligned}
&\int e^x \cos x \, dx + i \int e^x \sin x \, dx \\
&= \int e^x (\cos x + i \sin x) \, dx \\
&= \int e^x e^{ix} \, dx \\
&= \int e^{(1+i)x} \, dx \\
&= \frac{e^{(1+i)x}}{1+i} + c \\
&= \frac{(1-i)e^{(1+i)x}}{(1-i)(1+i)} + c \\
&= \frac{(1-i)e^{x+ix}}{1-i^2} + c \\
&= \frac{(1-i)e^x e^{ix}}{2} + c \\
&= \frac{e^x}{2}(1-i) \operatorname{cis} x + c \\
&= \frac{e^x}{2}(1-i)(\cos x + i \sin x) + c \\
&= \frac{e^x}{2} \left((\cos x + i \sin x) - i(\cos x + i \sin x)\right) + c \\
&= \frac{e^x}{2} \left(\cos x + i \sin x - i \cos x - i^2 \sin x)\right) + c \\
&= \frac{e^x}{2} \left(\cos x + i \sin x - i \cos x + \sin x)\right) + c \\
&= \frac{e^x(\sin x + \cos x)}{2} + i\frac{e^x(\sin x - \cos x)}{2} + c
\end{aligned}$$

Equating real and imaginary parts (and bearing in mind that the constant of integration can also have real and imaginary parts) gives

$$\int e^x \cos x \, dx = \frac{e^x(\sin x + \cos x)}{2} + c$$

$$\text{and} \quad \int e^x \sin x \, dx = \frac{e^x(\sin x - \cos x)}{2} + c$$

## Miscellaneous Exercise 2

1.
$$\begin{aligned}
\cos\theta + i\sin\theta &= e^{i\theta} \\
\therefore \quad \cos n\theta + i \sin n\theta &= e^{in\theta} \\
&= \left(e^i\theta\right)^n \\
&= (\cos\theta + i\sin\theta)^n
\end{aligned}$$

2.
$$\begin{aligned}
\cos\theta + i\sin\theta &= e^{i\theta} \\
\therefore \quad \cos(-n\theta) + i\sin(-n\theta) &= e^{-in\theta} \\
&= \left(e^i\theta\right)^{-n} \\
&= (\cos\theta + i\sin\theta)^{-n}
\end{aligned}$$

3. (a)
$$\begin{aligned}
z + \bar{z} &= a + bi + a - bi \\
&= 2a
\end{aligned}$$

(b)
$$\begin{aligned}
z + \bar{z} &= a + bi - (a - bi) \\
&= 2bi
\end{aligned}$$

(c)
$$\begin{aligned}
z\bar{z} &= (a + bi)(a - bi) \\
&= a^2 - b^2i^2 \\
&= a^2 + b^2
\end{aligned}$$

(d)
$$\begin{aligned}
\frac{z}{\bar{z}} &= \frac{z^2}{z\bar{z}} \\
&= \frac{(a+bi)^2}{a^2+b^2} \\
&= \frac{a^2 + 2abi + b^2i^2}{a^2+b^2} \\
&= \frac{a^2 - b^2 + 2abii^2}{a^2+b^2}
\end{aligned}$$

4.
$$\begin{aligned}
\frac{z}{\bar{z}} &= \frac{re^{i\theta}}{re^{-i\theta}} \\
&= \frac{e^{i\theta}}{e^{-i\theta}} \\
&= e^{i\theta + i\theta} \\
&= e^{2i\theta}
\end{aligned}$$

5. Most of these require no working. About half of them need the chain rule, but in such a straightforward way that you should be able to differentiate them in a single line.

(i) $$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x+1}{x-1} = \frac{(x-1)-(x+1)}{(x-1)^2} = -\frac{2}{(x-1)^2}$$

6. (a) $$3y + 3x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 7$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}(3x+2y) = 7 - 3y$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{7-3y}{3x+2y}$$

(b) $$2xy + x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 3x^2 = y + x\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} - x\frac{\mathrm{d}y}{\mathrm{d}x} = y - 2xy - 3x^2$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}(x^2 - x) = y - 2xy - 3x^2$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y-2xy-3x^2}{x(x-1)}$$

(c) $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}$$

(d) $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{3t^2}{6t-2}$$

7. Let $P(n)$ be the *proposition* that $5^n + 7\times 13^n$ is a multiple of 8. The initial case, where $n = 1$:

$$5^1 + 7\times 13^1 = 5 + 7\times 13 = 96 = 8(12)$$

The statement is true for the initial case: we have established $P(1)$.

Assume $P(k)$, that is, that the statement is true for $n = k$, so.

$$5^k + 7\times 13^k = 8a$$

for some integer $a$.

Then for $n = k+1$

$$\begin{aligned}5^{k+1} &+ 7\times 13^{k+1}\\ &= 5\times 5^k + 13\times 7\times 13^k\\ &= 5\times 5^k + (5+8)\times 7\times 13^k\\ &= 5\times 5^k + 5\times 7\times 13^k + 8\times 7\times 13^k\\ &= 5(5^k + 7\times 13^k) + 8\times 7\times 13^k\\ &= 5(8a) + 8\times 7\times 13^k\\ &= 8(5a + 7\times 13^k)\end{aligned}$$

which is a multiple of 8.

Thus $P(k) \implies P(k+1)$ (i.e. if the statement is true for $n = k$ it is also true for $n = k+1$).

Hence since the statement is true for $n = 1$ it follows by mathematical induction that it is true for all integer $n \geq 1$. □

8. $u = 2x + 3 \qquad x = \frac{u-3}{2}$

$\mathrm{d}u = 2\,\mathrm{d}x \qquad \mathrm{d}x = \frac{\mathrm{d}u}{2}$

$$\begin{aligned}\int \frac{5x}{\sqrt{2x+3}}\,\mathrm{d}x &= \int \frac{5(u-3)}{2\sqrt{u}}\frac{\mathrm{d}u}{2}\\ &= \frac{5}{4}\int \frac{u-3}{\sqrt{u}}\,\mathrm{d}u\\ &= \frac{5}{4}\int\left(\sqrt{u} - \frac{3}{\sqrt{u}}\right)\mathrm{d}u\\ &= \frac{5}{4}\left(\frac{2}{3}u^{\frac{3}{2}} - 3\left(2u^{\frac{1}{2}}\right)\right) + c\\ &= \frac{5}{4}\left(\frac{2}{3}u^{\frac{3}{2}} - 3\left(2u^{\frac{1}{2}}\right)\right) + c\\ &= \frac{5}{6}\left(u^{\frac{3}{2}} - 9\sqrt{u}\right) + c\\ &= \frac{5}{6}\sqrt{u}(u-9) + c\\ &= \frac{5}{6}\sqrt{2x+3}(2x+3-9) + c\\ &= \frac{5}{6}\sqrt{2x+3}(2x-6) + c\\ &= \frac{5}{3}\sqrt{2x+3}(x-3) + c\end{aligned}$$

9. $$\begin{aligned}3|z-5| &= 2|z+5\mathrm{i}|\\ 3|x+\mathrm{i}y-5| &= 2|x+\mathrm{i}y+5\mathrm{i}|\\ 3|x-5+\mathrm{i}y| &= 2|x+(y+5)\mathrm{i}|\\ 3^2|x-5+\mathrm{i}y|^2 &= 2^2|x+(y+5)\mathrm{i}|^2\\ 9\left((x-5)^2+y^2\right) &= 4\left(x^2+(y+5)^2\right)\\ 9\left(x^2-10x+25+y^2\right) &= 4\left(x^2+y^2+10y+25\right)\\ 9x^2-90x+225+9y^2 &= 4x^2+4y^2+40y+100\\ 5x^2-90x+5y^2-40y &= -125\\ x^2-18x+y^2-8y &= -25\\ (x-9)^2-81+(y-4)^2-16 &= -25\\ (x-9)^2+(y-4)^2 &= 72\end{aligned}$$

□

# Chapter 3

## Exercise 3A

1. No working needed.
2. No working needed.
3. (a) A + B cannot be determined because the matrices are not the same size.

   (b) $A + C = \begin{bmatrix} 1+2 & 2-3 \\ 0+1 & -4-5 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & -9 \end{bmatrix}$

   (c) $C - A = \begin{bmatrix} 2-1 & -3-2 \\ 1-0 & -5+4 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix}$

   (d) $2D = \begin{bmatrix} 2\times 3 \\ 2\times 1 \\ 2\times -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ -4 \end{bmatrix}$

   (e) $3B = \begin{bmatrix} 3\times 3 & 3\times -1 \\ 3\times 2 & 3\times 4 \\ 3\times 0 & 3\times 3 \end{bmatrix} = \begin{bmatrix} 9 & -3 \\ 6 & 12 \\ 0 & 9 \end{bmatrix}$

   (f) B + D cannot be determined because the matrices are not the same size.

   (g) $2A = \begin{bmatrix} 2\times 1 & 2\times 2 \\ 2\times 0 & 2\times -4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & -8 \end{bmatrix}$

   (h) $2A - C = \begin{bmatrix} 2-2 & 4+3 \\ 0-1 & -8+5 \end{bmatrix}$
   $= \begin{bmatrix} 0 & 7 \\ -1 & -3 \end{bmatrix}$

4. (a) $P + Q = \begin{bmatrix} 3+2 & 2+1 & -1+0 \\ 1+0 & 4-1 & 3+0 \end{bmatrix}$
   $= \begin{bmatrix} 5 & 3 & -1 \\ 1 & 3 & 3 \end{bmatrix}$

   (b) $Q - P = \begin{bmatrix} 2-3 & 1-2 & 0+1 \\ 0-1 & -1-4 & 0-3 \end{bmatrix}$
   $= \begin{bmatrix} -1 & -1 & 1 \\ -1 & -5 & -3 \end{bmatrix}$

   (c) $3R = \begin{bmatrix} 3\times 1 & 3\times 2 & 3\times 1 \\ 3\times 2 & & \\ 3\times 1 & 3\times 2 & \end{bmatrix}$
   $= \begin{bmatrix} 3 & 6 & 3 \\ 6 & 3 & 6 \end{bmatrix}$

   (d) $3P - 2Q$
   $= \begin{bmatrix} 3\times 3-2\times 2 & 3\times 2-2\times 1 & 3\times -1-2\times 0 \\ 3\times 1-2\times 0 & 3\times 4-2\times -1 & 3\times 3-2\times 0 \end{bmatrix}$
   $= \begin{bmatrix} 5 & 4 & -3 \\ 3 & 14 & 9 \end{bmatrix}$

5. (a) A + B cannot be determined because the matrices are not the same size.

   (b) $3A = \begin{bmatrix} 3\times 2 & 3\times 4 \\ 3\times 1 & 3\times 3 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ 3 & 9 \end{bmatrix}$

   (c) B+2C
   $= \begin{bmatrix} 2+2\times 3 & 1+2\times 1 & 3+2\times 4 \end{bmatrix}$
   $= \begin{bmatrix} 8 & 3 & 11 \end{bmatrix}$

   (d) C + D cannot be determined because the matrices are not the same size.

6. (a) A + B cannot be determined because the matrices are not the same size.

   (b) A+C
   $= \begin{bmatrix} 1+5 & 3+1 & 0+3 & 1-1 \\ 0+2 & 1+1 & 2+4 & 3+3 \\ 0+1 & 0+5 & 1+2 & 4+0 \end{bmatrix}$
   $= \begin{bmatrix} 6 & 4 & 3 & 0 \\ 2 & 2 & 6 & 6 \\ 1 & 5 & 3 & 4 \end{bmatrix}$

   (c) $2B = \begin{bmatrix} 2\times 3 & 2\times 1 & 2\times 4 \\ 2\times 2 & 2\times 1 & 2\times -3 \\ 2\times 0 & 2\times 1 & 2\times 2 \\ 2\times 1 & 2\times 0 & 2\times 0 \end{bmatrix}$
   $= \begin{bmatrix} 6 & 2 & 8 \\ 4 & 2 & -6 \\ 0 & 2 & 4 \\ 2 & 0 & 0 \end{bmatrix}$

   (d) $5A - C$
   $= \begin{bmatrix} 5\times 1-5 & 5\times 3-1 & 5\times 0-3 & 5\times 1+1 \\ 5\times 0-2 & 5\times 1-1 & 5\times 2-4 & 5\times 3-3 \\ 5\times 0-1 & 5\times 0-5 & 5\times 1-2 & 5\times 4-0 \end{bmatrix}$
   $= \begin{bmatrix} 0 & 14 & -3 & 6 \\ -2 & 4 & 6 & 12 \\ -1 & -5 & 3 & 20 \end{bmatrix}$

7. No working required. ‘Yes’ or ‘No’ for additions or subtractions is determined by whether the matrices specified are the same size. Multiplication by a scalar can always be determined.

8. No working required.

9. No working required.

10. $3A - 2C = B$
    $2C = 3A - B$
    $C = \frac{1}{2}(3A - B)$
    $= \frac{1}{2}\begin{bmatrix} 2 & 4 & -6 \\ 2 & 0 & -4 \end{bmatrix}$
    $= \begin{bmatrix} 1 & 2 & -3 \\ 1 & 0 & -2 \end{bmatrix}$

11. (a) Add the four individual game matrices.

    (b) Multiply the result from (a) by $\frac{1}{4}$.

12. Let the two matrices provided by A and B. The

required forecast is given by

$$\begin{aligned} &1.1(\mathrm{A}+\mathrm{B}) \\ &= 1.1\begin{bmatrix} 5600 & 1750 & 2320 & 1770 & 4250 \\ 2840 & 1270 & 1370 & 1020 & 2720 \\ 5050 & 1470 & 2820 & 1280 & 2700 \\ 2190 & 940 & 1520 & 840 & 1780 \end{bmatrix} \\ &= \begin{bmatrix} 6160 & 1925 & 2552 & 1947 & 4675 \\ 3124 & 1397 & 1507 & 1122 & 2992 \\ 5555 & 1617 & 3102 & 1408 & 2970 \\ 2409 & 1034 & 1672 & 924 & 1958 \end{bmatrix} \end{aligned}$$

13. No working needed. (Just substitute row and column into the expression to find the value for each element.)

14. No working needed. (Just substitute row and column into the expression to find the value for each element.)

## Exercise 3B

1. $$\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1\times2+2\times1 & 1\times3+2\times3 \end{bmatrix} = \begin{bmatrix} 4 & 9 \end{bmatrix}$$

2. Not possible: the number of columns in the first matrix (2) does not equal the number of rows in the second (1).

3. $$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 2\times1-1\times0 & 2\times4-1\times-2 \\ 1\times1+0\times0 & 1\times4+0\times-2 \end{bmatrix} = \begin{bmatrix} 2 & 10 \\ 1 & 4 \end{bmatrix}$$

4. $$\begin{bmatrix} 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3\times1+1\times4 \end{bmatrix} = \begin{bmatrix} 7 \end{bmatrix}$$

5. $$\begin{bmatrix} 1 \\ 4 \end{bmatrix}\begin{bmatrix} 3 & 1 \end{bmatrix} = \begin{bmatrix} 1\times3 & 1\times1 \\ 4\times3 & 4\times1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 12 & 4 \end{bmatrix}$$

6. $$\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2\times2-3\times-3 & 2\times1-3\times2 \\ -1\times2+4\times-3 & -1\times1+4\times2 \end{bmatrix} = \begin{bmatrix} 13 & -4 \\ -14 & 7 \end{bmatrix}$$

7. $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1\times2+0\times1 & 1\times3+0\times-1 \\ 0\times2+1\times1 & 0\times3+1\times-1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$$

8. $$\begin{bmatrix} 1 & 4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\times1+4\times0 & 1\times0+4\times1 \\ -1\times1+4\times0 & -1\times0+3\times1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -1 & 3 \end{bmatrix}$$

9. $$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 0\times2+0\times4 & 0\times1+0\times5 \\ 0\times2+0\times4 & 0\times1+0\times5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

10. $$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 3\times2+1\times-5 & 3\times-1+1\times3 \\ 5\times2+2\times-5 & 5\times-1+2\times3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

11. $$\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 8\times2-5\times3 & 8\times5-5\times8 \\ -3\times2+2\times3 & -3\times5+2\times8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

12. $$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 1.5 \end{bmatrix} = \begin{bmatrix} 3\times0.5+1\times-0.5 & 3\times-0.5+1\times1.5 \\ 1\times0.5+1\times-0.5 & 1\times-0.5+1\times1.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

13. $\begin{bmatrix} 1 & 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$
$= \begin{bmatrix} 1\times 2+2\times 1+1\times 2+2\times 1 \end{bmatrix}$
$= \begin{bmatrix} 8 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 3 & -1 & 0 \\ 2 & 2 & 2 \\ 1 & 4 & 1 \end{bmatrix}$
$= \begin{bmatrix} 1+2 & 0+2 & 1+2 \\ 3+1 & -1+4 & 0+1 \end{bmatrix}$
$= \begin{bmatrix} 3 & 2 & 3 \\ 4 & 3 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 5 \\ 5 & 1 & -1 \end{bmatrix}$
$= \begin{bmatrix} 1+0 & 0+0 & 5+0 \\ 0+10 & 0+2 & 0-2 \\ 1+5 & 0+1 & 5-1 \end{bmatrix}$
$= \begin{bmatrix} 1 & 0 & 5 \\ 10 & 2 & -2 \\ 6 & 1 & 4 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 3 & 1 \\ 3 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 1 \\ -3 & -2 \end{bmatrix}$
$= \begin{bmatrix} 1+12-3 & 2+3-2 \\ 3+0+6 & 6+0+4 \end{bmatrix}$
$= \begin{bmatrix} 10 & 3 \\ 9 & 10 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+4+9 \\ 4+10+18 \end{bmatrix}$
$= \begin{bmatrix} 14 \\ 32 \end{bmatrix}$

18. $\begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ 3 & 1 & 4 \end{bmatrix}$
$= \begin{bmatrix} 2+0+0 & 2+2+0 & -2+3+0 \\ -1+0+6 & -1+6+2 & 1+9+8 \\ 0+0+12 & 0+4+4 & 0+6+16 \end{bmatrix}$
$= \begin{bmatrix} 2 & 4 & 1 \\ 5 & 7 & 18 \\ 12 & 8 & 22 \end{bmatrix}$

19. (a) $AB = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$
$= \begin{bmatrix} 0+0+0 & 1+0+1 & 2+0-1 \\ 0+0+0 & 2+0-1 & 4+0+1 \\ 0+2+0 & 0+1-1 & 0+0+1 \end{bmatrix}$
$= \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 5 \\ 2 & 0 & 1 \end{bmatrix}$

(b) $BA = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$
$= \begin{bmatrix} 0+2+0 & 0+0+2 & 0+1+2 \\ 2+2+0 & 0+0+0 & -2+1+0 \\ 0-2+0 & 0+0+1 & 0-1+1 \end{bmatrix}$
$= \begin{bmatrix} 2 & 2 & 3 \\ 4 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix}$

(c) $A^2 = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$
$= \begin{bmatrix} 1+0+0 & 0+0-1 & -1+0-1 \\ 2+0+0 & 0+0+1 & -2+0+1 \\ 0+2+0 & 0+0+1 & 0+1+1 \end{bmatrix}$
$= \begin{bmatrix} 1 & -1 & -2 \\ 2 & 1 & -1 \\ 2 & 1 & 2 \end{bmatrix}$

(d) $B^2 = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$
$= \begin{bmatrix} 0+2+0 & 0+1-2 & 0+0+2 \\ 0+2+0 & 2+1+0 & 4+0+0 \\ 0-2+0 & 0-1-1 & 0+0+1 \end{bmatrix}$
$= \begin{bmatrix} 2 & -1 & 2 \\ 2 & 3 & 4 \\ -2 & -2 & 1 \end{bmatrix}$

20. It's probably simplest to refer to 19(a) and 19(b) above.

21. (a) $(AB)C = \begin{bmatrix} 3 & -1 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$
$= \begin{bmatrix} 4 & 5 \\ -2 & -7 \end{bmatrix}$
$A(BC) = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 7 \\ 1 & -1 \end{bmatrix}$
$= \begin{bmatrix} 4 & 5 \\ -2 & -7 \end{bmatrix}$

(b) $(AB)C = \begin{bmatrix} 5 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$
$= \begin{bmatrix} 4 & 5 \end{bmatrix}$
$A(BC) = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$
$= \begin{bmatrix} 4 & 5 \end{bmatrix}$

22. (a) $A(B+C) = \begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$
$= \begin{bmatrix} 1 & 8 \\ 4 & 8 \end{bmatrix}$
$AB + AC = \begin{bmatrix} -2 & 3 \\ -4 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 8 & 4 \end{bmatrix}$
$= \begin{bmatrix} 1 & 8 \\ 4 & 8 \end{bmatrix}$

(b) $\text{A(B+C)} = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

$= \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

$\text{AB+AC} = \begin{bmatrix} 6 \\ -7 \end{bmatrix} + \begin{bmatrix} -2 \\ 7 \end{bmatrix}$

$= \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

23. $$(k\text{A})\text{B} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} kae+kbg & kag+kah \\ kce+kdg & kcg+kdh \end{bmatrix}$$
$$\text{A}(k\text{B}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} ke & kf \\ kg & kh \end{bmatrix} = \begin{bmatrix} kae+kbg & kag+kah \\ kce+kdg & kcg+kdh \end{bmatrix}$$
$$k(\text{AB}) = k\begin{bmatrix} ae+bg & ag+ah \\ ce+dg & cg+dh \end{bmatrix} = \begin{bmatrix} kae+kbg & kag+kah \\ kce+kdg & kcg+kdh \end{bmatrix}$$
$$\therefore \quad (k\text{A})\text{B} = \text{A}(k\text{B}) = k(\text{AB})$$

□

24. No working required.

25. No working required. (Write down the dimensions of each matrix, then this question becomes a repeat of the previous one.)

26. No working required.

27. Consider each possible permutation of two matrices:

| | A | B | C |
|---|---|---|---|
| A | AA | AB | AC |
| B | BA | BB | BC |
| C | CA | CB | CC |

then simply decide which products have dimensions that allow multiplication:

| | A | B | C |
|---|---|---|---|
| A | $(2\times2)(2\times2)$ | ~~$(2\times2)(1\times2)$~~ | $(2\times2)(2\times1)$ |
| B | $(1\times2)(2\times2)$ | ~~$(1\times2)(1\times2)$~~ | $(1\times2)(2\times1)$ |
| C | ~~$(2\times1)(2\times2)$~~ | $(2\times1)(1\times2)$ | ~~$(2\times1)(2\times1)$~~ |

Note: BC *is* a valid product even though not listed in Mr Sadler's solution. (It results in a $1\times1$ matrix.)

28. (a) $\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 4 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 7 & -3 \end{bmatrix}$

29. (a) $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 3 & 3 \\ 1 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 18 \\ 12 \\ 11 \\ 13 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 3 & 3 \\ 1 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 15 \\ 15 \\ 10 \\ 14 \end{bmatrix}$

30. Initially:
$$\begin{bmatrix} 1\,000 & 5\,000 & 400 & 270 \\ 500 & 8\,000 & 500 & 250 \\ 500 & 3\,000 & 500 & 500 \end{bmatrix} \begin{bmatrix} 5 \\ 0.5 \\ 12 \\ 10 \end{bmatrix} = \begin{bmatrix} 15\,000 \\ 15\,000 \\ 15\,000 \end{bmatrix}$$
All client portfolios are initially worth \$15 000.

Two years later:
$$\begin{bmatrix} 1\,000 & 5\,000 & 400 & 270 \\ 500 & 8\,000 & 500 & 250 \\ 500 & 3\,000 & 500 & 500 \end{bmatrix} \begin{bmatrix} 4 \\ 0.6 \\ 20 \\ 10 \end{bmatrix} = \begin{bmatrix} 17\,700 \\ 19\,300 \\ 18\,800 \end{bmatrix}$$
The portfolios of Client 1, Client 2 and Client 3 are worth \$17 700, \$19 300 and \$18 800 respectively.

31. $\begin{bmatrix} 15 & 10 \end{bmatrix} \begin{bmatrix} 375 & 1 \\ 1250 & 4 \end{bmatrix} = \begin{bmatrix} 18\,125 & 55 \end{bmatrix}$
The order requires 18 125mL of drink and 55 burgers.

32. (a) P is $3\times3$ and Q is $1\times3$ so QP is possible and PQ is not.

(b) $\text{QP} = \begin{bmatrix} 75 & 125 & 180 \end{bmatrix} \begin{bmatrix} 15 & 5 & 5 \\ 25 & 25 & 14 \\ 2 & 1 & 3 \end{bmatrix}$
$= \begin{bmatrix} 4\,610 & 3\,680 & 2\,665 \end{bmatrix}$
The product shows the income per night for each hotel when all rooms are in use.

(c) Refer to the answer in Sadler.

33. No working required.

34. No working required.

35. Remember that the multiplication must make sense. We need to multiply the times for model A (i.e. the first row of P) with the number of orders for A; it makes no sense to multiply any of the times for A with the number of orders for B. PQ has as its first element (cutting A)×(orders A)+(assembling A)×(orders B)+(packing A)×(orders C) and thus makes no sense. In contrast, RP has as its first element (orders A)×(cutting A)+(orders B)×(cutting B)+(orders C)×(cutting C): the models are not mixed and the product makes sense. This should also make it clear what this first element is: the total cutting time. The meaning of the remainder of the RP should be obvious.

## Exercise 3C

1–8 No working required.

9. $\dfrac{1}{2\times 1-1\times 1}\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}=\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}$

10. $\dfrac{1}{3\times 3-2\times 4}\begin{bmatrix}3 & -2\\ -4 & 3\end{bmatrix}=\begin{bmatrix}3 & -2\\ -4 & 3\end{bmatrix}$

11. $\dfrac{1}{2\times 1-1\times -1}\begin{bmatrix}1 & -1\\ 1 & 2\end{bmatrix}=\dfrac{1}{3}\begin{bmatrix}1 & -1\\ 1 & 2\end{bmatrix}$

12. $\dfrac{1}{4\times 2-3\times 1}\begin{bmatrix}2 & -3\\ -1 & 4\end{bmatrix}=\dfrac{1}{5}\begin{bmatrix}2 & -3\\ -1 & 4\end{bmatrix}$

13. $\dfrac{1}{3\times 3+1\times 1}\begin{bmatrix}3 & 1\\ -1 & 3\end{bmatrix}=\dfrac{1}{10}\begin{bmatrix}3 & 1\\ -1 & 3\end{bmatrix}$

14. $\dfrac{1}{9+1}\begin{bmatrix}-3 & -1\\ 1 & -3\end{bmatrix}=\dfrac{1}{10}\begin{bmatrix}-3 & -1\\ 1 & -3\end{bmatrix}$

15. Matrix is singular and so has no inverse.

16. Matrix is singular and so has no inverse.

17. Matrix is singular and so has no inverse.

18. $\dfrac{1}{x\times 1-y\times 0}\begin{bmatrix}1 & -y\\ 0 & x\end{bmatrix}=\dfrac{1}{x}\begin{bmatrix}1 & -y\\ 0 & x\end{bmatrix}$

19. $\dfrac{1}{1-0}\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$

20. $\dfrac{1}{-1-0}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}=-1\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$

$$=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

21. No working required.

22.
$$\begin{bmatrix}5 & 3\\ 3 & 2\end{bmatrix}\mathrm{A}=\begin{bmatrix}2 & 11\\ 1 & 6\end{bmatrix}$$
$$\begin{bmatrix}2 & -3\\ -3 & 5\end{bmatrix}\begin{bmatrix}5 & 3\\ 3 & 2\end{bmatrix}\mathrm{A}=\begin{bmatrix}2 & -3\\ -3 & 5\end{bmatrix}\begin{bmatrix}2 & 11\\ 1 & 6\end{bmatrix}$$
$$\mathrm{A}=\begin{bmatrix}1 & 4\\ -1 & -3\end{bmatrix}$$

23.
$$\mathrm{B}\begin{bmatrix}5 & 1\\ 3 & 1\end{bmatrix}\&=\begin{bmatrix}7 & 1\\ 15 & 3\end{bmatrix}$$
$$\mathrm{B}\begin{bmatrix}5 & 1\\ 3 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ -3 & 5\end{bmatrix}=\begin{bmatrix}7 & 1\\ 15 & 3\end{bmatrix}\begin{bmatrix}1 & -1\\ -3 & 5\end{bmatrix}$$
$$\mathrm{B}=\begin{bmatrix}2 & -1\\ 3 & 0\end{bmatrix}$$

24.
$$\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}\mathrm{C}=\begin{bmatrix}-1 & 4 & -2\\ 1 & -1 & 1\end{bmatrix}$$
$$-1\begin{bmatrix}-1 & -2\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}\mathrm{C}=-1\begin{bmatrix}-1 & -2\\ 0 & 1\end{bmatrix}\begin{bmatrix}-1 & 4 & -2\\ 1 & -1 & 1\end{bmatrix}$$
$$\mathrm{C}=-1\begin{bmatrix}-1 & -2 & 0\\ 1 & -1 & 1\end{bmatrix}$$
$$=\begin{bmatrix}1 & 2 & 0\\ -1 & 1 & -1\end{bmatrix}$$

25.
$$\mathrm{D}\begin{bmatrix}4 & -3\\ 2 & 1\end{bmatrix}\&=\begin{bmatrix}-2 & -6\end{bmatrix}$$
$$\mathrm{D}\begin{bmatrix}4 & -3\\ 2 & 1\end{bmatrix}\frac{1}{10}\begin{bmatrix}1 & 3\\ -2 & 4\end{bmatrix}=\begin{bmatrix}-2 & -6\end{bmatrix}\frac{1}{10}\begin{bmatrix}1 & 3\\ -2 & 4\end{bmatrix}$$
$$\mathrm{D}=\frac{1}{10}\begin{bmatrix}10 & -30\end{bmatrix}$$
$$=\begin{bmatrix}1 & -3\end{bmatrix}$$

26.
$$\mathrm{A}^2-2\mathrm{A}=\begin{bmatrix}13 & 8\\ 2 & 5\end{bmatrix}-\begin{bmatrix}6 & 8\\ 2 & -2\end{bmatrix}$$
$$=\begin{bmatrix}7 & 0\\ 0 & 7\end{bmatrix}$$
$$=7\mathrm{I}$$
$$\therefore\quad k=7$$

27.
$$\mathrm{A}^{-1}=\frac{1}{0k+10}\begin{bmatrix}0 & 2\\ -5 & k\end{bmatrix}$$
$$\therefore\quad 10\mathrm{A}^{-1}=\begin{bmatrix}0 & 2\\ -5 & k\end{bmatrix}$$
$$\mathrm{A}+10\mathrm{A}^{-1}=\begin{bmatrix}k & -2\\ 5 & 0\end{bmatrix}+\begin{bmatrix}0 & 2\\ -5 & k\end{bmatrix}$$
$$=\begin{bmatrix}k & 0\\ 0 & k\end{bmatrix}$$
$$=k\mathrm{I}$$
$$\therefore\quad k=5$$

28. (a) No working required.

(b) $16\times 5+5\times -14=10$

(c)
$$\mathrm{A}^{-1}=\frac{1}{3-2}\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}$$
$$=\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}$$

(d) No working required (since we've already calculated the determinant of B).

(e)
$$\mathrm{AC}=\mathrm{B}$$
$$\mathrm{A}^{-1}\mathrm{AC}=\mathrm{A}^{-1}\mathrm{B}$$
$$\mathrm{C}=\mathrm{A}^{-1}\mathrm{B}$$
$$=\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}\begin{bmatrix}16 & -5\\ -14 & 5\end{bmatrix}$$
$$=\begin{bmatrix}2 & 0\\ -10 & 5\end{bmatrix}$$

(f)
$$\mathrm{DA}=\mathrm{B}$$
$$\mathrm{DAA}^{-1}=\mathrm{BA}^{-1}$$
$$\mathrm{D}=\mathrm{BA}^{-1}$$
$$=\begin{bmatrix}16 & -5\\ -14 & 5\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}$$
$$=\begin{bmatrix}6 & 1\\ -4 & 1\end{bmatrix}$$

29. (a) No working required.

(b)
$$\mathrm{P}^{-1}=\frac{1}{-4+3}\begin{bmatrix}1 & -1\\ -3 & -4\end{bmatrix}$$
$$=\begin{bmatrix}-1 & 1\\ 3 & 4\end{bmatrix}$$

(c)
$$\mathrm{Q}^{-1}=\frac{1}{6-0}\begin{bmatrix}1 & -2\\ 0 & 6\end{bmatrix}$$
$$=\frac{1}{6}\begin{bmatrix}1 & -2\\ 0 & 6\end{bmatrix}$$

(d) $$(\mathrm{P}+\mathrm{Q})^{-1} = \frac{1}{4-3}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

(e) $$\begin{aligned} \mathrm{R}(\mathrm{P}+\mathrm{Q}) &= \mathrm{Q} \\ \mathrm{R}(\mathrm{P}+\mathrm{Q})(\mathrm{P}+\mathrm{Q})^{-1} &= \mathrm{Q}(\mathrm{P}+\mathrm{Q})^{-1} \\ \mathrm{R} &= \mathrm{Q}(\mathrm{P}+\mathrm{Q})^{-1} \\ &= \begin{bmatrix} 6 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -2 \\ -3 & 2 \end{bmatrix} \end{aligned}$$

30. $$\mathrm{A} = (\mathrm{AB})\mathrm{B}^{-1} = \begin{bmatrix} 2 & -1 \\ 17 & -9 \end{bmatrix}$$

31. $$\mathrm{D} = \mathrm{C}^{-1}(\mathrm{CD}) = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$$

32. (a) $$3x - 24 = 0 \qquad x = 8$$

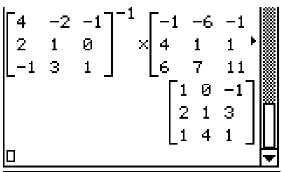
(b) $$x^2 - 16 = 0 \qquad x = \pm 4$$

(c) $$\begin{aligned} x^2 - x - 20 &= 0 \\ (x-5)(x+4) &= 0 \\ x &= 5 \\ \text{or} \quad x &= -4 \end{aligned}$$

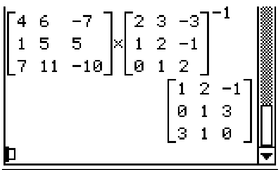
33. $$\begin{aligned} \mathrm{F} &= \mathrm{E}^{-1}(\mathrm{EF}) \\ &= \frac{1}{2}\begin{bmatrix} 2 & 6 \\ 4 & -6 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ 2 & -3 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathrm{G} &= (\mathrm{GE})\mathrm{E}^{-1} \\ &= \frac{1}{2}\begin{bmatrix} 6 & -8 \\ 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -4 \\ 0 & 2 \end{bmatrix} \end{aligned}$$

34. $$\mathrm{AC} = \mathrm{B} \qquad \mathrm{C} = \mathrm{A}^{-1}\mathrm{B}$$




35. $$\mathrm{CA} = \mathrm{B} \qquad \mathrm{C} = \mathrm{BA}^{-1}$$




36. (a) No working required.

(b) $$\begin{aligned} \mathrm{CA} &= \mathrm{B} \\ \mathrm{C} &= \mathrm{BA}^{-1} \\ &= \begin{bmatrix} 24 & 56 \\ 16 & 36 \end{bmatrix}\frac{1}{30\times 36 - 70\times 16}\begin{bmatrix} 36 & -70 \\ -16 & 30 \end{bmatrix} \\ &= -\frac{1}{40}\begin{bmatrix} 24 & 56 \\ 16 & 36 \end{bmatrix}\begin{bmatrix} 36 & -70 \\ -16 & 30 \end{bmatrix} \\ &= -\frac{1}{40}\begin{bmatrix} -32 & 0 \\ 0 & -40 \end{bmatrix} \\ &= \begin{bmatrix} 0.8 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

37. $$\begin{aligned} \mathrm{C} &= \mathrm{A} - \mathrm{CB} \\ \mathrm{C} + \mathrm{CB} &= \mathrm{A} \\ \mathrm{CI} + \mathrm{CB} &= \mathrm{A} \\ \mathrm{C}(\mathrm{I}+\mathrm{B}) &= \mathrm{A} \\ \mathrm{C}(\mathrm{I}+\mathrm{B})(\mathrm{I}+\mathrm{B})^{-1} &= \mathrm{A}(\mathrm{I}+\mathrm{B})^{-1} \\ \mathrm{C} &= \mathrm{A}(\mathrm{I}+\mathrm{B})^{-1} \end{aligned}$$

□

$$\begin{aligned} \mathrm{C} &= \mathrm{A}(\mathrm{I}+\mathrm{B})^{-1} \\ &= \begin{bmatrix} -1 & 6 \\ 11 & 4 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -5 & 1 \end{bmatrix}\right)^{-1} \\ &= \begin{bmatrix} -1 & 6 \\ 11 & 4 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -5 & 2 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} -1 & 6 \\ 11 & 4 \end{bmatrix}\frac{1}{14}\begin{bmatrix} 2 & -2 \\ 5 & 2 \end{bmatrix} \\ &= \frac{1}{14}\begin{bmatrix} 28 & 14 \\ 42 & -14 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \end{aligned}$$

38. $$\begin{aligned} \mathrm{A} &= \mathrm{BC} - \mathrm{AC} \\ &= (\mathrm{B}-\mathrm{A})\mathrm{C} \\ (\mathrm{B}-\mathrm{A})^{-1}\mathrm{A} &= (\mathrm{B}-\mathrm{A})^{-1}(\mathrm{B}-\mathrm{A})\mathrm{C} \\ \mathrm{C} &= (\mathrm{B}-\mathrm{A})^{-1}\mathrm{A} \\ \mathrm{B}-\mathrm{A} &= \begin{bmatrix} -1 & 0 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -3 & 5 \\ 1 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \\ (\mathrm{B}-\mathrm{A})^{-1} &= \begin{bmatrix} -2 & 5 \\ -1 & 2 \end{bmatrix} \\ \mathrm{C} &= \begin{bmatrix} -2 & 5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -3 & 5 \\ 1 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 20 \\ 5 & 7 \end{bmatrix} \end{aligned}$$

39.
$$\begin{aligned}
\mathrm{P} - \mathrm{PQ} - \mathrm{PQ}^2 &= \mathrm{Q}\\
\mathrm{P}(\mathrm{I} - \mathrm{Q} - \mathrm{Q}^2) &= \mathrm{Q}\\
\mathrm{P} &= \mathrm{Q}(\mathrm{I} - \mathrm{Q} - \mathrm{Q}^2)^{-1}\\
\mathrm{Q}^2 &= \begin{bmatrix} 1 & 0 \\ -15 & 4 \end{bmatrix}\\
\mathrm{I} - \mathrm{Q} - \mathrm{Q}^2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 5 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -15 & 4 \end{bmatrix}\\
&= \begin{bmatrix} 1 & 0 \\ 10 & -1 \end{bmatrix}\\
(\mathrm{I} - \mathrm{Q} - \mathrm{Q}^2)^{-1} &= -1\begin{bmatrix} -1 & 0 \\ -10 & 1 \end{bmatrix}\\
&= \begin{bmatrix} 1 & 0 \\ 10 & -1 \end{bmatrix}\\
\mathrm{P} &= \begin{bmatrix} -1 & 0 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 10 & -1 \end{bmatrix}\\
&= \begin{bmatrix} -1 & 0 \\ -15 & 2 \end{bmatrix}
\end{aligned}$$

40. (a) No working required.

(b) No working required.

(c)
$$\begin{aligned}
\mathrm{BA} &= \begin{bmatrix} 860 & 740 \end{bmatrix}\\
\mathrm{BAA}^{-1} &= \begin{bmatrix} 860 & 740 \end{bmatrix}\mathrm{A}^{-1}\\
\mathrm{B} &= \begin{bmatrix} 860 & 740 \end{bmatrix}\begin{bmatrix} 6 & 5 \\ 8 & 7 \end{bmatrix}^{-1}\\
&= \begin{bmatrix} 860 & 740 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 7 & -5 \\ -8 & 6 \end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix} 50 & 70 \end{bmatrix}\\
\begin{bmatrix} x & y \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} 50 & 70 \end{bmatrix}\\
x &= 50\\
y &= 70
\end{aligned}$$

## Exercise 3D

1–6 No working required.

7. (a) No working required. (By this stage you should be able to find the inverse of a $2 \times 2$ matrix in a single step.)

(b)
$$\begin{aligned}
\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 4 \\ -9 \end{bmatrix}\\
\begin{bmatrix} x \\ y \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} 4 & 2 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ -9 \end{bmatrix}\\
&= \begin{bmatrix} -1 \\ -\frac{7}{2} \end{bmatrix}\\
x &= -1\\
y &= -\frac{7}{2}
\end{aligned}$$

8. (a) No working required.

(b)
$$\begin{aligned}
\begin{bmatrix} -2 & 1 & -2 \\ 2 & -1 & 3 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 3 \\ -1 \\ 9 \end{bmatrix}\\
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} -5 & -4 & 1 \\ -4 & -4 & 2 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \\ 9 \end{bmatrix}\\
&= \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}\\
x &= -1\\
y &= 5\\
z &= 2
\end{aligned}$$

9. (a)
$$\begin{aligned}
\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 2 \\ 1 \end{bmatrix}\\
\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}\\
&= \begin{bmatrix} 3 \\ -7 \end{bmatrix}\\
x &= 3\\
y &= -7
\end{aligned}$$

(b)
$$\begin{aligned}
\begin{bmatrix} 3 & 1 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 8 \\ 13 \end{bmatrix}\\
\begin{bmatrix} x \\ y \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ 13 \end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix} 11 \\ -17 \end{bmatrix}\\
x &= 5.5\\
y &= -8.5
\end{aligned}$$

10. (a)
$$\begin{aligned}
\mathrm{AB} &= \begin{bmatrix} 8 - 11 + 10 & -4 + 2 + 2 & 2 - 8 + 6 \\ -4 - 11 + 15 & 2 + 2 + 3 & -1 - 8 + 9 \\ -12 + 22 - 10 & 6 - 4 - 2 & -3 + 16 - 6 \end{bmatrix}\\
&= \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}\\
&= 7\mathrm{I}
\end{aligned}$$

(b)
$$\begin{aligned}
7\mathrm{I} &= \mathrm{AB}\\
\mathrm{I} &= \frac{1}{7}\mathrm{AB}\\
\mathrm{A}^{-1}\mathrm{I} &= \mathrm{A}^{-1}\frac{1}{7}\mathrm{AB}\\
\mathrm{A}^{-1} &= \frac{1}{7}\mathrm{B}
\end{aligned}$$

(c) $\text{A}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \\ 5 \end{bmatrix}$

$$\begin{aligned}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \text{A}^{-1}\begin{bmatrix} -3 \\ 7 \\ 5 \end{bmatrix} \\
&= \frac{1}{7}\text{B}\begin{bmatrix} -3 \\ 7 \\ 5 \end{bmatrix} \\
&= \frac{1}{7}\begin{bmatrix} 12+14-5 \\ -33-14+40 \\ -15+7+15 \end{bmatrix} \\
&= \frac{1}{7}\begin{bmatrix} 21 \\ -7 \\ 7 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}
\end{aligned}$$

$x = 3$
$y = -1$
$z = 1$

11. No working required for (a) and (b) is purely calculator work to determine $\text{X} = \text{A}^{-1}\text{B}$, then interpret the result.

## Miscellaneous Exercise 3

1. No working required.

2. (a) $2z = 2 \times 3 \operatorname{cis} \frac{5\pi}{6} = 6 \operatorname{cis} \frac{5\pi}{6}$

   (b) $3w = 3 \times 2 \operatorname{cis}\left(-\frac{2\pi}{3}\right) = 6 \operatorname{cis}\left(-\frac{2\pi}{3}\right)$

   (c) $zw = 3 \times 2 \operatorname{cis}\left(\frac{5\pi}{6} - \frac{2\pi}{3}\right) = 6 \operatorname{cis} \frac{\pi}{6}$

   (d) $$\begin{aligned}\frac{z}{w} &= \frac{3}{2} \operatorname{cis}\left(\frac{5\pi}{6} + \frac{2\pi}{3}\right) \\ &= 1.5 \operatorname{cis} \frac{9\pi}{6} \\ &= 1.5 \operatorname{cis}\left(\frac{3\pi}{2} - 2\pi\right) \\ &= 1.5 \operatorname{cis}\left(-\frac{\pi}{2}\right)\end{aligned}$$

   (e) $$\begin{aligned}\mathrm{i}z &= 3 \operatorname{cis}\left(\frac{5\pi}{6} + \frac{\pi}{2}\right) \\ &= 3 \operatorname{cis} \frac{8\pi}{6} \\ &= 3 \operatorname{cis}\left(\frac{4\pi}{3} - 2\pi\right) \\ &= 3 \operatorname{cis}\left(-\frac{2\pi}{3}\right)\end{aligned}$$

   (f) $-w = 2 \operatorname{cis}\left(-\frac{2\pi}{3} + \pi\right) = 2 \operatorname{cis} \frac{\pi}{3}$

   (g) No working required.

   (h) No working required (using the answer to (c) as the starting point).

   (i) No working required (because $\bar{z}\bar{w} = \overline{zw}$).

   (j) $$\begin{aligned}z^2w^3 &= 3^2 \operatorname{cis}\left(\frac{5\pi}{6} \times 2\right) \times 2^3 \operatorname{cis}\left(-\frac{2\pi}{3} \times 3\right) \\ &= 9 \times 8 \operatorname{cis}\left(\frac{5\pi}{3} - 2\pi\right) \\ &= 72 \operatorname{cis}\left(-\frac{\pi}{3}\right)\end{aligned}$$

   It is useful to remember that

   - $\mathrm{i} = \operatorname{cis} \frac{\pi}{2}$
   - $-1 = \operatorname{cis} \pi = \operatorname{cis}(-\pi)$
   - $-\mathrm{i} = \operatorname{cis}\left(-\frac{\pi}{2}\right)$

3. (a) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \end{bmatrix}$

4. Write the dimensions for each matrix and rearrange so that adjacent numbers match:

   | B | A | C |
   |---|---|---|
   | $1 \times 2$ | $2 \times 3$ | $3 \times 4$ |

   Before doing any calculations, it should be clear that this will result in a $1 \times 4$ matrix.

$$\begin{aligned}
\text{BA} &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 2 & 1 \end{bmatrix} \\
\text{BAC} &= \begin{bmatrix} 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 3 \\ 3 & 1 & 4 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 5 & 2 & 4 \end{bmatrix}
\end{aligned}$$

5. $$\begin{aligned}
A &= \int_3^6 x^2 - 2x + 3\,\mathrm{d}x \\
&= \left[\frac{x^3}{3} - x^2 + 3x\right]_3^6 \\
&= (72 - 36 + 18) - (9 - 9 + 9) \\
&= 54 - 9 \\
&= 45 \text{ units}^2
\end{aligned}$$

6. (a) $$\begin{aligned}
\text{LHS} &= \mathrm{e}^{\mathrm{i}(\alpha+\beta)} \\
&= \mathrm{e}^{\mathrm{i}\alpha+\mathrm{i}\beta} \\
&= \mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\beta} \\
&= \operatorname{cis}\alpha \operatorname{cis}\beta \\
&= \text{RHS}
\end{aligned}$$

□

(b) $$\begin{aligned}
\text{LHS} &= \cos(\alpha+\beta) + \mathrm{i}\sin(\alpha+\beta) \\
&= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
&\quad + \mathrm{i}(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \\
&= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
&\quad + \mathrm{i}\sin\alpha\cos\beta + \mathrm{i}\cos\alpha\sin\beta \\
&= \cos\alpha\cos\beta + \mathrm{i}\cos\alpha\sin\beta \\
&\quad - \sin\alpha\sin\beta + \mathrm{i}\sin\alpha\cos\beta \\
&= \cos\alpha(\cos\beta + \mathrm{i}\sin\beta) \\
&\quad + \mathrm{i}^2\sin\alpha\sin\beta + \mathrm{i}\sin\alpha\cos\beta \\
&= \cos\alpha(\cos\beta + \mathrm{i}\sin\beta) \\
&\quad + \mathrm{i}\sin\alpha(\mathrm{i}\sin\beta + \cos\beta) \\
&= (\cos\alpha + \mathrm{i}\sin\alpha)(\cos\beta + \mathrm{i}\sin\beta) \\
&= \operatorname{cis}\alpha\operatorname{cis}\beta \\
&= \text{RHS}
\end{aligned}$$

□

7. No working required.
(For (c), think $4\mathrm{e}^{\mathrm{i}\pi x} = 4\operatorname{cis}(\pi x)$ so the set described is the points $z = r\operatorname{cis}\theta$ having modulus $r = 4$ and argument $0 < \theta \leq \frac{\pi}{2}$.)

8. $$\begin{aligned}
A &= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 2\sin^2 x\,\mathrm{d}x \\
&= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} -(1 - 2\sin^2 x) + 1\,\mathrm{d}x \\
&= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} -\cos 2x + 1\,\mathrm{d}x \\
&= \left[-\frac{\sin 2x}{2} + x\right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \\
&= \left(-\frac{\sin\frac{4\pi}{3}}{2} + \frac{2\pi}{3}\right) - \left(-\frac{\sin\frac{2\pi}{3}}{2} + \frac{\pi}{3}\right) \\
&= -\frac{-\frac{\sqrt{3}}{2}}{2} + \frac{\frac{\sqrt{3}}{2}}{2} + \frac{2\pi}{3} - \frac{\pi}{3} \\
&= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} + \frac{\pi}{3} \\
&= \frac{\sqrt{3}}{2} + \frac{\pi}{3} \text{ units}^2
\end{aligned}$$

9. $$\begin{aligned}
r &= \sqrt{3+1} \\
&= 2 \\
\tan\theta &= \frac{1}{-\sqrt{3}} \qquad \text{(second quadrant)} \\
\theta &= \frac{5\pi}{6} \\
\therefore \quad -\sqrt{3} + \mathrm{i} &= 2\operatorname{cis}\frac{5\pi}{6} \\
\left(-\sqrt{3} + \mathrm{i}\right)^{12} &= 2^{12}\operatorname{cis}\left(\frac{5\pi}{6} \times 12\right) \\
&= 4096\operatorname{cis}(10\pi) \\
&= 4096\operatorname{cis}0 \\
&= 4096
\end{aligned}$$

10. $$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= 6x - 1 \\
&= 11 \\
y - y_1 &= m(x - x_1) \\
y - 5 &= 11(x - 2) \\
y &= 11x - 17
\end{aligned}$$

11. $$\begin{aligned}
2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
2y\frac{\mathrm{d}y}{\mathrm{d}x} &= -2x \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x}{y} \\
&= \frac{3}{4} \\
y - y_1 &= m(x - x_1) \\
y + 4 &= \frac{3}{4}(x - 3) \\
4y + 16 &= 3x - 9 \\
3x - 4y &= 25
\end{aligned}$$

12. Points where $x = -2$:

$$\begin{aligned} -2y + y^2 - (-2)^3 &= 11 \\ y^2 - 2y + 8 &= 11 \\ y^2 - 2y - 3 &= 0 \\ (y-3)(y+1) &= 0 \end{aligned}$$

The points are $(-2,-1)$ and $(-2,3)$.
Differentiating:

$$\begin{aligned} y + x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 3x^2 &= 0 \\ x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 3x^2 - y \\ \frac{\mathrm{d}y}{\mathrm{d}x}(x+2y) &= 3x^2 - y \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3x^2 - y}{x+2y} \end{aligned}$$

At $(-2,-1)$:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{12+1}{-2-2} \\ &= -\frac{13}{4} \\ y - y_1 &= m(x - x_1) \\ y + 1 &= -\frac{13}{4}(x+2) \\ 4y + 4 &= -13x - 26 \\ 13x + 4y &= -30 \end{aligned}$$

At $(-2,3)$:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{12-3}{-2+6} \\ &= \frac{9}{4} \\ y - y_1 &= m(x - x_1) \\ y - 3 &= \frac{9}{4}(x+2) \\ 4y - 12 &= 9x + 18 \\ 9x - 4y &= -30 \end{aligned}$$

13. (a) The column matrix Y is useful. When forming the product XY, the number of units of the commodities is multiplied by the cost of the corresponding commodity.

(b) $$\begin{aligned} \mathrm{XY} &= \begin{bmatrix} 100+120+200 \\ 150+60+200 \\ 50+180+200 \end{bmatrix} \\ &= \begin{bmatrix} 420 \\ 410 \\ 430 \end{bmatrix} \end{aligned}$$

(c) The product gives the total cost for each of the three models.

14. The initial case, where $n = 1$:

$$\begin{aligned} \text{R.H.S.} &= \frac{r(r^1 - 1)}{r-1} \\ &= r \\ &= \text{L.H.S.} \end{aligned}$$

The statement is true for the initial case.

Assume the statement is true for $n = k$, i.e.

$$r + r^2 + r^3 + \ldots + r^k = \frac{r(r^k - 1)}{r - 1}$$

Then for $n = k + 1$

$$\begin{aligned} \text{L.H.S.} &= r + r^2 + r^3 + \ldots + r^k + r^{k+1} \\ &= \frac{r(r^k-1)}{r-1} + r^{k+1} \\ &= \frac{r(r^k-1)}{r-1} + \frac{r^{k+1}(r-1)}{r-1} \\ &= \frac{r(r^k-1)}{r-1} + \frac{rr^k(r-1)}{r-1} \\ &= \frac{r(r^k-1)}{r-1} + \frac{r(r^{k+1}-r^k)}{r-1} \\ &= \frac{r(r^k - 1 + r^{k+1} - r^k)}{r-1} \\ &= \frac{r(r^{k+1}-1)}{r-1} \\ &= \text{R.H.S.} \end{aligned}$$

Thus if the statement is true for $n = k$ it is also true for $n = k + 1$.

Hence since the statement is true for $n = 1$ it follows by induction that it is true for all integer $n \geq 1$. □

15. For the matrix to be singular, the determinant must be zero. For this matrix the determinant is $2x^2 + 4$. This quadratic has no real roots, so the matrix cannot be singular for $x \in \Re$.

16.
$$\begin{aligned} \mathrm{A}^2 &= \begin{bmatrix} k & 4 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} k & 4 \\ -3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} k^2-12 & 4k-4 \\ -3k+3 & -11 \end{bmatrix} \\ \mathrm{A}^2 + \mathrm{A} &= \begin{bmatrix} k^2-12 & 4k-4 \\ -3k+3 & -11 \end{bmatrix} + \begin{bmatrix} k & 4 \\ -3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} k^2+k-12 & 4k \\ -3k & -12 \end{bmatrix} \\ \begin{bmatrix} 0 & p \\ q & -12 \end{bmatrix} &= \begin{bmatrix} k^2+k-12 & 4k \\ -3k & -12 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} k^2 + k - 12 &= 0 \\ (k+4)(k-3) &= 0 \\ p &= 4k \\ q &= -3k \end{aligned}$$

$$\begin{aligned} k &= -4 \\ p &= -16 \\ q &= 12 \end{aligned}$$

$$\begin{aligned} \text{or } k &= 3 \\ p &= 12 \\ q &= -9 \end{aligned}$$

Reject the first solution set because it does not satisfy $p > 0$ and conclude

$$\begin{aligned} k &= 3 \\ p &= 12 \\ q &= -9 \end{aligned}$$

17. (a)
$$\begin{aligned} \frac{\ln x}{x} &= 0 \\ \ln x &= 0 \\ x &= 1 \end{aligned}$$

The coordinates of A are $(1, 0)$.

(b)
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\left(\frac{1}{x}\right)(x) - (\ln x)(1)}{x^2} \\ &= \frac{1 - \ln x}{x^2} \end{aligned}$$

At B,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \frac{1 - \ln x}{x^2} &= 0 \\ 1 - \ln x &= 0 \\ \ln x &= 1 \\ x &= \mathrm{e} \\ y &= \frac{\ln x}{x} \\ &= \frac{\ln \mathrm{e}}{\mathrm{e}} \\ &= \frac{1}{\mathrm{e}} \end{aligned}$$

The coordinates of B are $(\mathrm{e}, \frac{1}{\mathrm{e}})$

(c)
$$\begin{aligned} \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\left(-\frac{1}{x}\right)(x^2) - (1 - \ln x)(2x)}{x^4} \\ &= \frac{-x - 2x + 2x \ln x}{x^4} \\ &= \frac{2x \ln x - 3x}{x^4} \\ &= \frac{2 \ln x - 3}{x^3} \end{aligned}$$

At C,

$$\begin{aligned} \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 0 \\ \frac{2 \ln x - 3}{x^3} &= 0 \\ 2 \ln x - 3 &= 0 \\ \ln x &= 1.5 \\ x &= \mathrm{e}^{1.5} \\ y &= \frac{\ln x}{x} \\ &= \frac{1.5}{\mathrm{e}^1.5} \\ &= 1.5\mathrm{e}^{-1.5} \end{aligned}$$

The coordinates of C are $(\mathrm{e}^{1.5}, 1.5\mathrm{e}^{-1.5})$

18. Because of the symmetry, we need only consider one quadrant. The positive $x-$intercept is at

$$\begin{aligned} 8x - 16 &= 0 \\ x &= 2. \end{aligned}$$

Rewriting the right bound in the first quadrant as a function of $x$ gives

$$y = \sqrt{8x - 16}$$

The $x-$coordinate of the top-right point is given by the intersection of the top and right curves:

$$\begin{aligned} \sqrt{8x - 16} &= 2 + \frac{1}{8}x^2 \\ 8x - 16 &= 4 + \frac{1}{2}x^2 + \frac{1}{64}x^4 \end{aligned}$$

This looks messy to solve, but there's a simpler approach: based on the symmetry we know that this point also intersects the line $y = x$, giving us

$$\begin{aligned} 2 + \frac{1}{8}x^2 &= x \\ 16 + x^2 &= 8x \\ x^2 - 8x + 16 &= 0 \\ (x - 4)^2 &= 0 \\ x &= 4 \end{aligned}$$

The area in the first quadrant is given by

$$A = \int_0^4 2 + \frac{1}{8}x^2 \,\mathrm{d}x - \int_2^4 \sqrt{8x - 16} \,\mathrm{d}x$$

Although we should be able to calculate this, there is again a simpler approach. Rather than use the square root function for the lower bound of the region we will find only the area above the line $y = x$. Based on the symmetry, we know that this will give us half the area of the first quadrant (or one eighth of the total area).

$$\begin{aligned} A &= 2\int_0^4 2 + \frac{1}{8}x^2 - x \,\mathrm{d}x \\ &= \int_0^4 4 + \frac{1}{4}x^2 - 2x \,\mathrm{d}x \\ &= \left[4x + \frac{x^3}{12} - x^2\right]_0^4 \\ &= \left[x(4 + \frac{x^2}{12} - x)\right]_0^4 \\ &= 4(4 + \frac{4^2}{12} - 4) - 0 \\ &= \frac{16}{3} \end{aligned}$$

Thus the total area is $\frac{64}{3} \approx 21.33\text{cm}^2$.

(Note the correct units for this answer are square centimetres, not square units as shown in Sadler.)

It should be noted that questions like this have several different paths to the correct answer, some much simpler than others. You should always be on the lookout for a simpler approach, even if it means changing track part way through the problem as I have done here.

Another approach to this problem, possibly simpler still, would be to consider the area as an $8\times 8$ square with four parabolic 'bites' taken out of it and determine the area of each of these bites as

$$\begin{aligned}
A &= \int_{-4}^{4} 4-(2+\frac{1}{8}x^2)\,\mathrm{d}x\\
&= \int_{-4}^{4} 2-\frac{1}{8}x^2\,\mathrm{d}x\\
&= \left[2x-\frac{x^3}{24}\right]_{-4}^{4}\\
&= \left(8-\frac{16}{6}\right)-\left(-8+\frac{16}{6}\right)\\
&= 16-\frac{16}{3}\\
\text{Total area} &= 8^2-4\left(16-\frac{16}{3}\right)\\
&= \frac{64}{3}
\end{aligned}$$

If I were preparing this as a "display answer", I would include only the determination of the points of intersection and then this last approach. I leave the other work in this solution simply to show students the kind of thinking process that a capable student would go through for a complex problem like this.

19. (a) The curve intersects the $x-$axis in the given domain at $x=0$ and $x=\frac{\pi}{2}$. The area enclosed by the curve and the axis is then (using the calculator)

$$\begin{aligned}
A &= \int_{0}^{\frac{\pi}{2}} |3x^2(1-\sin x)|\\
&= 0.451 \quad (3\text{ d.p.})
\end{aligned}$$

But this is not the exact value we require, so we'll use the calculator to give us the *indefinite* integral, then substitute to get the exact area:

$$\begin{aligned}
A &= \left[x^3+3x^2\cos x-6x\sin x-6\cos x\right]_0^{\frac{\pi}{2}}\\
&= \left(\frac{\pi^3}{8}-3\pi\right)-(-6)\\
&= \frac{\pi^3}{8}-3\pi+6
\end{aligned}$$

(b) The curve intersects the line in the given domain where

$$\frac{25\pi(2x-\pi)}{1}6 = 3x^2(1-\sin x)$$

for which the calculator gives solutions at $x=1.571$ and $x=2.618$. However, these are not the exact values we will need for the bounds, so it needs further work. The first solution looks like $\frac{\pi}{2}$ which we can confirm:

$$\begin{aligned}
\frac{25\pi(2x-\pi)}{1}6 &= 3x^2(1-\sin x)\\
\frac{25\pi\left(2\left(\frac{\pi}{2}\right)\pi-\pi\right)}{1}6 &= 0\\
3\left(\frac{\pi}{2}\right)^2\left(1-\sin\frac{\pi}{2}\right) &= 0
\end{aligned}$$

Guessing that the second intercept is also a multiple of $\pi$ we obtain $2.618\approx 0.8333\pi = \frac{5\pi}{6}$. We can also confirm this:

$$\begin{aligned}
\frac{25\pi(2x-\pi)}{1}6 &= 3x^2(1-\sin x)\\
\frac{25\pi\left(2\left(\frac{5\pi}{6}\right)\pi-\pi\right)}{1}6 &= \frac{25\pi^2}{24}\\
3\left(\frac{5\pi}{6}\right)^2\left(1-\sin\frac{5\pi}{6}\right) &= \frac{25\pi^2}{24}
\end{aligned}$$

The area enclosed by the curve and the line is then

$$\begin{aligned}
A &= \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \left|\frac{25\pi(2x-\pi)}{1}6-3x^2(1-\sin x)\right|\\
&= 2.355 \quad (3\text{ d.p.})
\end{aligned}$$

Again, this is not the exact value we want so we'll again use the calculator to obtain an indefinite ingtegral and proceed from there.

$$\begin{aligned}
A &= \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \frac{25\pi(2x-\pi)}{1}6-3x^2(1-\sin x)\\
&= \left[\begin{matrix}\frac{-16x^3+46x^2\cos(x)-25x^2\pi}{16}\\ +\frac{-96x\sin x-96\cos x+25x\pi^2}{16}\end{matrix}\right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}\\
&= \frac{-\left(\frac{1375\pi^3}{108}-40\pi-\frac{50\sqrt{3}\pi^2}{3}+48\sqrt{3}\right)}{16}\\
&\quad+\frac{\frac{33\pi^3}{4}-48\pi}{16}\\
&= -\frac{121\pi^3}{432}-\frac{\pi}{2}+\frac{25\sqrt{3}\pi^2}{24}-3\sqrt{3}
\end{aligned}$$

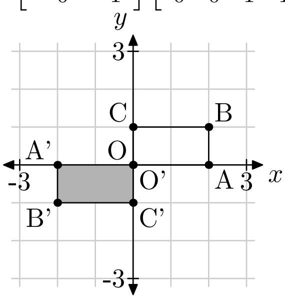
This is a challenging calculator task. Calculator skills involved: graphing, solving equations, finding definite and indefinite integrals, evaluating an expression in $x$ given a particular value of $x$, storing intermediate expressions in variables, defining functions, etc.

# Chapter 4

## Exercise 4A

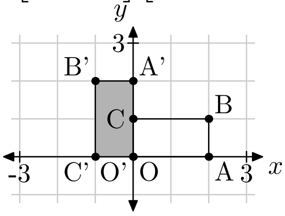
1–12 For these questions, rather than pre-multiply each of O, A, B and C by the given matrix, I will assemble [OABC] into a 2 × 4 matrix $\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ and do the matrix multiplication in a single step.

1. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -2 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}$



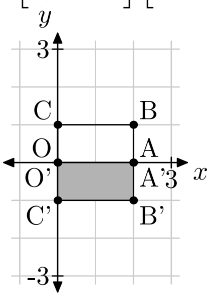

This represents a 180° rotation.

2. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 2 & 2 & 0 \end{bmatrix}$



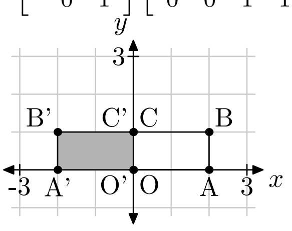

This represents a 90° anticlockwise rotation.

3. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}$



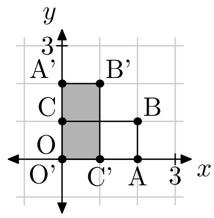

This represents a reflection in the $x-$axis.

4. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$



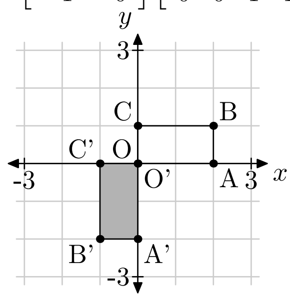

This represents a reflection in the $y-$axis.

5. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 2 & 0 \end{bmatrix}$



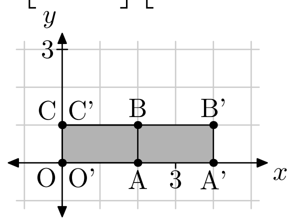

This represents a reflection in the line $y = x$.

6. $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & -2 & -2 & 0 \end{bmatrix}$



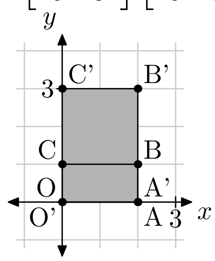

This represents a reflection in the line $y = -x$.

7. $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 4 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$



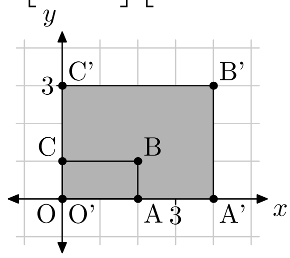

This represents a horizontal dilation of factor 2.

8. $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 3 & 3 \end{bmatrix}$



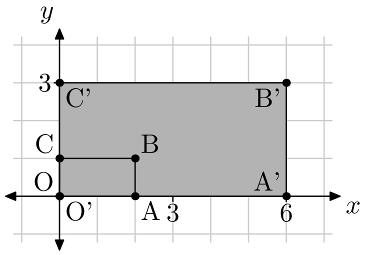

This represents a vertical dilation of factor 3.

9. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 4 & 0 \\ 0 & 0 & 3 & 3 \end{bmatrix}$



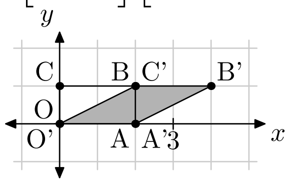

This represents a dilation with a horizontal scale factor of 2 and vertical scale factor of 3.

10. $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 6 & 0 \\ 0 & 0 & 3 & 3 \end{bmatrix}$



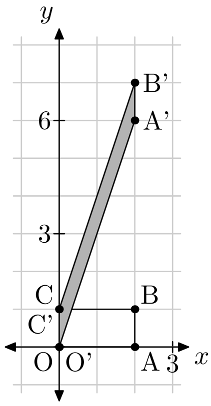

This represents a dilation with uniform scale factor of 3.

11. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 4 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$




This represents a shear parallel to the $x-$axis with scale factor of 2.

12. $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 6 & 7 & 1 \end{bmatrix}$




This represents a shear parallel to the $y-$axis with scale factor of 3.

13. The working needed here is quite straightforward. I present a worked solution for the first matrix only.

 (a) $\det \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -1 \times -1 + 0 \times 0 = 1$

 (b) $$\begin{aligned} \text{Area OABC} &= 2 \times 1 = 2 \\ \text{Area O}'\text{A}'\text{B}'\text{C}' &= 2 \times 1 = 2 \\ \frac{\text{Area O}'\text{A}'\text{B}'\text{C}'}{\text{Area OABC}} &= \frac{2}{2} = 1 \end{aligned}$$

## Exercise 4B

1. (a) For matrix A, $(1, 0)$ maps to $(0, -1)$ and $(0, 1)$ maps to $(1, 0)$; the required matrix is

$$\text{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

 For matrix B, $(1, 0)$ maps to $(-1, 0)$ and $(0, 1)$ maps to $(0, -1)$; the required matrix is

$$\text{B} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

 For matrix C, $(1, 0)$ maps to $(0, 1)$ and $(0, 1)$ maps to $(-1, 0)$; the required matrix is

$$\text{C} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

 (b) $$\begin{aligned} \text{A}^2 &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2 \\ &= \begin{bmatrix} 0^2 + 1 \times -1 & 0 \times 1 + 1 \times 0 \\ -1 \times 0 + 0 \times -1 & -1 \times 1 + 0^2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \text{B} \end{aligned}$$

 (c) $$\begin{aligned} \text{C}^2 &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 \\ &= \begin{bmatrix} 0^2 - 1 \times 1 & 0 \times -1 - 1 \times 0 \\ 1 \times 0 + 0 \times 1 & 1 \times -1 + 0^2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \text{B} \end{aligned}$$

 (d) $$\begin{aligned} \text{A}^3 &= \text{A}^2\text{A} \\ &= \text{BA} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 \times 0 + 0 \times -1 & -1 \times 1 + 0^2 \\ 0^2 + (-1)^2 & 0 \times 1 - 1 \times 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \text{C} \end{aligned}$$

(e) $B^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^2$
$= \begin{bmatrix} (-1)^2 + 0^2 & -1 \times 0 + 0 \times -1 \\ 0 \times -1 - 1 \times 0 & 0^2 + (-1)^2 \end{bmatrix}$
$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$= I$

(f) $A^{-1} = \dfrac{1}{0^2 - (-1 \times 1)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
$= \dfrac{1}{1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
$= C$
(Alternatively, show that $AC = I$)

(g) $B^{-1} = \dfrac{1}{(-1)^2 - 0^2} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
$= \dfrac{1}{1} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$= B$
Alternatively, since we have already shown that $B^2 = I$,

$$\begin{aligned} B^2 &= I \\ B^{-1}B^2 &= B^{-1}I \\ (B^{-1}B)B &= B^{-1} \\ IB &= B^{-1} \\ B &= B^{-1} \end{aligned}$$

2. (a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ maps to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ maps to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$
The transformation matrix is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ maps to $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$
$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ maps to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
The transformation matrix is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ maps to $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$
$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ maps to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$
The transformation matrix is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(d) A reflection in the $x-$axis followed by a reflection in the $y-$axis is represented by premultiplying the matrix for the first reflection by the matrix for the second, i.e.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

A reflection in the $y-$axis followed by a reflection in the $x-$axis is represented by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

(e) Compare the results from (d) and (e).

3. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ maps to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$
$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ maps to $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$
The transformation matrix is $P = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
If $P$ is its own inverse, then $P^2 = I$.

$$\begin{aligned} P^2 &= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I \end{aligned}$$

4. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ maps to $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$
$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ maps to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
The transformation matrix is $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$
The determinant of this matrix is $3 \times 1 - 0 \times 0 = 3$ as expected.

5. (a) No working needed.
(b) No working needed. (A, B, C and D are the columns of the second matrix and A′, B′, C′ and D′ are the columns of the product.)

6.
$$\begin{aligned} TA &= A' \\ T^{-1}TA &= T^{-1}A' \\ A &= T^{-1}A' \\ T^{-1} &= \frac{1}{1 \times 1 - 2 \times 0}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \\ A &= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ B &= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ C &= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ -3 \end{bmatrix} \end{aligned}$$

A, B and C have coordinates $(1, 3)$, $(1, 1)$ and $(4, -3)$ respectively.

7. $$\begin{aligned} T^{-1} &= \frac{1}{2\times 1 - 0 \times -3}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \\ A &= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 2 \\ 6 \end{bmatrix} \\ B &= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} -2 \\ 4 \end{bmatrix} \\ C &= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 0 \\ 4 \end{bmatrix} \end{aligned}$$

A, B and C have coordinates $(1, 3)$, $(-1, 2)$ and $(0, 2)$ respectively.

8. $$\begin{aligned} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} P &= P' \\ \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} P' &= P'' \\ \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} P &= P'' \\ \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix} P &= P'' \end{aligned}$$

Matrix $\begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}$ will transform PQR directly to P″Q″R″.

9. $$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$$
$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}^{-1} &= \frac{1}{-1-0}\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix} \end{aligned}$$

Matrix $\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$ will transform PQR directly to P″Q″R″.

Matrix $\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$ will transform P″Q″R″ directly to PQR. (The matrix is its own inverse.)

10. A shear parallel to the $y$-axis, scale factor 3, transforms $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and so is represented by $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$.

A clockwise rotation of 90° about the origin transforms $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and so is represented by $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

The single matrix to perform both these transformations in sequence is

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}$$

11. These are the same transformations as in the previous question, simply applied in the opposite order, so the single matrix to perform both these transformations in this new sequence is

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$$

12. Post-multiply both sides of the equation with the inverse of $\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}$ to eliminate it from the LHS:

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix} &= \begin{bmatrix} 12 & -1 \\ 7 & 0 \end{bmatrix} \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 12 & -1 \\ 7 & 0 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}^{-1} \\ \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}^{-1} &= \frac{1}{1+6}\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 12 & -1 \\ 7 & 0 \end{bmatrix}\frac{1}{7}\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} 14 & 35 \\ 7 & 21 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \end{aligned}$$

so $a = 2$, $b = 5$, $c = 1$ and $d = 3$.

13. (a) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 9 \\ -1 & -4 \end{bmatrix}$

(c) $$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1} &= \frac{1}{1-0}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \end{aligned}$$

(d) First, to transform $A_3B_3C_3D_3$ to $A_2B_2C_2D_2$

$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} &= \frac{1}{0+1}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

then to further transform the result to $A_1B_1C_1D_1$ we use the matrix we obtained in (c), so the single matrix that combines both is

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$$

14. A reflection in the $x$-axis transforms $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and so is represented by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

A reflection in the line $y = x$ transforms $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and so is represented by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

A 90° clockwise rotation is represented by $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (see question 10).

The matrix that represents these three transformations in sequence is

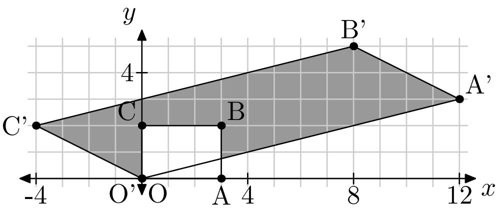
$$\begin{aligned}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\end{aligned}$$

which is the identity matrix, resulting in the original shape in the original position.

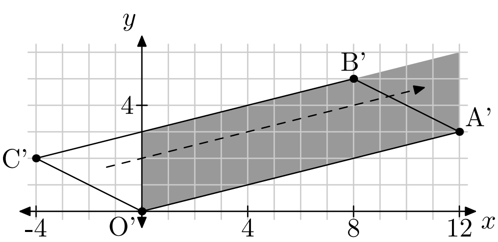
15. (a) $\det \mathrm{T} = 4 \times 1 - (-2) \times 1 = 6$. Given that the area of OABC is 6 units$^2$, the area of O′A′B′C′ is $6 \times 6 = 36$ units$^2$.

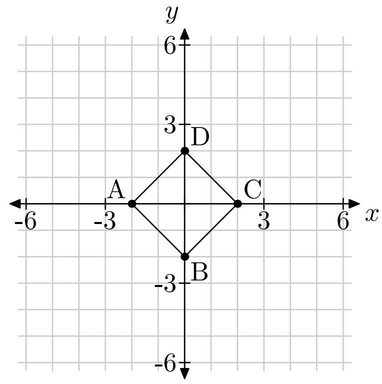
(b) $\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 12 & 8 & -4 \\ 0 & 3 & 5 & 2 \end{bmatrix}$

The coordinates of O′, A′, B′ and C′ are $(0, 0)$, $(12, 3)$, $(8, 5)$ and $(-4, 2)$ respectively.

(c) 


(d) There are number of straightforward ways of determining the area of the parallelogram. For example if we slice off the part of the parallelogram that is left of the $y$-axis and slide it to the other end (as shown below), we get a parallelogram with a (vertical) base of 3 and (horizontal) perpendicular height of 12, yielding an area of 36.

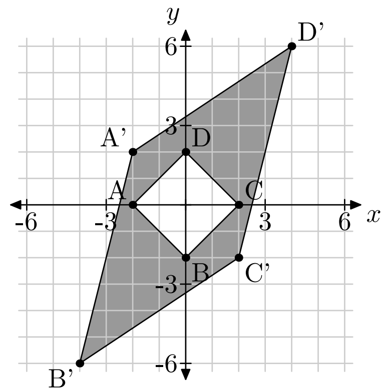



16. (a) 


(b) Area = 8 units$^2$ (area of any square, rhombus or kite is half the product of its diagonals).

(c) $\det \mathrm{M} = 1 \times 3 - 2 \times -1 = 5$. The area of A′B′C′D′ is $5 \times 8 = 40$ units$^2$.

(d) $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -4 & 2 & 4 \\ 2 & -6 & -2 & 6 \end{bmatrix}$




Area = 40 units$^2$.

17. Every point on the line $y = 2x + 3$ can be represented by $\begin{bmatrix} x \\ 2x+3 \end{bmatrix}$.

To prove:

$$\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ 2x+3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

for all $x$.

Proof:

$$\begin{aligned}\text{LHS} &= \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ 2x+3 \end{bmatrix} \\ &= \begin{bmatrix} 2(x) - (2x+3) \\ -2(x) + (2x+3) \end{bmatrix} \\ &= \begin{bmatrix} -3 \\ 3 \end{bmatrix} \\ &= \text{RHS}\end{aligned}$$

□

Notice that the matrix $\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$ is singular (i.e. it has a determinant of zero) and therefore is not invertable. This is a requirement of any matrix that transforms two or more distinct points to the same position in the image.

18. Every point on the line $y = x - 1$ can be represented by $\begin{bmatrix} x \\ x-1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ x-1 \end{bmatrix} = \begin{bmatrix} x \\ 2(x) + (x-1) \end{bmatrix} = \begin{bmatrix} x \\ 3x - 1 \end{bmatrix}$$

The equation of the image line is $y = 3x - 1$.

19. To prove:

$$\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x \\ 3x \end{bmatrix}$$

for all $a$, $b$ and for some relationship between $x$ and $a$ and $b$.

Proof:

$$\begin{aligned} \text{LHS} &= \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \\ &= \begin{bmatrix} a + 3b \\ 3a + 9b \end{bmatrix} \\ &= \begin{bmatrix} a + 3b \\ 3(a + 3b) \end{bmatrix} \\ \text{Let } x &= a + 3b \\ \text{then LHS} &= \begin{bmatrix} x \\ 3x \end{bmatrix} \\ &= \text{RHS} \end{aligned}$$

□

20. (a) $$\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ 5 - 3x \end{bmatrix} = \begin{bmatrix} 6(x) + 2(5 - 3x) \\ 3(x) + (5 - 3x) \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

The line $y = 5 - 3x$ is transformed to the point $(10, 5)$.

(b) $$\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 6a + 2b \\ 3a + b \end{bmatrix}$$

Let $x = 6a + 2b$

$$\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x \\ \frac{x}{2} \end{bmatrix}$$

Points on the $x$-$y$ plane are transformed to the line $y = \frac{x}{2}$ or $2y = x$.

21. Let $(a, b)$ be an arbitrary point before transformation and $(a', b')$ the corresponding point after transformation.

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a \\ 2a + b \end{bmatrix}$$

If the point before transformation lies on the line $y = m_1x + p$ then $b = m_1a + p$ and the transformed point is

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} 3a \\ 2a + (m_1a + p) \end{bmatrix} = \begin{bmatrix} 3a \\ (m_1 + 2)a + p \end{bmatrix}$$

We can turn this into a pair of parametric equations then convert that to a Cartesian equation of a line:

$$\begin{aligned} x &= 3a \\ y &= (m_1 + 2)a + p \\ &= \frac{(m_1 + 2)(3a)}{3} + p \\ &= \frac{m_1 + 2}{3}x + p \end{aligned}$$

which is in the form $y = m_2x + p$ where $m_2 = \frac{m_1+2}{3}$, as required.

Now consider two lines perpendicular to each other both before and after transformation.

Let $q$ be the gradient of the first line before transformation.

Since the lines are perpendicular, the gradient of the second line is $-\frac{1}{q}$.

Transforming the first line results in a gradient of $\frac{q+2}{3}$.

Transforming the second line results in a gradient of $\frac{-\frac{1}{q}+2}{3} = \frac{-1+2q}{3q}$.

Since the lines a perpendicular after transformation,

$$\begin{aligned} \frac{q+2}{3} &= -\frac{3q}{-1 + 2q} \\ &= \frac{3q}{1 - 2q} \\ (q + 2)(1 - 2q) &= 9q \\ q - 2q^2 + 2 - 4q &= 9q \\ -2q^2 + 2 - 12q &= 0 \\ q^2 - 1 + 6q &= 0 \\ q^2 + 6q - 1 &= 0 \\ (q + 3)^2 - 9 - 1 &= 0 \\ (q + 3)^2 &= 10 \\ q + 3 &= \pm\sqrt{10} \\ q &= -3 \pm \sqrt{10} \end{aligned}$$

Hence the gradients of the two lines before transformation are $-3 + \sqrt{10}$ and $-3 - \sqrt{10}$.

## Exercise 4C

1. (a) $\begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$
$= \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$

(b) $\begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$
$= \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$
$= \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$

(d) $\begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

(e) Two consecutive 30° anticlockwise rotations about the origin are represented by

$$\frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}\frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$
$$= \frac{1}{4}\begin{bmatrix} 3-1 & -2\sqrt{3} \\ 2\sqrt{3} & -1+3 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

which is a 60° anticlockwise rotation about the origin.

(f) A 30° anticlockwise rotation about the origin followed by a 60° anticlockwise rotation about the origin is represented by

$$\frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$
$$= \frac{1}{4}\begin{bmatrix} \sqrt{3}-\sqrt{3} & -1-3 \\ 3+1 & -\sqrt{3}+\sqrt{3} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

which is a 90° anticlockwise rotation about the origin.

(g) Two consecutive 45° anticlockwise rotations about the origin are represented by

$$\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$
$$= \frac{2}{4}\begin{bmatrix} 1-1 & -1-1 \\ 1+1 & -1+1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

which is a 90° anticlockwise rotation about the origin.

2. (a) $\begin{bmatrix} \cos(2\times 30) & \sin(2\times 30) \\ \sin(2\times 30) & -\cos(2\times 30) \end{bmatrix}$
$= \begin{bmatrix} \cos 60 & \sin 60 \\ \sin 60 & -\cos 60 \end{bmatrix}$
$= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$
$= \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$

(b) $\begin{bmatrix} \cos(2\times 60) & \sin(2\times 60) \\ \sin(2\times 60) & -\cos(2\times 60) \end{bmatrix}$
$= \begin{bmatrix} \cos 120 & \sin 120 \\ \sin 120 & -\cos 120 \end{bmatrix}$
$= \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$
$= \frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$

For (a),

$$\left(\frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}\right)^2 = \frac{1}{4}\begin{bmatrix} 1+3 & \sqrt{3}-\sqrt{3} \\ \sqrt{3}-\sqrt{3} & 3+1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similarly for (b),

$$\left(\frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\right)^2 = \frac{1}{4}\begin{bmatrix} 1+3 & -\sqrt{3}+\sqrt{3} \\ -\sqrt{3}+\sqrt{3} & 3+1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Any reflection must be its own inverse since reflecting a reflection restores the original. Consider the general form for a reflection:

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}^2$$
$$= \begin{bmatrix} \cos^2 2\theta + \sin^2 2\theta & \cos 2\theta \sin 2\theta - \cos 2\theta \sin 2\theta \\ \cos 2\theta \sin 2\theta - \cos 2\theta \sin 2\theta & \sin^2 2\theta + \cos^2 2\theta \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3. $\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

4. A rotation of angle $A$ followed by a rotation of angle $B$ is equivalent to a rotation of angle $A+B$.

A rotation of angle $A$ followed by a rotation of angle $B$ is represented by $\begin{bmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{bmatrix}\begin{bmatrix} \cos B & -\sin B \\ \sin B & \cos B \end{bmatrix}$
$= \begin{bmatrix} \cos A\cos B - \sin A\sin B & -\cos A\sin B - \sin A\cos B \\ \sin A\cos B + \cos A\sin B & -\sin A\sin B + \cos A\cos B \end{bmatrix}$
$= \begin{bmatrix} \cos A\cos B - \sin A\sin B & -(\sin A\cos B + \cos A\sin B) \\ \sin A\cos B + \cos A\sin B & \cos A\cos B - \sin A\sin B \end{bmatrix}$

A single rotation of angle $A+B$ is represented by $\begin{bmatrix} \cos(A+B) & -\sin(A+B) \\ \sin(A+B) & \cos(A+B) \end{bmatrix}$

Equating these gives

$$\begin{bmatrix} \cos(A+B) & -\sin(A+B) \\ \sin(A+B) & \cos(A+B) \end{bmatrix}$$
$$= \begin{bmatrix} \cos A\cos B - \sin A\sin B & -(\sin A\cos B + \cos A\sin B) \\ \sin A\cos B + \cos A\sin B & \cos A\cos B - \sin A\sin B \end{bmatrix}$$

Equating corresponding matrix elements from any column or row gives:

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\text{and } \cos(A+B) = \cos A\cos B - \sin A\sin B$$

as required. □

5. (a) The 180° rotation is represented by

$$\begin{bmatrix} \cos 180° & -\sin 180° \\ \sin 180° & \cos 180° \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

This transformation leaves point O unchanged at the origin. We need to transform this point to $(6,4)$ so the total transformation is represented by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

(b) Let $\theta$ be the angle that the line OO′ makes with the $x$-axis. This is the angle that O′A′B′C′ must be rotated clockwise in order to transform O′ onto the $x$-axis. Since O′ has coordinates $(6,4)$,

$$\tan\theta = \frac{4}{6} = \frac{2}{3}$$
$$\sin\theta = \frac{4}{\sqrt{4^2+6^2}} = \frac{4}{\sqrt{52}} = \frac{4}{2\sqrt{13}} = \frac{2}{\sqrt{13}}$$
$$\cos\theta = \frac{6}{2\sqrt{13}} = \frac{3}{\sqrt{13}}$$

The matrix to achieve this clockwise rotation (see question 3) is

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{bmatrix}$$
$$= \frac{1}{\sqrt{13}}\begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$$

(c) $\frac{1}{\sqrt{13}}\begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 6 & 5 & 5 & 6 \\ 4 & 4 & 3 & 3 \end{bmatrix} = \frac{1}{\sqrt{13}}\begin{bmatrix} 26 & 23 & 21 & 24 \\ 0 & 2 & -1 & -3 \end{bmatrix}$

- O″$(\frac{26}{\sqrt{13}}, 0) = (2\sqrt{13}, 0)$,
- A″$(\frac{23}{\sqrt{13}}, \frac{2}{\sqrt{13}})$,
- B″$(\frac{21}{\sqrt{13}}, -\frac{1}{\sqrt{13}})$,
- C″$(\frac{24}{\sqrt{13}}, -\frac{3}{\sqrt{13}})$.

(You could, if preferred, give these with rational denominators and arrive at the same answers Sadler gives.)

## Miscellaneous Exercise 4

1. Choose every possibility where the number of columns in the first is equal to the number of rows in the second. Thus

   - A has 3 columns so it can pre-multiply every matrix with 3 rows, resulting in the products AC and AD.
   - B also has 3 columns, resulting in products BC and BD.
   - C has only 1 column so it can pre-multiply every matrix with one row: CB.
   - D has three columns, resulting in the products DC and $D^2$.

2. No working necessary for these questions.

3. XY cannot be formed – X has three columns but Y has only one row.

   YX cannot be formed – Y has three columns but X has five rows.

   XZ can be formed – X has three columns and Z has three rows.

   ZX cannot be formed – Z has one column and X has five rows.

   The product XZ has five rows (number of rows in X) and one column (number of columns in Z). Each row is the sum of the number of wins times the number of points per win, the number of draws times the number of points per draw and the number of losses times the number of points per loss, that is, the total points for the corresponding team.

4. (a) $$4e^{\pi i/6} = 4\operatorname{cis}\frac{\pi}{6}$$
$$= 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$
$$= 4\times\frac{\sqrt{3}}{2} + 4\times\frac{1}{2}i$$
$$= 2\sqrt{3} + 2i$$

(b) $$\begin{aligned} -20e^{\pi i/3} &= -20\,\text{cis}\,\frac{\pi}{3} \\ &= -20\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \\ &= -20\times\frac{1}{2} - 20\times\frac{\sqrt{3}}{2}i \\ &= -10 - 10\sqrt{3}i \end{aligned}$$

(c) $$\begin{aligned} 20e^{-\pi i/3} &= 20\,\text{cis}\left(-\frac{\pi}{3}\right) \\ &= 20\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) \\ &= 20\times\frac{1}{2} - 20\times\frac{\sqrt{3}}{2}i \\ &= 10 - 10\sqrt{3}i \end{aligned}$$

(d) $$\begin{aligned} 1 + e^{\pi i/2} &= 1 + \text{cis}\,\frac{\pi}{2} \\ &= 1 + \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \\ &= 1 + 0 + i \\ &= 1 + i \end{aligned}$$

5. $$\begin{aligned} P &= Q + PR \\ P - PR &= Q \\ P(I - R) &= Q \\ P &= Q(I - R)^{-1} \\ &= \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix}\right)^{-1} \\ &= \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ -1 & 2 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 5 & -7 \\ 5 & -8 \end{bmatrix} \end{aligned}$$

6. $$\begin{aligned} AB &= \begin{bmatrix} x & 2 \\ y & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 3x - 2 & x + 8 \\ 3y - 1 & y + 4 \end{bmatrix} \\ BA &= \begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x & 2 \\ y & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3x + y & 6 + 1 \\ -x + 4y & -2 + 4 \end{bmatrix} \\ &= \begin{bmatrix} 3x + y & 7 \\ -x + 4y & 2 \end{bmatrix} \end{aligned}$$

Given AB=BA, we can equate corresponding matrix elements. From element 1,2:

$$\begin{aligned} x + 8 &= 7 \\ x &= -1 \end{aligned}$$

From element 2,2:

$$\begin{aligned} y + 4 &= 2 \\ y &= -2 \end{aligned}$$

Confirm these results by substitution:

$$\begin{aligned} AB &= \begin{bmatrix} 3(-1) - 2 & 7 \\ 3(-2) - 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -5 & 7 \\ -7 & 2 \end{bmatrix} \\ BA &= \begin{bmatrix} 3(-1) + (-2) & 7 \\ -(-1) + 4(-2) & 2 \end{bmatrix} \\ &= \begin{bmatrix} -5 & 7 \\ -7 & 2 \end{bmatrix} \end{aligned}$$

Hence

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} -5 & 7 \\ -7 & 2 \end{bmatrix}$$

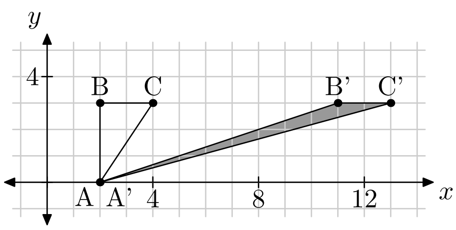
$$\therefore \quad \begin{aligned} p &= -5 \\ q &= 7 \\ r &= -7 \\ s &= 2 \end{aligned}$$

7. $$\begin{aligned} A\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix} \\ A &= \begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix}\frac{1}{4 - 3}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \end{aligned}$$

8. $$\begin{aligned} k &= z_1^4 \\ &= (2\,\text{cis}\,40°)^4 \\ &= 2^4\,\text{cis}(40\times 4)° \\ &= 16\,\text{cis}\,160°. \end{aligned}$$

The other roots are equal to $z_1$ rotated a multiple of 90°, i.e. $2\,\text{cis}(130°)$, $2\,\text{cis}(-50°)$ and $2\,\text{cis}(-140°)$.

9. $$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 4 \\ 0 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 11 & 13 \\ 0 & 3 & 3 \end{bmatrix}$$




This represents a shear parallel to the $x$-axis with scale factor 3.

10. To prove:

$$\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n}{4}(n+1)(n+2)(n+3)$$

Assume this is true for some $n = k$, then for $n = k+1$:

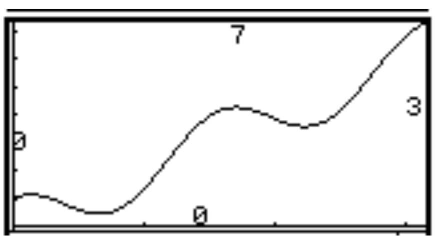
$$\begin{aligned}
\text{LHS} &= \sum_{i=1}^{k+1} i(i+1)(i+2) \\
&= \left(\sum_{i=1}^{k} i(i+1)(i+2)\right) + (k+1)(k+2)(k+3) \\
&= \frac{k}{4}(k+1)(k+2)(k+3) + (k+1)(k+2)(k+3) \\
&= (k+1)(k+2)(k+3)(\frac{k}{4}+1) \\
&= (k+1)(k+2)(k+3)(\frac{k+4}{4}) \\
&= \frac{1}{4}(k+1)(k+2)(k+3)(k+4) \\
&= \frac{k+1}{4}(k+2)(k+3)(k+4) \\
&= \text{RHS}
\end{aligned}$$

Therefore, if it is true for some $n = k$ then it is also true for $n = k+1$.

For $n = 1$,

$$\begin{aligned}
\text{LHS} &= \sum_{i=1}^{1} i(i+1)(i+2) \\
&= 1(2)(3) \\
&= (1)(2)(3)\frac{4}{4} \\
&= \frac{1}{4}(2)(3)(4) \\
&= \text{RHS}
\end{aligned}$$

Therefore the proposition is true for $n = 1$ and hence by mathematical induction, true for all $n, n \geq 1$. □

11. 


(a) From the graph, the global maximum appears to be where $x = \pi$ with coordinates $(\pi, 2\pi + \cos 4\pi) = (\pi, 2\pi + 1)$. (This is not actually a local maximum, since the gradient at that point is positive.)

(b) From the graph, the global minimum corresponds to the first local minimum which is the second stationary point.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - 4\sin 4x$$

at the stationary points,

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
2 - 4\sin 4x &= 0 \\
\sin 4x &= \frac{1}{2} \\
4x &= \frac{5\pi}{6}
\end{aligned}$$

(ignoring the first solution at $4x = \frac{\pi}{6}$ as this will yield the first stationary point and we want the second)

$$\begin{aligned}
x &= \frac{5\pi}{24} \\
y &= \frac{5\pi}{12} + \cos\frac{5\pi}{6} \\
&= \frac{5\pi}{12} - \frac{\sqrt{3}}{2} \\
&= \frac{5\pi - 6\sqrt{3}}{12}
\end{aligned}$$

Thus the coordinates of the global minimum are $(\frac{5\pi}{24}, \frac{5\pi-6\sqrt{3}}{12})$.

(c) The local minimum that is not the global minimum is the fourth stationary point:

$$\begin{aligned}
\sin 4x &= \frac{1}{2} \\
4x &= \frac{17\pi}{6} \\
x &= \frac{17\pi}{24} \\
y &= \frac{17\pi}{12} + \cos\frac{5\pi}{6} \\
&= \frac{17\pi}{12} - \frac{\sqrt{3}}{2} \\
&= \frac{17\pi - 6\sqrt{3}}{12}
\end{aligned}$$

Thus the coordinates of the local minimum are $(\frac{17\pi}{24}, \frac{17\pi-6\sqrt{3}}{12})$.

(d) The two local minima are the first and third stationary points

$$\begin{aligned}
\sin 4x &= \frac{1}{2} \\
4x &= \frac{\pi}{6} && \text{or} && 4x = \frac{13\pi}{6} \\
x &= \frac{\pi}{24} && \text{or} && x = \frac{13\pi}{24} \\
y &= \frac{\pi}{12} + \cos\frac{\pi}{6} && \text{or} && y = \frac{13\pi}{12} + \cos\frac{\pi}{6} \\
&= \frac{\pi}{12} + \frac{\sqrt{3}}{2} && && = \frac{13\pi}{12} + \frac{\sqrt{3}}{2} \\
&= \frac{\pi + 6\sqrt{3}}{12} && && = \frac{13\pi + 6\sqrt{3}}{12}
\end{aligned}$$

Thus the coordinates of the local minimum are $(\frac{\pi}{24}, \frac{\pi+6\sqrt{3}}{12})$ and $(\frac{13\pi}{24}, \frac{13\pi+6\sqrt{3}}{12})$.

12.
$$\begin{aligned}
\operatorname{cis} 4\theta &= (\operatorname{cis}\theta)^4 \\
&= (\cos\theta + \mathrm{i}\sin\theta)^4 \\
&= \cos^4\theta \\
&\quad + 4\cos^3\theta(\mathrm{i}\sin\theta) \\
&\quad + 6\cos^2\theta(\mathrm{i}^2\sin^2\theta) \\
&\quad + 4\cos\theta(\mathrm{i}^3\sin^3\theta) \\
&\quad + \mathrm{i}^4\sin^4\theta \\
&= \cos^4\theta \\
&\quad + 4\mathrm{i}\cos^3\theta\sin\theta \\
&\quad - 6\cos^2\theta\sin^2\theta \\
&\quad - 4\mathrm{i}\cos\theta\sin^3\theta \\
&\quad + \sin^4\theta \\
&= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) \\
&\quad + \mathrm{i}(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta) \\
\operatorname{Re}(\operatorname{cis}4\theta) &= \cos 4\theta \\
&= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\
\operatorname{Im}(\operatorname{cis}4\theta) &= \sin 4\theta \\
&= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta
\end{aligned}$$

13.
$$\begin{aligned}
\mathrm{AP + BP + P} &= \mathrm{Q} \\
\mathrm{AP + BP + IP} &= \mathrm{Q} \\
\mathrm{(A + B + I)P} &= \mathrm{Q} \\
\mathrm{P} &= \mathrm{(A+B+I)^{-1}Q} \\
\mathrm{A+B+I} &= \begin{bmatrix} 3 & -1 \\ 8 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 8 & 2 \\ 7 & 2 \end{bmatrix} \\
\mathrm{P} &= \begin{bmatrix} 8 & 2 \\ 7 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 2 & -2 \\ -7 & 8 \end{bmatrix} \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} -2 & 2 \\ 6 & -10 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 1 \\ 3 & -5 \end{bmatrix}
\end{aligned}$$

14. (a) To prove:

$$\mathrm{AB^{-1} = B^{-1}A}$$

Proof:

$$\begin{aligned}
\mathrm{LHS} &= \mathrm{AB^{-1}} \\
&= \mathrm{IAB^{-1}} \\
&= \mathrm{B^{-1}BAB^{-1}} \\
&= \mathrm{B^{-1}ABB^{-1}} \\
&= \mathrm{B^{-1}AI} \\
&= \mathrm{B^{-1}A} \\
&= \mathrm{RHS}
\end{aligned}$$

□

(b) To prove:

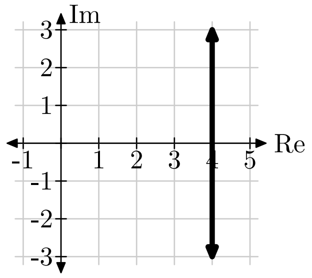
$$\mathrm{BA^{-1} = A^{-1}B}$$

Proof:

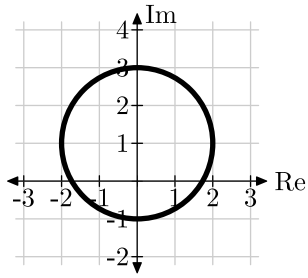
$$\begin{aligned}
\mathrm{LHS} &= \mathrm{BA^{-1}} \\
&= \mathrm{IBA^{-1}} \\
&= \mathrm{A^{-1}ABA^{-1}} \\
&= \mathrm{A^{-1}BAA^{-1}} \\
&= \mathrm{A^{-1}BI} \\
&= \mathrm{A^{-1}B} \\
&= \mathrm{RHS}
\end{aligned}$$

□

15. (a)
$$\begin{aligned}
z + \bar{z} &= 2\operatorname{Re}(z) \\
\text{so } z + \bar{z} &= 4 \\
\text{becomes } 2\operatorname{Re}(z) &= 4 \\
\operatorname{Re}(z) &= 2
\end{aligned}$$



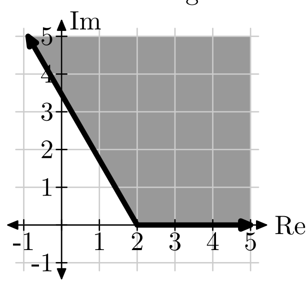

(b) This is a circle centred at $0 + \mathrm{i}$ and radius 2:




(c) This is a region the same shape as

$$0 \le \arg(z) \le \frac{2\pi}{3}$$

but translated 2 units right:




16. There is no conflict. The "proof" supposes that $\mathrm{A^{-1}}$ exists. In neither of the examples is this the case. In Example 1, A is not a square matrix, and hence no inverse exists. In Example 2, A is square, but it is singular so again no inverse exists.

What the proof actually shows is that if AB = AC **and** A **is a non-singular square matrix**, then B = C.

17. Let $\theta = \angle$AOB.

Let $s$ be the slant height of the resulting cone. This is equal to the radius of the original circle, that is, $s =$AO.

Let $r$ be the radius of the cone.

Let $h$ be the perpendicular height of the cone.

The circumference of the original circle is $2\pi s$. The fraction of this that is removed by sector AOB is $\frac{\theta}{2\pi}$, so the fraction remaining is

$$1 - \frac{\theta}{2\pi} = \frac{2\pi - \theta}{2\pi}$$

with the length of the remaining part of the circumference given by

$$l = \frac{2\pi - \theta}{2\pi}(2\pi s)$$

This becomes the circumference of the base of the cone, giving us the radius of the cone

$$\begin{aligned} r &= \frac{l}{2\pi} \\ &= \frac{(2\pi - \theta)s}{2\pi} \end{aligned}$$

The perpendicular height of the cone can be found using Pythagoras' theorem since the slant height, perpendicular height and radius of the cone form a right triangle:

$$\begin{aligned} h &= \sqrt{s^2 - r^2} \\ &= \sqrt{s^2 - \frac{(2\pi - \theta)^2 s^2}{4\pi^2}} \\ &= \sqrt{\frac{s^2}{4\pi^2}\left(4\pi^2 - (2\pi - \theta)^2\right)} \\ &= \frac{s}{2\pi}\sqrt{4\pi^2 - (2\pi - \theta)^2} \\ &= \frac{s}{2\pi}\sqrt{4\pi^2 - (4\pi^2 - 4\pi\theta + \theta^2)} \\ &= \frac{s}{2\pi}\sqrt{4\pi\theta - \theta^2} \end{aligned}$$

The volume of the cone is given by

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\left(\frac{(2\pi - \theta)s}{2\pi}\right)^2 \frac{s}{2\pi}\sqrt{4\pi\theta - \theta^2} \\ &= \frac{(1)\left((2\pi - \theta)s\right)^2 (s)\sqrt{4\pi\theta - \theta^2}}{(3)(2\pi)^2(2\pi)} \\ &= \frac{(2\pi - \theta)^2 s^2 (s)\sqrt{4\pi\theta - \theta^2}}{3(2\pi)^3} \\ &= \frac{(2\pi - \theta)^2 s^3\sqrt{4\pi\theta - \theta^2}}{3(2\pi)^3} \\ &= \frac{s^3}{3(2\pi)^3}(2\pi - \theta)^2\sqrt{4\pi\theta - \theta^2} \end{aligned}$$

Differentiating with respect to $\theta$ (and bearing in mind that $s$ is constant):

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}\theta} &= \frac{s^3}{3(2\pi)^3}\left(2(2\pi - \theta)(-1)\sqrt{4\pi\theta - \theta^2} + \frac{(2\pi - \theta)^2(4\pi - 2\theta)}{2\sqrt{4\pi\theta - \theta^2}}\right) \\ &= \frac{s^3}{3(2\pi)^3}\left(-2(2\pi - \theta)\sqrt{4\pi\theta - \theta^2} + \frac{(2\pi - \theta)^2 2(2\pi - \theta)}{2\sqrt{4\pi\theta - \theta^2}}\right) \\ &= \frac{s^3}{3(2\pi)^3}\left(-2(2\pi - \theta)\sqrt{4\pi\theta - \theta^2} + \frac{(2\pi - \theta)^3}{\sqrt{4\pi\theta - \theta^2}}\right) \\ &= \frac{s^3}{3(2\pi)^3}\,\frac{-2(2\pi - \theta)(4\pi\theta - \theta^2) + (2\pi - \theta)^3}{\sqrt{4\pi\theta - \theta^2}} \end{aligned}$$

At the maximum volume, this derivative is zero.

$$\begin{aligned} \frac{s^3}{3(2\pi)^3}\,\frac{-2(2\pi - \theta)(4\pi\theta - \theta^2) + (2\pi - \theta)^3}{\sqrt{4\pi\theta - \theta^2}} &= 0 \\ -2(2\pi - \theta)(4\pi\theta - \theta^2) + (2\pi - \theta)^3 &= 0 \\ -2(4\pi\theta - \theta^2) + (2\pi - \theta)^2 &= 0 \end{aligned}$$

(the previous step is only valid because we know $2\pi - \theta \neq 0$)

$$\begin{aligned} -8\pi\theta + 2\theta^2 + 4\pi^2 - 4\pi\theta + \theta^2 &= 0 \\ 3\theta^2 - 12\pi\theta + 4\pi^2 &= 0 \\ 3(\theta - 2\pi)^2 - 12\pi^2 + 4\pi^2 &= 0 \\ 3(\theta - 2\pi)^2 - 8\pi^2 &= 0 \\ 3(\theta - 2\pi)^2 &= 8\pi^2 \\ (\theta - 2\pi)^2 &= \frac{2}{3}(2\pi)^2 \\ \theta - 2\pi &= \pm\sqrt{\frac{2}{3}}(2\pi) \\ \theta &= 2\pi \pm \frac{\sqrt{6}}{3}(2\pi) \\ &= 2\pi\left(1 - \frac{\sqrt{6}}{3}\right) \\ &= \frac{2\pi(3 - \sqrt{6})}{3} \end{aligned}$$

(in the second last step, discarding the result that would result in $\theta > 2\pi$.)

Converting to degrees gives

$$\begin{aligned} \theta &= 120(3 - \sqrt{6}) \\ &= 66.1^\circ \text{ (1 d.p.)} \end{aligned}$$

# Chapter 5

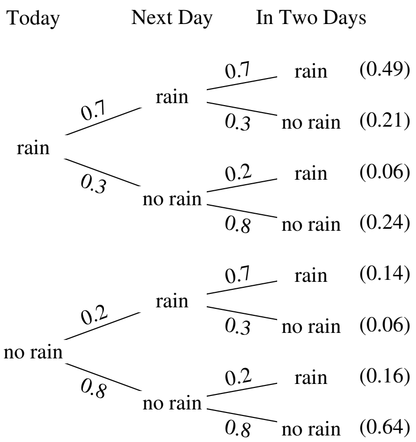
## Exercise 5A

1.

(a)

| | | From: Rain today | From: No rain today |
|---|---|---|---|
| To | Rain next day | 0.7 | 0.2 |
| | No rain next day | 0.3 | 0.8 |

(b) The tree diagram:




gives us the transition matrix

| | | From: Rain today | From: No rain today |
|---|---|---|---|
| To | Rain in 2 days | 0.49+0.06 | 0.14+0.16 |
| | No rain in 2 days | 0.21+0.24 | 0.64+0.06 |

which simplifies to

| | | From: Rain today | From: No rain today |
|---|---|---|---|
| To | Rain in 2 days | 0.55 | 0.30 |
| | No rain in 2 days | 0.45 | 0.70 |

and squaring the first matrix

$$\begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}^2 = \begin{bmatrix} 0.55 & 0.30 \\ 0.45 & 0.70 \end{bmatrix}$$

gives the same result.

2. (a)

| | | To: Tom has ball 1 pass later | To: Tim has ball 1 pass later | To: Tony has ball 1 pass later |
|---|---|---|---|---|
| From | Tom has ball now | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| | Tim has ball now | $\frac{2}{5}$ | 0 | $\frac{3}{5}$ |
| | Tony has ball now | $\frac{3}{4}$ | $\frac{1}{4}$ | 0 |

(b) For two passes later, the probabilities are

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix}^2 = \begin{bmatrix} \frac{23}{40} & \frac{1}{8} & \frac{3}{10} \\ \frac{9}{20} & \frac{7}{20} & \frac{1}{5} \\ \frac{1}{10} & \frac{3}{8} & \frac{21}{40} \end{bmatrix}$$

For three passes later, the probabilities are

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix}^3 = \begin{bmatrix} \frac{11}{40} & \frac{29}{80} & \frac{29}{80} \\ \frac{29}{100} & \frac{11}{40} & \frac{87}{200} \\ \frac{87}{160} & \frac{29}{160} & \frac{11}{40} \end{bmatrix}$$

Using these,

i. From Tom to Tony after two passes, refer to the cell in the first row (from Tom) and third column (to Tony): the probability is $\frac{3}{10}$.

ii. From Tony to Tim after two passes, refer to the cell in the third row (from Tony) and second column (to Tim): the probability is $\frac{3}{8}$.

iii. From Tim back to Tim after three passes, refer to the cell in the second row (from Tim) and second column (to Tim): the probability is $\frac{11}{40}$.

iv. From Tom back to Tom after three passes, refer to the cell in the first row (from Tom) and first column (to Tom): the probability that he will have the ball is $\frac{11}{40}$, so the probability that he will not have the ball is $\frac{29}{40}$.

3. Let $R$ be the transition matrix for Roz's coffee shop visits:

| | | To: A next week | To: B next week | To: C next week |
|---|---|---|---|---|
| From | A this week | $\frac{1}{10}$ | $\frac{1}{2}$ | $\frac{2}{5}$ |
| | B this week | $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{1}{2}$ |
| | C this week | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{2}$ |

(a) From cell (3,3) of $R$, $p = \frac{1}{2}$.

(b) $R^3 = \begin{bmatrix} 0.228 & 0.300 & 0.472 \\ 0.245 & 0.277 & 0.479 \\ 0.239 & 0.284 & 0.477 \end{bmatrix}$
so $p = 0.477$ (3 d.p.).

(c) $R^{10} = \begin{bmatrix} 0.238 & 0.286 & 0.476 \\ 0.238 & 0.286 & 0.476 \\ 0.238 & 0.286 & 0.476 \end{bmatrix}$
so $p = 0.476$ (3 d.p.).
Notice how all three rows are identical to three decimal places. This indicates that after ten weeks it makes no difference to the probabilities (at this level of precision) which coffee shop Roz started in. We have reached the long-range expectation for the three coffee shops.

4. The transition matrix $T$ is

| | | To: Labour next election | To: Conservative next election | To: Other next election |
|---|---|---|---|---|
| From | Labour this election | 0.81 | 0.07 | 0.12 |
| | Conservative this election | 0.13 | 0.78 | 0.09 |
| | Other this election | 0.13 | 0.12 | 0.75 |

(a) $T^3 = \begin{bmatrix} 0.59 & 0.17 & 0.24 \\ 0.28 & 0.52 & 0.20 \\ 0.28 & 0.24 & 0.49 \end{bmatrix}$

so the probability of a Labour voter in one election voting Labour three elections later is 0.59 (from cell (1,1)).

(b) To vote Labour in *each* of the next three elections, the probability is $0.81^3 = 0.53$.

(c) In addition to consistently voting Labour, answer (a) includes all combinations of voting behaviour that start with Labour, have votes for another party in the first and/or second subsequent elections and then return to Labour for the third election.

(d) $T^4 = \begin{bmatrix} 0.53 & 0.20 & 0.26 \\ 0.32 & 0.45 & 0.23 \\ 0.32 & 0.26 & 0.42 \end{bmatrix}$

The probability that a voter who votes Conservative in an election votes Other in four elections time is 0.23 (from cell (2,3)).

5. It's possible to answer (a) directly from the transition diagram (the probability of a transition from B to A is 0.2) and (b) and perhaps (c) could be answered using a tree diagram, but by far the simplest approach is to translate the transition diagram to a transition matrix $T$:

$$\begin{array}{cc} & \text{To} \\ & \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ \text{From} \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array} & \begin{bmatrix} 0.1 & 0.6 & 0.3 \\ 0.2 & 0.4 & 0.4 \\ 0.8 & 0 & 0.2 \end{bmatrix} \end{array}$$

(The transition matrix can also be written with the columns representing the initial state and the rows representing the resulting state.)

The probability of the situation being in state A given that it starts in state B is found in cell $(2,1)$

(a) $p = 0.2$

(b) $T^2 = \begin{bmatrix} 0.37 & 0.30 & 0.33 \\ 0.42 & 0.28 & 0.30 \\ 0.24 & 0.48 & 0.28 \end{bmatrix}$

$p = 0.42$

(c) $T^3 = \begin{bmatrix} 0.361 & 0.342 & 0.297 \\ 0.338 & 0.364 & 0.298 \\ 0.344 & 0.336 & 0.320 \end{bmatrix}$

$p = 0.338$

(d) $T^5 = \begin{bmatrix} 0.348 & 0.347 & 0.305 \\ 0.349 & 0.346 & 0.304 \\ 0.345 & 0.351 & 0.304 \end{bmatrix}$

$p = 0.349$

(e) $T^{10} = \begin{bmatrix} 0.348 & 0.348 & 0.304 \\ 0.348 & 0.348 & 0.304 \\ 0.348 & 0.348 & 0.304 \end{bmatrix}$

$p = 0.348$

6. The transition diagram translates to the following transition matrix $T$:

$$\begin{array}{cc} & \text{From} \\ & \begin{array}{ccc} \text{P} & \text{Q} & \text{R} \end{array} \\ \text{To} \begin{array}{c} \text{P} \\ \text{Q} \\ \text{R} \end{array} & \begin{bmatrix} 0.2 & 0.4 & 0.1 \\ 0.8 & 0 & 0.9 \\ 0 & 0.6 & 0 \end{bmatrix} \end{array}$$

After starting in state R, the probability of being in state P is found in cell $(1,3)$, the probability of being in state Q is in cell $(2,3)$ and the probability of being in state R is in $(3,3)$.

(a) $p(\text{P}) = 0.1$

(b) $T^2 = \begin{bmatrix} 0.36 & 0.14 & 0.38 \\ 0.16 & 0.86 & 0.08 \\ 0.48 & 0 & 0.54 \end{bmatrix}$

$p(\text{Q}) = 0.08$

(c) $T^3 = \begin{bmatrix} 0.184 & 0.372 & 0.162 \\ 0.720 & 0.112 & 0.790 \\ 0.096 & 0.516 & 0.048 \end{bmatrix}$

$p(\text{R}) = 0.048$

(d) $T^{10} = \begin{bmatrix} 0.295 & 0.224 & 0.303 \\ 0.362 & 0.590 & 0.336 \\ 0.343 & 0.186 & 0.361 \end{bmatrix}$

$p(\text{P}) = 0.303$

(e) $T^{20} = \begin{bmatrix} 0.272 & 0.254 & 0.274 \\ 0.436 & 0.492 & 0.429 \\ 0.292 & 0.254 & 0.297 \end{bmatrix}$

$p(\text{Q}) = 0.429$

## Exercise 5B

1. Transition matrix $T$:

$$\text{To}\ \begin{matrix} \text{City in 5 years} \\ \text{Country in 5 years} \end{matrix} \overset{\begin{matrix}\text{From} \\ \text{City now} \quad \text{Country now}\end{matrix}}{\begin{bmatrix} 0.95 & 0.18 \\ 0.05 & 0.82 \end{bmatrix}}$$

(a) i. $\begin{bmatrix} 0.95 & 0.18 \\ 0.05 & 0.82 \end{bmatrix}\begin{bmatrix} 765 \\ 511 \end{bmatrix} = \begin{bmatrix} 819 \\ 457 \end{bmatrix}$
The model predicts 819 000 will live in the city and 457 000 in the country in five years time.

ii. $\begin{bmatrix} 0.95 & 0.18 \\ 0.05 & 0.82 \end{bmatrix}^2\begin{bmatrix} 765 \\ 511 \end{bmatrix} = \begin{bmatrix} 860 \\ 416 \end{bmatrix}$
The model predicts 860 000 will live in the city and 416 000 in the country in five years time.

(b) Experiment with increasingly high powers of $T$ until both columns are equal, to two decimals:

$$T^{20} = \begin{bmatrix} 0.78 & 0.78 \\ 0.22 & 0.22 \end{bmatrix}$$

In the long term 78 per cent of the population will live in the city and 22 per cent in the country.

(c) I postmultiplied the transition matrix by the initial state matrix because of the way I set up my transition matrix with the columns representing the 'From' state and the rows representing the 'To' state.

2. Transition matrix $T$:

$$\text{From}\ \begin{matrix} A \\ B \end{matrix} \overset{\begin{matrix}\text{To} \\ A \quad\quad B\end{matrix}}{\begin{bmatrix} 0.98 & 0.02 \\ 0.05 & 0.95 \end{bmatrix}}$$

(a) $\begin{bmatrix} 260 & 138 \end{bmatrix}\begin{bmatrix} 0.98 & 0.02 \\ 0.05 & 0.95 \end{bmatrix}^2 = \begin{bmatrix} 263 & 135 \end{bmatrix}$
The model predicts that 263 of the 398 staff, or 66% will be at A in two years time, with 34% at B.

(b) $\begin{bmatrix} 260 & 138 \end{bmatrix}\begin{bmatrix} 0.98 & 0.02 \\ 0.05 & 0.95 \end{bmatrix}^5 = \begin{bmatrix} 267 & 131 \end{bmatrix}$
The model predicts that 267 of the 398 staff, or 67% will be at A in five years time, with 33% at B.

(c) Experiment with increasingly high powers of $T$ until both rows are equal, to two decimals:

$$T^{100} = \begin{bmatrix} 0.71 & 0.29 \\ 0.71 & 0.29 \end{bmatrix}$$

and conclude that the long term expectation is that 71% will be at A and 29% at B.
Alternatively, if an exact value is required solve the steady state equation

$$\begin{bmatrix} a & 1-a \end{bmatrix}\begin{bmatrix} 0.98 & 0.02 \\ 0.05 & 0.95 \end{bmatrix} = \begin{bmatrix} a & 1-a \end{bmatrix}$$
$$\begin{bmatrix} 0.98a+0.05(1-a) & 0.02a+0.95(1-a) \end{bmatrix} = \begin{bmatrix} a & 1-a \end{bmatrix}$$

equating the first elements,

$$\begin{aligned} 0.98a + 0.05(1-a) &= a \\ 0.05 - 0.05a &= 0.02a \\ 0.07a &= 0.05 \\ a &= \frac{5}{7} \\ &\approx 0.71 \end{aligned}$$

similarly the second elements

$$\begin{aligned} 0.02a + 0.95(1-a) &= 1-a \\ 0.02a &= 0.05(1-a) \\ 0.07a &= 0.05 \\ a &= \frac{5}{7} \\ &\approx 0.71 \end{aligned}$$

3. (a) Cell $(1,1)$: $p = \frac{1}{2}$.

(b) $\begin{bmatrix} \frac{1}{2} & \frac{1}{5} & \frac{3}{10} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \end{bmatrix}^2 = \begin{bmatrix} \frac{121}{300} & \frac{13}{50} & \frac{101}{300} \\ \frac{3}{10} & \frac{7}{20} & \frac{7}{20} \\ \frac{59}{150} & \frac{13}{50} & \frac{26}{75} \end{bmatrix}$
Cell $(2,3)$: $p = \frac{7}{20}$

(c) To two decimal places,

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{5} & \frac{3}{10} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \end{bmatrix}^5 = \begin{bmatrix} 0.37 & 0.29 & 0.34 \\ 0.37 & 0.29 & 0.34 \\ 0.37 & 0.29 & 0.34 \end{bmatrix}$$

In the long term, the team expects to win 37%, lose 29% and draw 34%.

4. First present the information as initial matrix $I$ and transition matrix $T$:

$$I = \begin{matrix} \text{Tea} \\ \text{Coffee} \\ \text{Juice} \\ \text{No drink} \end{matrix} \begin{bmatrix} 52 \\ 93 \\ 84 \\ 21 \end{bmatrix}$$

$$T = \begin{matrix} \\ \text{Next} \\ \text{day} \\ \\ \end{matrix} \begin{matrix} \text{Tea} \\ \text{Coffee} \\ \text{Juice} \\ \text{No drink} \end{matrix} \overset{\begin{matrix}\text{Today} \\ \text{Tea} \quad \text{Coffee} \quad \text{Juice} \quad \text{No drink}\end{matrix}}{\begin{bmatrix} 0.65 & 0.08 & 0.04 & 0.16 \\ 0.22 & 0.75 & 0.08 & 0.24 \\ 0.09 & 0.15 & 0.82 & 0.25 \\ 0.04 & 0.02 & 0.06 & 0.35 \end{bmatrix}}$$

(a) $TI = \begin{bmatrix} 48 \\ 93 \\ 93 \\ 16 \end{bmatrix}$
The probabilities suggest 48 tea, 93 coffee, 93 juice and 16 no drink the next day.

(b) $T^{30} = \begin{bmatrix} 0.16 & 0.16 & 0.16 & 0.16 \\ 0.34 & 0.34 & 0.34 & 0.34 \\ 0.44 & 0.44 & 0.44 & 0.44 \\ 0.06 & 0.06 & 0.06 & 0.06 \end{bmatrix}$

The probabilities suggest the long term percentages are 16% tea, 34% coffee, 44% juice and 6% no drink.

5. Since the question does not spell out exactly what "randomly select" means, we must assume equal probability for each available path. A person at A or D has three available paths so we assign each the probability of $\frac{1}{3}$ while a person at B or C has only two available paths so we assign each of these the probability of $\frac{1}{2}$. This gives the transition matrix

$$T = \text{From}\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\overset{\text{To}}{\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix} \end{array}}$$

$$\begin{bmatrix} 22 & 22 & 22 & 22 \end{bmatrix} T^{10} = \begin{bmatrix} 27 & 16 & 24 & 21 \end{bmatrix}$$

$$\begin{bmatrix} 88 & 0 & 0 & 0 \end{bmatrix} T^{10} = \begin{bmatrix} 27 & 16 & 24 & 21 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 44 & 0 & 44 \end{bmatrix} T^{10} = \begin{bmatrix} 27 & 16 & 24 & 21 \end{bmatrix}$$

etc.

In the long term, after (say) ten moves, it makes no difference where the 88 people are initially stationed, they will end up with 27 on A, 16 on B, 24 on C and 21 on D.

(These are, of course, only expected outcomes. Because of the random nature of the moves the actual numbers would vary either side of these expected values.)

6. First complete the transition matrix so each column totals 1.0:

$$\begin{bmatrix} 0.61 & 0.28 & 0.19 \\ 0.35 & 0.64 & 0.72 \\ 0.04 & 0.08 & 0.09 \end{bmatrix}$$

(a) $$\begin{bmatrix} 0.61 & 0.28 & 0.19 \\ 0.35 & 0.64 & 0.72 \\ 0.04 & 0.08 & 0.09 \end{bmatrix}\begin{bmatrix} 35 \\ 48 \\ 12 \end{bmatrix} = \begin{bmatrix} 37 \\ 52 \\ 6 \end{bmatrix}$$

The table suggests that 37 will have school, 52 a normal degree and 6 a higher degree as their highest level of education.

(b) $$\begin{bmatrix} 0.61 & 0.28 & 0.19 \\ 0.35 & 0.64 & 0.72 \\ 0.04 & 0.08 & 0.09 \end{bmatrix}^6 = \begin{bmatrix} 0.41 & 0.41 & 0.41 \\ 0.53 & 0.53 & 0.53 \\ 0.06 & 0.06 & 0.06 \end{bmatrix}$$

If the trends continue, in the long term 41% will have school, 53% a normal degree and 6% a higher degree as their highest level of education. (However, given the nature of the data, it might be considered very unlikely for these trends to continue unchanged for the several generations needed for the long-term outcome to be meaningful. Also note other potential issues with the experimental design mentioned in Sadler's answers.)

7. Completing the matrix gives

$$\text{To}\begin{array}{c} \\ A \\ B \\ C \end{array}\overset{\text{From}}{\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{2} \end{bmatrix} \end{array}}$$

For the long-term probabilities,

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{2} \end{bmatrix}^{15} = \begin{bmatrix} 0.286 & 0.286 & 0.286 \\ 0.429 & 0.429 & 0.429 \\ 0.286 & 0.286 & 0.286 \end{bmatrix}$$

$p(A) = 0.286$; $p(B) = 0.429$; and $p(C) = 0.286$. Intuitively we would expect the probability of finding the sentry at B to be greater than that of finding him at A or C, and we would have expected to find the symmetry with A and C having equal probability, but we would not have predicted the actual probabilities intuitively.

8. The transition matrix is

$$\text{To}\begin{array}{c} \\ X \\ Y \\ Z \end{array}\overset{\text{From}}{\begin{array}{c} \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{2}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{2}{3} & 0 & 0 \end{bmatrix} \end{array}}$$

For the long-term probabilities,

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{2}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{2}{3} & 0 & 0 \end{bmatrix}^{15} = \begin{bmatrix} 0.474 & 0.474 & 0.474 \\ 0.211 & 0.211 & 0.211 \\ 0.316 & 0.316 & 0.316 \end{bmatrix}$$

$p(X) = 0.474$; $p(Y) = 0.211$; and $p(Z) = 0.316$.

(These probabilities correct to three decimal places appear to add to 1.001. This is, of course, simply an artefact of the rounding and we should not be disconcerted by it.)

## Exercise 5C

1. (a) $L^2P = \begin{bmatrix} 3\,888 \\ 3\,318 \\ 1\,792 \\ 864 \end{bmatrix}$

$L^5P = \begin{bmatrix} 6\,179 \\ 4\,130 \\ 2\,977 \\ 1\,306 \end{bmatrix}$

$L^{10}P = \begin{bmatrix} 11\,807 \\ 7\,477 \\ 5\,196 \\ 2\,751 \end{bmatrix}$

$L^{20}P = \begin{bmatrix} 40\,108 \\ 24\,867 \\ 17\,607 \\ 9\,370 \end{bmatrix}$

$L^{50}P = \begin{bmatrix} 1\,530\,091 \\ 948\,631 \\ 672\,154 \\ 357\,192 \end{bmatrix}$

(b) The product is a $1 \times 1$ matrix where the cell value is the total population.

(c) $TL^5P = \begin{bmatrix} 14\,593 \end{bmatrix}$ $x_5 = 14\,593$

$TL^6P = \begin{bmatrix} 17\,003 \end{bmatrix}$ $x_6 = 17\,003$

$TL^7P = \begin{bmatrix} 18\,942 \end{bmatrix}$ $x_7 = 18\,942$

$TL^8P = \begin{bmatrix} 21\,335 \end{bmatrix}$ $x_8 = 21\,335$

$TL^9P = \begin{bmatrix} 24\,312 \end{bmatrix}$ $x_9 = 24\,312$

$TL^{10}P = \begin{bmatrix} 27\,231 \end{bmatrix}$ $x_{10} = 27\,231$

$TL^{19}P = \begin{bmatrix} 81\,436 \end{bmatrix}$ $x_{19} = 81\,436$

$TL^{20}P = \begin{bmatrix} 91\,952 \end{bmatrix}$ $x_{20} = 91\,952$

$TL^{29}P = \begin{bmatrix} 274\,149 \end{bmatrix}$ $x_{29} = 274\,149$

$TL^{30}P = \begin{bmatrix} 309\,534 \end{bmatrix}$ $x_{30} = 309\,534$

$TL^{39}P = \begin{bmatrix} 922\,931 \end{bmatrix}$ $x_{39} = 922\,931$

$TL^{40}P = \begin{bmatrix} 1\,042\,047 \end{bmatrix}$ $x_{40} = 1\,042\,047$

$TL^{49}P = \begin{bmatrix} 3\,107\,062 \end{bmatrix}$ $x_{49} = 3\,107\,062$

$TL^{50}P = \begin{bmatrix} 3\,508\,068 \end{bmatrix}$ $x_{50} = 3\,508\,068$

(d) $\dfrac{x_6}{x_5} = 1.165$

$\dfrac{x_7}{x_6} = 1.114$

$\dfrac{x_8}{x_7} = 1.126$

$\dfrac{x_9}{x_8} = 1.140$

$\dfrac{x_{10}}{x_9} = 1.120$

$\dfrac{x_{20}}{x_{19}} = 1.129$

$\dfrac{x_{30}}{x_{29}} = 1.129$

$\dfrac{x_{40}}{x_{39}} = 1.129$

$\dfrac{x_{50}}{x_{49}} = 1.129$

These results suggest a long-term steady growth rate of 12.9% per generation.

2. (a) The 0.3 is multiplied by the Youngster population and the product contributes to the Infant population in the next generation: that is, it is the reproduction rate of Youngsters.

(b) The 0.9 is multiplied by the Prime population and the product gives the Elderly population in the next generation: that is, it is the survival rate of the Prime population.

(c) $P_1 = \begin{bmatrix} 0 & 0.3 & 1.8 & 0.5 & 0 \\ 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 250 \\ 540 \\ 620 \\ 280 \\ 140 \end{bmatrix}$

$= \begin{bmatrix} 1420 \\ 175 \\ 430 \\ 560 \\ 140 \end{bmatrix}$

Translate into the terms of the question:

| Gen. | Infant | Youngster | Prime | Elderly | Aged |
|---|---|---|---|---|---|
| Pop'n | 1420 | 175 | 430 | 560 | 140 |

(Note: these are rounded figures.)

(d) $P_5 = \begin{bmatrix} 0 & 0.3 & 1.8 & 0.5 & 0 \\ 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \end{bmatrix}^5 \begin{bmatrix} 250 \\ 540 \\ 620 \\ 280 \\ 140 \end{bmatrix}$

$= \begin{bmatrix} 1630 \\ 1210 \\ 420 \\ 560 \\ 360 \end{bmatrix}$

| Gen. | Infant | Youngster | Prime | Elderly | Aged |
|---|---|---|---|---|---|
| Pop'n | 1630 | 1210 | 420 | 560 | 360 |

(e) $$P_{25} = \begin{bmatrix} 0 & 0.3 & 1.8 & 0.5 & 0 \\ 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \end{bmatrix}^{25} \begin{bmatrix} 250 \\ 540 \\ 620 \\ 280 \\ 140 \end{bmatrix} = \begin{bmatrix} 20\,100 \\ 12\,300 \\ 8\,800 \\ 6\,900 \\ 3\,000 \end{bmatrix}$$

| Gen. | Infant | Youngster | Prime | Elderly | Aged |
|---|---|---|---|---|---|
| Pop'n | 20 100 | 12 300 | 8 800 | 6 900 | 3 000 |

(f) $$\begin{bmatrix} 0 & 0.3 & 1.8 & 0.5 & 0 \\ 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \end{bmatrix}^{10} \begin{bmatrix} 250 \\ 540 \\ 620 \\ 280 \\ 140 \end{bmatrix} = \begin{bmatrix} 3\,100 \\ 1\,600 \\ 1\,300 \\ 1\,100 \\ 350 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3\,100 \\ 1\,600 \\ 1\,300 \\ 1\,100 \\ 350 \end{bmatrix} = \begin{bmatrix} 7600 \end{bmatrix}$$

The total population after 20 years (10 generations) is 7 600 (to the nearest hundred).

3. (a) No working required. Put reproduction rates in the first row and survival rates (not death rates!) in the first diagonal below the prime diagonal.

(b) $$\begin{bmatrix} 0 & 2.3 & 1.7 & 0.1 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}^{10} \begin{bmatrix} 2300 \\ 2800 \\ 3200 \\ 1800 \end{bmatrix} = \begin{bmatrix} 308\,000 \\ 143\,000 \\ 76\,000 \\ 21\,000 \end{bmatrix}$$

(rounded to the nearest thousand).

(c) $$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2.3 & 1.7 & 0.1 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}^{25} \begin{bmatrix} 2300 \\ 2800 \\ 3200 \\ 1800 \end{bmatrix} = \begin{bmatrix} 243\,000\,000 \end{bmatrix}$$

(rounded to the nearest million).

(d) From generation 25 to generation 26 the total population increases by

$$\frac{364961828}{243089970} = 1.5013$$

From generation 26 to generation 27 the total population increases by

$$\frac{364961828}{243089970} = 1.5013$$

Since these growth factors are equal to four decimal places, we can assume that we have reached the long-term growth rate of 50% every two years.

(e) I would expect the investigation to proceed somewhat along these lines:
Initial population:

$$P_0 = \begin{bmatrix} 2300 \\ 2800 \\ 3200 \\ 1800 \end{bmatrix} = 101 \begin{bmatrix} 0.23 \\ 0.28 \\ 0.32 \\ 0.18 \end{bmatrix}$$

Subsequent generations:

$$P_1 = \begin{bmatrix} 0 & 2.3 & 1.7 & 0.1 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix} \begin{bmatrix} 2300 \\ 2800 \\ 3200 \\ 1800 \end{bmatrix} = \begin{bmatrix} 12060 \\ 1610 \\ 2240 \\ 1280 \end{bmatrix} = 17190 \begin{bmatrix} 0.70 \\ 0.09 \\ 0.13 \\ 0.07 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0 & 2.3 & 1.7 & 0.1 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}^2 \begin{bmatrix} 2300 \\ 2800 \\ 3200 \\ 1800 \end{bmatrix} = \begin{bmatrix} 7639 \\ 8442 \\ 1288 \\ 896 \end{bmatrix} = 18256 \begin{bmatrix} 0.42 \\ 0.46 \\ 0.07 \\ 0.05 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 0 & 2.3 & 1.7 & 0.1 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}^3 \begin{bmatrix} 2300 \\ 2800 \\ 3200 \\ 1800 \end{bmatrix} = \begin{bmatrix} 21696 \\ 5347 \\ 6754 \\ 515 \end{bmatrix} = 34312 \begin{bmatrix} 0.63 \\ 0.16 \\ 0.20 \\ 0.02 \end{bmatrix}$$

$$P_4 = \begin{bmatrix} 0 & 2.3 & 1.7 & 0.1 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}^4 \begin{bmatrix} 2300 \\ 2800 \\ 3200 \\ 1800 \end{bmatrix} = \begin{bmatrix} 23831 \\ 15187 \\ 4278 \\ 2701 \end{bmatrix} = 45997 \begin{bmatrix} 0.52 \\ 0.33 \\ 0.09 \\ 0.06 \end{bmatrix}$$

Proceeding in this manner and tabulating the proportions in each generation gives

| Gen | $0 < x < 2$ | $2 \le x < 4$ | $4 \le x < 6$ | $6 \le x < 8$ |
|---|---|---|---|---|
| 0 | 0.23 | 0.28 | 0.32 | 0.18 |
| 1 | 0.70 | 0.09 | 0.13 | 0.07 |
| 2 | 0.42 | 0.46 | 0.07 | 0.05 |
| 3 | 0.63 | 0.16 | 0.20 | 0.02 |
| 4 | 0.52 | 0.33 | 0.09 | 0.06 |
| 5 | 0.58 | 0.23 | 0.17 | 0.02 |
| 6 | 0.55 | 0.28 | 0.12 | 0.05 |
| 7 | 0.56 | 0.26 | 0.15 | 0.03 |
| 8 | 0.56 | 0.26 | 0.14 | 0.04 |
| 9 | 0.56 | 0.26 | 0.14 | 0.04 |
| 10 | 0.56 | 0.26 | 0.14 | 0.04 |
| 11 | 0.56 | 0.26 | 0.14 | 0.04 |
| 12 | 0.56 | 0.26 | 0.14 | 0.04 |

At first the proportions vary wildly, but after a few generations they begin to settle down and after 8 generations there is no change (at least at this level of precision) from 56% in the youngest age group, 26% in the second, 14% in the third and 4% in the oldest.

## Exercise 5D

1. (a)

$$L = \begin{bmatrix} 0 & 0 & 0.8 & 0.4 & 0.1 \\ 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 \end{bmatrix}$$

$$P_0 = \begin{bmatrix} 350 \\ 420 \\ 330 \\ 140 \\ 70 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} TP_1 &= TLP_0 \\ &= \begin{bmatrix} 989 \end{bmatrix} \\ TP_2 &= TL^2P_0 \\ &= \begin{bmatrix} 770 \end{bmatrix} \\ TP_2 &= TL^3P_0 \\ &= \begin{bmatrix} 517 \end{bmatrix} \\ TP_2 &= TL^4P_0 \\ &= \begin{bmatrix} 375 \end{bmatrix} \\ TP_2 &= TL^5P_0 \\ &= \begin{bmatrix} 289 \end{bmatrix} \\ TP_2 &= TL^{10}P_0 \\ &= \begin{bmatrix} 55 \end{bmatrix} \end{aligned}$$

The total population in 1, 2, 3, 4, 5 and 10 generations time is predicted by the model to be 989, 770, 517, 375, 289 and 55.

(b)

$$L = \begin{bmatrix} 0 & 0 & 0.8 & 0.4 & 0.1 \\ 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \end{bmatrix}$$

$$\begin{aligned} TP_1 &= TLP_0 \\ &= \begin{bmatrix} 1319 \end{bmatrix} \\ TP_2 &= TL^2P_0 \\ &= \begin{bmatrix} 1363 \end{bmatrix} \\ TP_2 &= TL^3P_0 \\ &= \begin{bmatrix} 1256 \end{bmatrix} \\ TP_2 &= TL^4P_0 \\ &= \begin{bmatrix} 1141 \end{bmatrix} \\ TP_2 &= TL^5P_0 \\ &= \begin{bmatrix} 1127 \end{bmatrix} \\ TP_2 &= TL^{10}P_0 \\ &= \begin{bmatrix} 848 \end{bmatrix} \end{aligned}$$

The total population in 1, 2, 3, 4, 5 and 10 generations time is predicted by the revised model to be 1319, 1363, 1256, 1141, 1127 and 848.

(c) $$L = \begin{bmatrix} 0 & 0.2 & 0.9 & 0.5 & 0.1 \\ 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \end{bmatrix}$$

$$\begin{aligned} TP_1 &= TLP_0 \\ &= \begin{bmatrix} 1450 \end{bmatrix} \\ TP_2 &= TL^2P_0 \\ &= \begin{bmatrix} 1588 \end{bmatrix} \\ TP_2 &= TL^3P_0 \\ &= \begin{bmatrix} 1575 \end{bmatrix} \\ TP_2 &= TL^4P_0 \\ &= \begin{bmatrix} 1620 \end{bmatrix} \\ TP_2 &= TL^5P_0 \\ &= \begin{bmatrix} 1736 \end{bmatrix} \\ TP_2 &= TL^{10}P_0 \\ &= \begin{bmatrix} 2054 \end{bmatrix} \end{aligned}$$

The total population in 1, 2, 3, 4, 5 and 10 generations time is predicted by the second revised model to be 1450, 1588, 1575, 1620, 1736 and 2054.

2. (a) No working required.

(b) No working required.

(c) i. $$\begin{bmatrix} 0 & 0.5 & 0.9 & 1.5 & 0 \\ 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 \end{bmatrix} \begin{bmatrix} 350 \\ 620 \\ 750 \\ 180 \\ 60 \end{bmatrix} = \begin{bmatrix} 1255 \\ 245 \\ 496 \\ 675 \\ 108 \end{bmatrix}$$

ii. $$\begin{bmatrix} 0 & 0.5 & 0.9 & 1.5 & 0 \\ 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 \end{bmatrix}^{10} \begin{bmatrix} 350 \\ 620 \\ 750 \\ 180 \\ 60 \end{bmatrix} = \begin{bmatrix} 4082 \\ 2597 \\ 1695 \\ 1246 \\ 711 \end{bmatrix}$$

(d) Let the current population figures be represented by

$$P_0 = \begin{bmatrix} 350 \\ 620 \\ 750 \\ 180 \\ 60 \end{bmatrix}$$

and let U be given by

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

then the total population after $n$ years is given by

$$P_n = UL^nP_0$$

i. $UL^{19}P_0 = \begin{bmatrix} 39\,380 \end{bmatrix}$

ii. $UL^{20}P_0 = \begin{bmatrix} 45\,714 \end{bmatrix}$

iii. $UL^{29}P_0 = \begin{bmatrix} 174\,005 \end{bmatrix}$

iv. $UL^{30}P_0 = \begin{bmatrix} 201\,888 \end{bmatrix}$

(e) $\dfrac{45\,714}{39\,380} \approx 1.16$ The long term annual per-
$\dfrac{201\,888}{174\,005} \approx 1.16$
centage growth rate is about 16%.

(f) $\frac{1}{1.16} = 0.86$ so the annual harvesting rate should be about 14%.

(g) $$(0.95L)^5P_0 = \begin{bmatrix} 1699 \\ 786 \\ 557 \\ 617 \\ 294 \end{bmatrix} \approx \begin{bmatrix} 1700 \\ 800 \\ 550 \\ 600 \\ 300 \end{bmatrix}$$

3. (a) $$L = \begin{bmatrix} 0 & 0.8 & 1.6 & 0.3 \\ 0.6 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.7 & 0 \end{bmatrix}$$

(b) Let

$$P_0 = \begin{bmatrix} 850 \\ 750 \\ 600 \\ 400 \end{bmatrix}$$

and let

$$I = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

then

i. $IL^9P_0 = \begin{bmatrix} 7250 \end{bmatrix}$

ii. $IL^{10}P_0 = \begin{bmatrix} 8134 \end{bmatrix}$

iii. $IL^{19}P_0 = \begin{bmatrix} 21968 \end{bmatrix}$

iv. $IL^{20}P_0 = \begin{bmatrix} 24542 \end{bmatrix}$

v. $IL^{29}P_0 = \begin{bmatrix} 66558 \end{bmatrix}$

vi. $IL^{30}P_0 = \begin{bmatrix} 74359 \end{bmatrix}$

(c) $\dfrac{P_{10}}{P_9} = 1.122$

$\dfrac{P_{20}}{P_{19}} = 1.119$

$\dfrac{P_{30}}{P_{29}} = 1.117$

The data suggests the long term annual growth rate will be about 12%.

(d) $\dfrac{1}{1.117} = 0.895$

$1 - 0.895 = 0.105$

The annual harvesting rate should be between 10% and 11%.

To estimate the long term steady population of each year group, consider $P_{30}$ with 10.5% harvesting:

$$P_{30} = (0.895L)^{30}\,P_0 = \begin{bmatrix} 1230 \\ 660 \\ 470 \\ 300 \end{bmatrix}$$

(answers rounded to the nearest 10.)

(e) $\dfrac{0.95}{1.117} = 0.850$

$1 - 0.850 = 0.15$

The annual harvesting rate should be about 15%.

$$P_{10} = (0.850L)^{10}\,P_0 = \begin{bmatrix} 740 \\ 400 \\ 280 \\ 180 \end{bmatrix}$$

(answers rounded to the nearest 10.)

## Miscellaneous Exercise 5

1. (a) AB cannot be determined because it would require the number of columns of A (1) to equal the number of rows of B (2).

   (b) $\text{BA} = \begin{bmatrix} -1\times 3+2\times 1 \\ 1\times 3+4\times 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$

   (c) $\text{BC} = \begin{bmatrix} -3 & 3 & -3 \\ -3 & 3 & -3 \end{bmatrix}$

   (d) CD cannot be determined because it would require the number of columns of C (3) to equal the number of rows of D (2).

   (e) $\text{BD} = \begin{bmatrix} 0 & 1 & 2 \\ 6 & 5 & 4 \end{bmatrix}$

2. (a) $\text{AB} = \begin{bmatrix} 1\times 2-2\times 0+2\times -1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$

   (b) $\text{BA} = \begin{bmatrix} 2\times 1 & 2\times -2 & 2\times 2 \\ 0\times 1 & 0\times -2 & 0\times 2 \\ -1\times 1 & -1\times -2 & -1\times 2 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 4 \\ 0 & 0 & 0 \\ -1 & 2 & -2 \end{bmatrix}$

3. $$\begin{aligned} \text{AC} &= \text{B} \\ \text{A}^{-1}\text{AC} &= \text{A}^{-1}\text{B} \\ \text{IC} &= \text{A}^{-1}\text{B} \\ \text{C} &= \text{A}^{-1}\text{B} \\ &= \frac{1}{2\times 4-2\times -1}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 21 \\ 9 & 17 \end{bmatrix} \\ &= \frac{1}{11}\begin{bmatrix} 16-27 & 84-51 \\ 4+18 & 21+34 \end{bmatrix} \\ &= \frac{1}{11}\begin{bmatrix} -11 & 33 \\ 22 & 55 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 3 \\ 2 & 5 \end{bmatrix} \end{aligned}$$

4. (a) XY (2×2)(2×1) and ZX (1×2)(2×2) can be formed since these have the correct match of the number of columns in the first matrix with the number of rows in the second.

   (b) XY does not make sense. Consider the calculation of the value in the first cell: (No. of Aus stamps in the Mainly Aus pack × No. of Mainly Aus packs) + (No. of World stamps in the Mainly Aus pack × No. of Rest of World packs). Although the first term makes sense (giving the total number of Aus stamps in the Mainly Aus pack), the second term makes no sense since it multiplies two unrelated quantities.

   ZX does make sense. Consider again the calculation of the value in the first cell: (No. of Aus stamps in the Mainly Aus pack × No. of Mainly Aus packs) + (No. of Aus stamps in the Rest of World pack × No. of Rest of World packs). The two terms here give the total number of Aus stamps in the Mainly Aus pack and the total number of Aus stamps in the Rest of World pack, and their sum gives the total number of Aus stamps required. Similarly the value in the second cell gives the total number of Rest of World stamps required.

   (c) $\text{ZX} = \begin{bmatrix} 210 & 120 \end{bmatrix}\begin{bmatrix} 75 & 25 \\ 20 & 80 \end{bmatrix} = \begin{bmatrix} 18150 & 14850 \end{bmatrix}$

   18150 Australian stamps and 14850 Rest of World stamps will be required in order to supply the requests.

5. From matrix element (1,1):

$$\begin{aligned} 45-x^2 &= 4x \\ x^2+4x-45 &= 0 \end{aligned}$$

From matrix element (2,2):

$$\begin{aligned} 6x-5 &= x^2 \\ x^2-6x+5 &= 0 \end{aligned}$$

Combining these:

$$\begin{aligned} x^2+4x-45 &= x^2-6x+5 \\ 10x &= 50 \\ x &= 5 \end{aligned}$$

Similarly, from matrix element (1,2):

$$\begin{aligned} y^2-y &= 4-y \\ y^2-4 &= 0 \end{aligned}$$

From matrix element (2,1):

$$\begin{aligned} y^2+5y &= -6 \\ y^2-5y+6 &= 0 \end{aligned}$$

Combining these:

$$\begin{aligned} y^2-4 &= y^2-5y+6 \\ -5y &= -10 \\ y &= 2 \end{aligned}$$

6.
$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{6x(x-2)}{2y-3}\\
\int (2y-3)dy &= \int (6x(x-2))\,dx\\
&= \int \left(6x^2-12x\right)dx\\
\text{at } (3,4):\quad y^2-3y &= 2x^3-6x^2+c\\
(4)^2-3(4) &= 2(3)^3-6(3)^2+c\\
16-12 &= 54-54+c\\
c &= 4\\
y^2-3y &= 2x^3-6x^2+4\\
\text{given } x=1,\quad y^2-3y &= 2(1)^3-6(1)^2+4\\
&= 2-6+4\\
&= 0\\
y(y-3) &= 0\\
y &= 0\\
\text{or}\quad y &= 3
\end{aligned}$$

7. (a)
$$\begin{aligned}
\mathrm{T} &= \begin{bmatrix} \cos(90°) & -\sin(90°)\\ \sin(90°) & \cos(90°)\end{bmatrix}\\
&= \begin{bmatrix} 0 & -1\\ 1 & 0\end{bmatrix}\\
|\det(\mathrm{T})| &= |0\times 0-(-1)\times 1|\\
&= 1
\end{aligned}$$

(b)
$$\begin{aligned}
\mathrm{T} &= \begin{bmatrix} \cos(180°) & -\sin(180°)\\ \sin(180°) & \cos(180°)\end{bmatrix}\\
&= \begin{bmatrix} -1 & 0\\ 0 & -1\end{bmatrix}\\
|\det(\mathrm{T})| &= |-1\times -1-0\times 0|\\
&= 1
\end{aligned}$$

(c)
$$\begin{aligned}
\mathrm{T} &= \begin{bmatrix} 1 & 0\\ 0 & -1\end{bmatrix}\\
|\det(\mathrm{T})| &= |1\times -1-0\times 0|\\
&= 1
\end{aligned}$$

(d)
$$\begin{aligned}
\mathrm{T} &= \begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix}\\
|\det(\mathrm{T})| &= |0\times 0-1\times 1|\\
&= 1
\end{aligned}$$

(e)
$$\begin{aligned}
\mathrm{T} &= \begin{bmatrix} 1 & 4\\ 0 & 1\end{bmatrix}\\
|\det(\mathrm{T})| &= |1\times 1-4\times 0|\\
&= 1
\end{aligned}$$

(f)
$$\begin{aligned}
\mathrm{T} &= \begin{bmatrix} 1 & 0\\ 3 & 1\end{bmatrix}\\
|\det(\mathrm{T})| &= |1\times 1-0\times 3|\\
&= 1
\end{aligned}$$

8.
$$\begin{aligned}
\mathrm{A}^2 &= \begin{bmatrix} x & 1\\ 0 & 3\end{bmatrix}\begin{bmatrix} x & 1\\ 0 & 3\end{bmatrix}\\
&= \begin{bmatrix} x^2 & x+3\\ 0 & 9\end{bmatrix}\\
\mathrm{A}^2+\mathrm{A} &= \begin{bmatrix} x^2 & x+3\\ 0 & 9\end{bmatrix}+\begin{bmatrix} x & 1\\ 0 & 3\end{bmatrix}\\
&= \begin{bmatrix} x^2+x & x+4\\ 0 & 12\end{bmatrix}\\
\mathrm{A}^2+\mathrm{A} &= \begin{bmatrix} 6 & x^2-8\\ p & q\end{bmatrix}\\
\therefore\quad \begin{bmatrix} x^2+x & x+4\\ 0 & 12\end{bmatrix} &= \begin{bmatrix} 6 & x^2-8\\ p & q\end{bmatrix}
\end{aligned}$$

From element $(2,1)$, $p=0$.

From element $(2,2)$, $q=12$.

From element $(1,1)$,

$$\begin{aligned}
x^2+x &= 6\\
x^2+x-6 &= 0
\end{aligned}$$

and from element $(1,2)$:

$$\begin{aligned}
x+4 &= x^2-8\\
x^2-x-12 &= 0\\
\text{so}\qquad x^2+x-6 &= x^2-x-12\\
2x &= -6\\
x &= -3
\end{aligned}$$

9. (a)
$$\begin{aligned}
\mathrm{AB} &= \begin{bmatrix} -5-9 & 0-6\\ 1+3 & 0+2\end{bmatrix}\\
&= \begin{bmatrix} -14 & -6\\ 4 & 2\end{bmatrix}
\end{aligned}$$

(b)
$$\begin{aligned}
\mathrm{BA} &= \begin{bmatrix} -5+0 & 3+0\\ 15-2 & -9+2\end{bmatrix}\\
&= \begin{bmatrix} -5 & 3\\ 13 & -7\end{bmatrix}
\end{aligned}$$

(c)
$$\begin{aligned}
\mathrm{A}^{-1} &= \frac{1}{5-3}\begin{bmatrix} 1 & 3\\ 1 & 5\end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix} 1 & 3\\ 1 & 5\end{bmatrix}
\end{aligned}$$

(d)
$$\begin{aligned}
\mathrm{B}^{-1} &= \frac{1}{-2-0}\begin{bmatrix} 2 & 0\\ -3 & -1\end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix} -2 & 0\\ 3 & 1\end{bmatrix}
\end{aligned}$$

(e)
$$\begin{aligned}
\mathrm{AC} &= \mathrm{B}\\
\mathrm{C} &= \mathrm{A}^{-1}\mathrm{B}\\
&= \frac{1}{2}\begin{bmatrix} 1 & 3\\ 1 & 5\end{bmatrix}\begin{bmatrix} -1 & 0\\ 3 & 2\end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix} -1+9 & 0+6\\ -1+15 & 0+10\end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix} 8 & 6\\ 14 & 10\end{bmatrix}\\
&= \begin{bmatrix} 4 & 3\\ 7 & 5\end{bmatrix}
\end{aligned}$$

(f) $DA = B$

$$\begin{aligned} D &= BA^{-1} \\ &= \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -1+0 & -3+0 \\ 3+2 & 9+10 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -1 & -3 \\ 5 & 19 \end{bmatrix} \end{aligned}$$

Note: the solution in Sadler is incorrectly missing the $\frac{1}{2}$.

10.
$$AA^{-1} = I$$
$$\begin{bmatrix} a & -1 & a \\ b & 3 & c \\ d & 1 & -1 \end{bmatrix} \begin{bmatrix} -3 & e & f \\ 1 & 0 & g \\ 4 & -1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} -3a-1+4a & ae+0-a & af-g+7a \\ -3b+3+4c & be+0-c & bf+3g+7c \\ -3d+1-4 & de+0+1 & df+g-7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From matrix element (3,1),

$$\begin{aligned} -3d-3 &= 0 \\ d &= -1 \end{aligned}$$
$$\begin{bmatrix} a-1 & ae-a & af-g+7a \\ -3b+3+4c & be-c & bf+3g+7c \\ 0 & -e+1 & -f+g-7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now from element (3,2),

$$\begin{aligned} -e+1 &= 0 \\ e &= 1 \end{aligned}$$
$$\begin{bmatrix} a-1 & 0 & af-g+7a \\ -3b+3+4c & b-c & bf+3g+7c \\ 0 & 0 & -f+g-7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now from element (1,1),

$$\begin{aligned} a-1 &= 1 \\ a &= 2 \end{aligned}$$
$$\begin{bmatrix} 1 & 0 & 2f-g+14 \\ -3b+3+4c & b-c & bf+3g+7c \\ 0 & 0 & -f+g-7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now from element (2,2),

$$\begin{aligned} b-c &= 1 \\ c &= b-1 \end{aligned}$$
$$\begin{bmatrix} 1 & 0 & 2f-g+14 \\ -3b+3+4(b-1) & 1 & bf+3g+7(b-1) \\ 0 & 0 & -f+g-7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now from element (2,1),

$$\begin{aligned} -3b+3+4(b-1) &= 0 \\ b-1 &= 0 \\ b &= 1 \\ \text{hence } c &= b-1 \\ c &= 0 \end{aligned}$$
$$\begin{bmatrix} 1 & 0 & 2f-g+14 \\ 0 & 1 & f+3g \\ 0 & 0 & -f+g-7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now from elements (2,3) and (3,2),

$$\begin{aligned} f+3g &= 0 \\ -f+g-7 &= 1 \\ -f+g &= 8 \\ 4g &= 8 \\ g &= 2 \\ -f+2 &= 8 \\ f &= -6 \end{aligned}$$

Check that this works for the remaining element: $2(-6)-2+14 = 0$ is correct.

Therefore, $a = 2$, $b = 1$, $c = 0$, $d = -1$, $e = 1$, $f = -6$ and $g = 2$.

11. Let the transition matrix T be defined as

| | | Received by P | M | T |
|---|---|---|---|---|
| Thrown by | Phillipe | 0 | 0.6 | 0.4 |
| | Marlon | 0.5 | 0 | 0.5 |
| | Tony | 0.7 | 0.3 | 0 |

There are two ways we can find the long term percentage of passes each will receive. (Note that the percentage of passes each receives is equal to the percentage throws each gives and equal to the percentage each has possession of the ball.)

First, empirically. Suppose (without loss of generality) that Phillipe starts with the ball. This gives us an initial state matrix

$$S_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Now after $n$ passes, the probability of each having the ball is given by

$$S_n = S_0 T^n$$

Calculate this for increasingly large values of $n$ until there is no significant change and interpret the results.

After 20 throws,

$$S_0 T^{20} = \begin{bmatrix} 0.374 & 0.317 & 0.308 \end{bmatrix}$$

Phillipe received 37% of passes, Marlon 32% and Tony 31%.

The more analytical approach is to find the state matrix S such that $ST = S$, that is

$$\begin{bmatrix} p & m & t \end{bmatrix} \begin{bmatrix} 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \\ 0.7 & 0.3 & 0 \end{bmatrix} = \begin{bmatrix} p & m & t \end{bmatrix}$$
$$\begin{bmatrix} 0.5m+0.7t & 0.6p+0.3t & 0.4p+0.5m \end{bmatrix} = \begin{bmatrix} p & m & t \end{bmatrix}$$
$$\begin{aligned} -p+0.5m+0.7t &= 0 \\ 0.6p-m+0.3t &= 0 \\ 0.4p+0.5m-t &= 0 \end{aligned}$$

Solve any two of these simultaneously together with

$$p+m+t = 1$$

to give the same answers as we found empirically. This second approach can be used to find exact values, but this is seldom of relevance with processes that are probabilistic in nature.

12. $$\frac{27}{i} = -27i$$
$$= 27\,\text{cis}(-\frac{\pi}{2})$$

This gives us a principal solution of

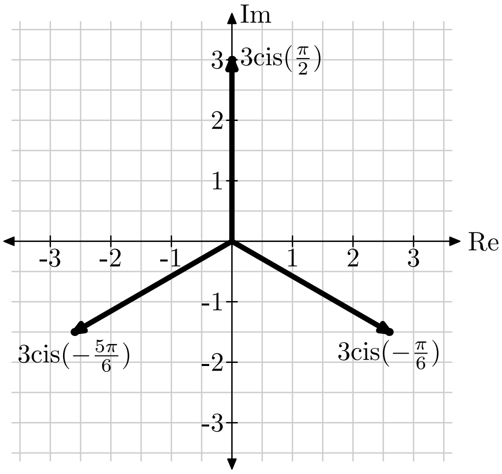
$$z = 3\,\text{cis}(-\frac{\pi}{6})$$

The three solutions are separated by $\frac{2\pi}{3}$ so the other two solutions are

$$z = 3\,\text{cis}(-\frac{\pi}{6} - \frac{2\pi}{3}) = 3\,\text{cis}(-\frac{5\pi}{6})$$

and

$$z = 3\,\text{cis}(-\frac{\pi}{6} + \frac{2\pi}{3}) = 3\,\text{cis}(\frac{\pi}{2})$$




13. (a) The Leslie matrix is

$$L = \begin{bmatrix} 0 & 1.2 & 0.9 \\ 0.6 & 0 & 0 \\ 0 & 0.8 & 0 \end{bmatrix}$$

If P is the population vector from the previous year and Q is the current population, then

$$Q = \begin{bmatrix} 2778 \\ 1572 \\ 1192 \end{bmatrix}$$
$$LP = Q$$
$$L^{-1}LP = L^{-1}Q$$
$$P = \begin{bmatrix} 2620 \\ 1490 \\ 1100 \end{bmatrix}$$

(using calculator for the last step).

Alternatively, we can simply work backward. Second generation population is given by the survival from the previous year's first generation: $q_2 = 0.6p_1$ so $p_1 = \frac{q_2}{0.6} = \frac{1572}{0.6} = 2620$.

Similarly third generation population is given by the survival from the previous year's second generation: $q_3 = 0.8p_2$ so $p_2 = \frac{q_3}{0.8} = \frac{1192}{0.8} = 1490$.

The first generation population is given by the reproduction rates from the previous year:

$$q_1 = 1.2p_2 + 0.9p_3$$
$$p_3 = \frac{q_1 - 1.2p_2}{0.9}$$
$$= \frac{2778 - 1.2 \times 1490}{0.9}$$
$$= 1100$$

(b) The Leslie matrix becomes

$$L = \begin{bmatrix} 0 & 1.2 & 0 \\ 0.6 & 0 & 0 \\ 0 & 0.8 & 0 \end{bmatrix}$$

This is not an invertable matrix so it is not possible to determine the previous year's population.

It *is* possible to work out the first and second generation population by working backward, as previously, but we can tell nothing about the third generation population.

$q_2 = 0.6p_1$ so $p_1 = \frac{q_2}{0.6} = \frac{1572}{0.6} = 2620$.

$q_3 = 0.8p_2$ so $p_2 = \frac{q_3}{0.8} = \frac{1852}{0.8} = 2315$.

The reproduction rate, however, now has no contribution from $p_3$: $q_1 = 1.2p_2$ which gives $p_2 = 2315$ which is consistent with the value given by the survival rates, but tells us nothing about $p_3$.

14. (a) $$\begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$

(b) $$\begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

(c) The gradient of the line of symmetry gives

$$\tan\theta = \frac{\sqrt{3}}{3}$$
$$\theta = 30°$$
$$2\theta = 60°$$

so the transformation matrix is

$$\begin{bmatrix} \cos 60° & \sin 60° \\ \sin 60° & -\cos 60° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$$

(d) The transformation matrix representing the combined reflection and rotation described is

$$\frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}\frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$
$$= \frac{1}{4}\begin{bmatrix} \sqrt{3}+\sqrt{3} & -1+3 \\ 3-1 & -\sqrt{3}-\sqrt{3} \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{bmatrix}$$

and this is not equal to the transformation matrix representing a 60° anticlockwise rotation about the origin.

The apparent equivalence in the diagram is caused by the symmetry of the square. (Refer to the solutions in Sadler for a more full explanation.)

15. $$\frac{\mathrm{d}}{\mathrm{d}x}2ex\ln x = 2e\ln x + 2ex\frac{1}{x}$$
$$= 2e\ln x + 2e$$
$$= 2e(\ln(x)+1)$$

At the stationary point

$$2e(\ln(x)+1) = 0$$
$$\ln x = -1$$
$$x = e^{-1}$$
$$= \frac{1}{e}$$

The second derivative is

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}2ex\ln x = \frac{\mathrm{d}}{\mathrm{d}x}2e(\ln(x)+1)$$
$$= \frac{2e}{x}$$

and at $x = \frac{1}{e}$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}2ex\ln x = 2e^2$$

which is positive, signifying that the gradient is increasing, so the stationary point is a minimum.

The minimum value is obtained by substituting $x = \frac{1}{e}$ into the original expression:

$$2e(\frac{1}{e})\ln\frac{1}{e} = 2\ln e^{-1}$$
$$= -2$$

16. $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{2}{x}(x^2) - 2\ln x(2x)}{x^4}$$
$$= \frac{2x - 4x\ln x}{x^4}$$
$$= \frac{2x(1-2\ln x)}{x^4}$$
$$= \frac{2(1-2\ln x)}{x^3}$$

At the stationary points

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\frac{2(1-2\ln x)}{x^3} = 0$$
$$1 - 2\ln x = 0$$
$$\ln x = \frac{1}{2}$$
$$x = e^{\frac{1}{2}}$$
$$= \sqrt{e}$$
$$y = \frac{2\ln e^{\frac{1}{2}}}{e}$$
$$= \frac{1}{e}$$

There is one stationary point, a maximum at $(\sqrt{e}, \frac{1}{e})$.

17. (a) De Moivre's theorem states that

$$(\cos\theta + \mathrm{i}\sin\theta)^k = \cos k\theta + \mathrm{i}\sin k\theta$$

so given

$$z = \cos\theta + \mathrm{i}\sin\theta$$
$$z^k = \cos k\theta + \mathrm{i}\sin k\theta$$

and

$$\frac{1}{z^k} = z^{-k}$$
$$= \cos(-k\theta) + \mathrm{i}\sin(-k\theta)$$
$$= \cos k\theta - \mathrm{i}\sin k\theta$$
$$\therefore \quad z^k + \frac{1}{z^k} = \cos k\theta + \mathrm{i}\sin k\theta$$
$$+ \cos k\theta - \mathrm{i}\sin k\theta$$
$$= 2\cos k\theta$$

□

(b) i. To prove:

$$\cos^3\theta = \frac{\cos(3\theta) + 3\cos\theta}{4}$$

Proof:

$$\text{LHS} = \cos^3\theta$$
$$= \left(\frac{z+\frac{1}{z}}{2}\right)^3$$
$$= \frac{z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}}{8}$$
$$= \frac{z^3 + \frac{1}{z^3} + 3z + \frac{3}{z}}{8}$$
$$= \frac{\left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right)}{8}$$
$$= \frac{2\cos(3\theta) + 6\cos\theta}{8}$$
$$= \frac{\cos(3\theta) + 3\cos\theta}{4}$$
$$= \text{RHS}$$

□

ii. To prove:

$$\cos^4\theta = \frac{\cos(4\theta) + 4\cos(2\theta) + 3}{8}$$

Proof:

$$\begin{aligned}
\text{LHS} &= \cos^4\theta \\
&= \left(\frac{z + \frac{1}{z}}{2}\right)^4 \\
&= \frac{z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}}{16} \\
&= \frac{z^4 + \frac{1}{z^4} + 4z^2 + \frac{4}{z^2} + 6}{16} \\
&= \frac{\left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6}{16} \\
&= \frac{2\cos(4\theta) + 8\cos(2\theta) + 6}{16} \\
&= \frac{\cos(4\theta) + 4\cos(2\theta) + 3}{8} \\
&= \text{RHS}
\end{aligned}$$

□

# Chapter 6

## Exercise 6A

1. (a) $N = -\log_{10}(6.4 \times 10^{-8}) = 7.19$

   (b) $$\begin{aligned} -N &= \log_{10}(2L) \\ 10^{-N} &= 2L \\ L &= \frac{10^{-N}}{2} \\ &= \frac{10^{-9.5}}{2} \\ &= 1.58 \times 10^{-10} \end{aligned}$$

   (Alternatively, use calculator skills to solve this.)

2. (a) $$\begin{aligned} x &= \frac{1}{\log 2} \times \log \frac{50}{20} \\ &= 1.32 \text{ octaves} \end{aligned}$$

   (b) $$\begin{aligned} 3 &= \frac{1}{\log 2} \times \log \frac{f_2}{f_1} \\ 3 \log 2 &= \log \frac{f_2}{f_1} \\ \log 2^3 &= \log \frac{f_2}{f_1} \\ 8 &= \frac{f_2}{f_1} \\ f_2 &= 8 f_1 \end{aligned}$$

3. (a) $$\begin{aligned} 7 &= -\log(\mathrm{H}^+) \\ \log(\mathrm{H}^+) &= -7 \\ \mathrm{H}^+ &= 10^{-7} \text{ moles per litre} \end{aligned}$$

   (b) $$\begin{aligned} \mathrm{pH} &= -\log(0.01) \\ &= 2 \end{aligned}$$

   (c) $$\begin{aligned} \mathrm{pH} &= -\log(4 \times 10^{-8}) \\ &= 7.40 \end{aligned}$$

4. (a) $$\begin{aligned} \mathrm{logit}(0.2) &= \ln\left(\frac{0.2}{0.8}\right) \\ &= -1.39 \end{aligned}$$

   (b) $$\begin{aligned} 4 &= \ln\left(\frac{p}{1-p}\right) \\ \frac{p}{1-p} &= e^4 \\ p &= e^4 - e^4 p \\ p + e^4 p &= e^4 \\ p(1+e^4) &= e^4 \\ p &= \frac{e^4}{1+e^4} \\ &= 0.98 \end{aligned}$$

   (c) If $p$ is negative, then

   $$\begin{aligned} \frac{p}{1-p} &< 1 \\ p &< 1 - p \\ 2p &< 1 \\ p &< 0.5 \end{aligned}$$

   which is to say that the event has a less than even chance of occurring.

   (d) $$\begin{aligned} \ln\left(\frac{x}{1-x}\right) &= k \\ \frac{x}{1-x} &= e^k \\ x &= e^k(1-x) \\ &= e^k + e^k x \\ x + e^k x &= e^k \\ x(1+e^k) &= e^k \\ x &= \frac{e^k}{1+e^k} \end{aligned}$$

   For real $k$, $e^k > 0$. From this we can conclude that the value of $x$ is positive, and that the denominator is greater than the numerator, hence

   $$0 < x < 1 \quad \forall k \in \Re$$

5. No working required.

## Exercise 6B

1. $$\begin{aligned} A &= A_0 e^{1.5t} \\ &= 100 e^{1.5t} \end{aligned}$$

   (a) $A = 100e^{1.5} = 488$

   (b) $A = 100e^{5 \times 1.5} = 180\,804 \approx 181\,000$

2. $$\begin{aligned} P &= P_0 e^{0.25t} \\ &= 5\,000 e^{0.25t} \end{aligned}$$

   (a) $A = 5,000e^{0.25 \times 5} = 17\,452 \approx 17\,000$

   (b) $A = 5,000e^{0.25 \times 25} = 2\,590\,064 \approx 2\,600\,000$

3. $A = 500e^{0.02t}$

   (a) $A = 500e^{0.2} = 611$

   (b) $A = 500e^{0.5} = 824$

4. $Q = 100\,000e^{-0.01t}$

   (a) $Q = 100\,000e^{-0.2} = 81\,873 \approx 82\,000$

   (b) $Q = 100\,000e^{-0.5} = 60\,653 \approx 61\,000$

5. $$\begin{aligned} X &= X_0e^{0.25t} \\ 5\times10^6 &= X_0e^1 \\ X_0 &= 1\,839\,397 \\ X &= 1\,839\,397e^{0.25t} \end{aligned}$$

   or alternatively

   $$\begin{aligned} X &= X_0e^{0.25t} \\ 5\times10^6 &= X_0e^1 \\ X_0 &= 5\times10^6e^{-1} \\ X &= 5\times10^6e^{-1}e^{0.25t} \\ &= 5\times10^6e^{0.25t-1} \end{aligned}$$

   (a) $X = 1\,839\,397e^{1.25} = 6\,420\,127 \approx 6.4$ million

   (b) $X = 5\times10^6e^{6.25-1} = 952\,831\,342 \approx 953$ million

6. $$\begin{aligned} Y &= Y_0e^{0.045t} \\ 25\,000 &= X_0e^{0.45} \\ X_0 &= 25\,000e^{-0.45} \\ X &= 25\,000e^{0.045t-0.45} \end{aligned}$$

   (a) $X = 25\,000e^{0.9-0.45} = 39\,208 \approx 39\,000$

   (b) $X = 25\,000e^{0.045\times25-0.45} = 49\,101 \approx 49\,000$

7. $$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} &= -0.08A \\ A &= A_0e^{-0.08t} \\ &= 5e^{-0.08\times25} \\ &= 5e^{-2} \\ &= 0.677\text{kg} \end{aligned}$$

8. $$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} &= -0.02A \\ A &= A_0e^{-0.02t} \\ &= 20e^{-0.02\times50} \\ &= 20e^{-1} \\ &= 7.36\text{kg} \end{aligned}$$

9. $$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}t} &= 0.025P \\ P &= P_0e^{0.025t} \end{aligned}$$

   (a) $P = 25e^{0.25} \approx 32$ million.

   (b) i. $t = 2030 - 1995 = 35$
   $P = 25e^{0.025\times35} \approx 60$ million

   ii. $t = 2060 - 1995 = 65$
   $P = 25e^{0.025\times65} \approx 127$ million

10. (a) $P = 100e^{-0.005} = 99.5\%$

    (b) $P = 100e^{-0.05} = 95\%$

    (c) $P = 100e^{-0.5} = 61\%$

    (d) $$\begin{aligned} 50 &= 100e^{-0.005t} \\ e^{-0.005t} &= 0.5 \\ -0.005t &= \ln 0.5 \\ t &= \frac{\ln 0.5}{-0.005} \\ &= 138.6 \end{aligned}$$

    The element has a half life of about 140 years.

11. $$\begin{aligned} 0.5A_0 &= A_0e^{-0.001t} \\ e^{-0.001t} &= 0.5 \\ e^{0.001t} &= 2 \\ 0.001t &= \ln 2 \\ t &= 1000\ln 2 \\ &= 693 \end{aligned}$$

    The element has a half life of about 690 years.

12. $$\begin{aligned} 0.0008t &= \ln 2 \\ t &= \frac{\ln 2}{0.0008} \\ &= 866 \end{aligned}$$

    The element has a half life of about 870 years.

    (After a few of these half-life problems, the pattern becomes clear and we can take some short-cuts.)

13. $$\begin{aligned} 200 &= 75e^{0.035t} \\ e^{0.035t} &= \frac{200}{75} \\ 0.035t &= \ln\frac{200}{75} \\ t &= \frac{\ln\frac{200}{75}}{0.035} \\ &= 28.02 \end{aligned}$$

    Population will reach 200 million in approximately 28 years.

14. $$\begin{aligned} t &= \frac{\ln 2}{0.0004} \\ &= 1733 \text{ years} \end{aligned}$$

15. $$\begin{aligned} t &= \frac{\ln 2}{0.009} \\ &= 77 \text{ years} \end{aligned}$$

16. (a) No calculations are needed (based on the definition of half-life, half the 20kg must be left after one half-life.)

    (b) This is similarly straightforward. 100 years is twice the half-life, so the amount has halved twice to 5kg.

(c) From the definition of half life,

$$\begin{aligned} e^{-50k} &= 0.5 \\ \therefore \quad 20e^{-75k} &= 20e^{-50k\times 1.5} \\ &= 20 \times 0.5^{1.5} \\ &= 7.07 \text{ kg} \end{aligned}$$

17. $P = 500e^{1.5t}$ or $t = \frac{\ln\left(\frac{P}{500}\right)}{1.5}$

(a) $$\begin{aligned} t &= \frac{\ln\left(\frac{1\,000\,000}{500}\right)}{1.5} \\ &= \frac{\ln(2\,000)}{1.5} \\ &= 5.07 \text{ hours} \\ &= 5 \text{ hours } 4 \text{ minutes} \end{aligned}$$

(b) $$\begin{aligned} t &= \frac{\ln\left(\frac{2\,000\,000}{500}\right)}{1.5} \\ &= \frac{\ln(4\,000)}{1.5} \\ &= 5.53 \text{ hours} \\ &= 5 \text{ hours } 32 \text{ minutes} \end{aligned}$$

The doubling time is the difference between the answers to (a) and (b), i.e. 28 minutes.

18. (a) Based on the half-life, 500g will remain after 30 years.

(b) This is two half lives, so the amount remaining will be

$$1000\left(\tfrac{1}{2}\right)^2 = 250 \text{ g}$$

(c) This is $\frac{4}{3}$ half lives, so the amount remaining will be

$$1000\left(\frac{1}{2}\right)^{\frac{4}{3}} = 397 \text{ g}$$

19. $$\begin{aligned} M &= M_0e^{-kt} \\ \frac{M}{M_0} &= e^{-kt} \\ e^{-250\,000k} &= 0.5 \\ \frac{M}{M_0} &= e^{-250\,000k\times\frac{t}{250\,000}} \\ &= \left(e^{-250\,000k}\right)^{\frac{t}{250\,000}} \\ &= 0.5^{\frac{t}{250\,000}} \\ &= 0.5^{\frac{5\,000}{250\,000}} \\ &= 0.986 \end{aligned}$$

98.6% remains after 5 000 years.

20. $$\begin{aligned} P &= P_0e^{kt} \\ 31\,250\,000 &= 18\,500\,000e^{15k} \\ k &= \frac{\ln\frac{31\,250\,000}{18\,500\,000}}{15} \\ &= 0.0349 \end{aligned}$$

The growth rate about is 3.5% per annum.

21. $$\begin{aligned} P &= P_0e^{kt} \\ 56 &= 325e^{8k} \\ k &= \frac{\ln\frac{56}{325}}{8} \\ &= -0.220 \end{aligned}$$

Population declined by about 22% per annum.

22. $$\begin{aligned} P_8 &= P_0e^{kt} \\ 1250 &= 200e^{8k} \\ k &= \frac{\ln\frac{1250}{200}}{8} \\ &= 0.229 P_{12} \quad = 200e^{12k} \\ &= 3125 \end{aligned}$$

23. $$\begin{aligned} e^{5k} &= 2 \\ k &= \frac{\ln 2}{5} \\ &= 0.139 \end{aligned}$$

The claim amounts to a 13.9%p.a. interest rate, compounding continuously.

24. $$\begin{aligned} \frac{P}{P_0} &= e^{-0.022t} \\ 0.6 &= e^{-0.022t} \\ t &= \frac{\ln(0.6)}{-0.022} \\ &= 23.22 \end{aligned}$$

A top-up dose will be required after 23 minutes.

25. $$\begin{aligned} \frac{C}{C_0} &= e^{kt} \\ 0.5 &= e^{5700k} \\ k &= \frac{\ln 0.5}{5700} \\ &= -0.0001216 \\ 0.6 &= e^{kt} \\ t &= \frac{\ln 0.6}{k} \\ &\approx 4\,200 \text{ years} \end{aligned}$$

26. $$\begin{aligned} \frac{M}{M_0} &= e^{kt} \\ 0.5 &= e^{30k} \\ k &= \frac{\ln(0.5)}{30} \\ &= -0.0231 \\ \frac{1}{15} &= e^{kt} \\ t &= \frac{\ln\frac{1}{15}}{k} \\ &= 117.2 \end{aligned}$$

The area should be considered unsafe for 118 years. (It becomes 'safe' a couple of months into the 118th year. In this situation it makes sense to round answers up rather than to the nearest year.)

27. (a)

$$\begin{aligned}
2 &= e^{\frac{p}{100}t} \\
t &= \frac{\ln 2}{\frac{p}{100}} \\
&= \frac{100\ln 2}{p} \\
100\ln 2 &\approx 69.3 \\
\therefore \quad t &\approx \frac{69.3}{p}
\end{aligned}$$

(b) Because 72 is a multiple of 2, 3, 4, 6, 8, 9, 12 and 18. This makes it easy to divide by common interest rates and this ease of calculation is important in a rule of thumb.

28.

$$\begin{aligned}
\frac{\mathrm{d}T}{\mathrm{d}t} &= -k(T-28) \\
\int \frac{dT}{T-28} &= -k\int dt \\
\ln(T-28) &= -kt + c
\end{aligned}$$

when $t = 0$

$$\begin{aligned}
c &= \ln(T_0 - 28) \\
e^c &= T_0 - 28 \\
\therefore \quad T - 28 &= e^{-kt+c} \\
&= e^c e^{-kt} \\
&= (T_0 - 28)e^{-kt} \\
T &= (T_0 - 28)e^{-kt} + 28
\end{aligned}$$

Let $t = 0$ represent the time the object was first placed. Let $t = x$ be the time of the first measurement of 135°C. The time of the second measurement of 91°C is then $t = x + 10$.

$$\begin{aligned}
135 - 28 &= (240 - 28)e^{-kx} \\
107 &= 212e^{-kx} \\
91 - 28 &= (240 - 28)e^{-k(x+10)} \\
63 &= 212e^{-kx-10k} \\
63 &= 212e^{-kx}e^{-10k} \\
63 &= 107e^{-10k} \\
e^{-10k} &= \frac{63}{107} \\
k &= \frac{\ln\frac{63}{107}}{-10} \\
&= 0.0530 \\
e^{-kx} &= \frac{107}{212} \\
x &= \frac{\ln\frac{107}{212}}{-k} \\
&= 12.91 \text{ minutes}
\end{aligned}$$

The item was in the 28°C environment for about 13 minutes before the 135°C temperature was recorded.

## Exercise 6C

1. With the product rule:

$$\begin{aligned}
y &= x^3(2x+1)^5 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= 3x^2(2x+1)^5 + x^3(5)(2x+1)^4(2) \\
&= 3x^2(2x+1)^5 + 10x^3(2x+1)^4 \\
&= x^2(2x+1)^4(3(2x+1) + 10x) \\
&= x^2(2x+1)^4(6x + 3 + 10x) \\
&= x^2(2x+1)^4(16x+3)
\end{aligned}$$

Using logarithmic differentiation:

$$\begin{aligned}
y &= x^3(2x+1)^5 \\
\ln y &= \ln(x^3(2x+1)^5) \\
&= \ln(x^3) + \ln((2x+1)^5) \\
&= 3\ln(x) + 5\ln(2x+1) \\
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3}{x} + \frac{5\times 2}{2x+1} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= y\left(\frac{3}{x} + \frac{10}{2x+1}\right) \\
&= y\left(\frac{3(2x+1)+10x}{x(2x+1)}\right) \\
&= (x^3(2x+1)^5)\left(\frac{16x+3}{x(2x+1)}\right) \\
&= x^2(2x+1)^4(16x+3)
\end{aligned}$$

2. With the chain rule:

$$\begin{aligned} y &= (3x^2-2)^5 \\ &= u^5 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \\ &= 5u^4(6x) \\ &= 30xu^4 \\ &= 30x(3x^2-2)^4 \end{aligned}$$

Using logarithmic differentiation:

$$\begin{aligned} y &= (3x^2-2)^5 \\ \ln y &= \ln((3x^2-2)^5) \\ &= 5\ln(3x^2-2) \\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{5}{3x^2-2}6x \\ &= \frac{30x}{3x^2-2} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{30xy}{3x^2-2} \\ &= \frac{30x(3x^2-2)^5}{3x^2-2} \\ &= 30x(3x^2-2)^4 \end{aligned}$$

3. With the quotient rule:

$$\begin{aligned} y &= \frac{x^3}{x^2+1} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3x^2(x^2+1)-x^3(2x)}{(x^2+1)^2} \\ &= \frac{3x^4+3x^2-2x^4}{(x^2+1)^2} \\ &= \frac{x^4+3x^2}{(x^2+1)^2} \end{aligned}$$

Using logarithmic differentiation:

$$\begin{aligned} y &= \frac{x^3}{x^2+1} \\ \ln y &= \ln\frac{x^3}{x^2+1} \\ &= \ln(x^3)-\ln(x^2+1) \\ &= 3\ln(x)-\ln(x^2+1) \\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3}{x}-\frac{2x}{x^2+1} \\ &= \frac{3(x^2+1)-2x^2}{x(x^2+1)} \\ &= \frac{3x^2+3-2x^2}{x(x^2+1)} \\ &= \frac{x^2+3}{x(x^2+1)} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= y\left(\frac{x^2+3}{x(x^2+1)}\right) \\ &= \frac{x^3}{x^2+1}\left(\frac{x^2+3}{x(x^2+1)}\right) \\ &= \frac{x^2(x^2+3)}{(x^2+1)^2} \\ &= \frac{x^4+3x^2}{(x^2+1)^2} \end{aligned}$$

4. (a)

$$\begin{aligned} y &= x^x \\ \ln y &= x\ln x \\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \ln(x)+\frac{x}{x} \\ &= \ln(x)+1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= y(\ln(x)+1) \\ &= x^x(\ln(x)+1) \end{aligned}$$

(b)

$$\begin{aligned} y &= x^{2x} \\ \ln y &= 2x\ln x \\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= 2\ln(x)+\frac{2x}{x} \\ &= 2\ln(x)+2 \\ &= 2(\ln(x)+1) \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= 2y(\ln(x)+1) \\ &= 2x^{2x}(\ln(x)+1) \end{aligned}$$

(c)

$$\begin{aligned} y &= x^{\cos x} \\ \ln y &= \cos(x)\ln x \\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\sin(x)\ln(x)+\frac{\cos(x)}{x} \\ &= \frac{\cos(x)-x\sin(x)\ln(x)}{x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y(\cos(x)-x\sin(x)\ln(x))}{x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x^{cos(x)}(\cos(x)-x\sin(x)\ln(x))}{x} \\ &= x^{cos(x)-1}(\cos(x)-x\sin(x)\ln(x)) \end{aligned}$$

(d)

$$\begin{aligned} y &= x^{\sin x} \\ \ln y &= \sin(x)\ln x \\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \cos(x)\ln(x)+\frac{\sin(x)}{x} \\ &= \frac{x\cos(x)\ln(x)+\sin(x)}{x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y(x\cos(x)\ln(x)+\sin(x))}{x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x^{sin(x)}(x\cos(x)\ln(x)+\sin(x))}{x} \\ &= x^{sin(x)-1}(x\cos(x)ln(x)+\sin(x)) \end{aligned}$$

(e)
$$\begin{aligned}
y &= \sqrt{\frac{3x+1}{3x-1}}\\
\ln y &= \frac{1}{2}\left(\ln(3x+1) - \ln(3x-1)\right)\\
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2}\left(\frac{3}{3x+1} - \frac{3}{3x-1}\right)\\
&= \frac{3}{2}\left(\frac{1}{3x+1} - \frac{1}{3x-1}\right)\\
&= \frac{3}{2}\left(\frac{(3x-1)-(3x+1)}{(3x+1)(3x-1)}\right)\\
&= \frac{3}{2}\left(\frac{-2}{(3x+1)(3x-1)}\right)\\
&= \frac{-3}{(3x+1)(3x-1)}\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= y\left(\frac{-3}{(3x+1)(3x-1)}\right)\\
&= \sqrt{\frac{3x+1}{3x-1}}\left(\frac{-3}{(3x+1)(3x-1)}\right)\\
&= \frac{-3\sqrt{3x+1}}{(3x+1)(3x-1)\sqrt{3x-1}}\\
&= \frac{-3\sqrt{(3x+1)(3x-1)}}{(3x+1)(3x-1)^2}\\
&= -3(3x+1)^{-0.5}(3x-1)^{-1.5}
\end{aligned}$$

(f)
$$\begin{aligned}
y &= \sqrt{\frac{1+x}{2-x}}\\
\ln y &= \frac{1}{2}\left(\ln(1+x) - \ln(2-x)\right)\\
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2}\left(\frac{1}{1+x} - \frac{-1}{2-x}\right)\\
&= \frac{1}{2}\left(\frac{(2-x)+(1+x)}{(1+x)(2-x)}\right)\\
&= \frac{3}{2(1+x)(2-x)}\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= y\left(\frac{3}{2(1+x)(2-x)}\right)\\
&= \sqrt{\frac{1+x}{2-x}}\left(\frac{3}{2(1+x)(2-x)}\right)\\
&= 1.5(1+x)^{-0.5}(2-x)^{-1.5}
\end{aligned}$$

## Miscellaneous Exercise 6

1. (a) No working needed.

(b) No working needed.

(c)
$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2(x+3)-(2x-1)}{(x+3)^2}\\
&= \frac{2x+6-2x+1}{(x+3)^2}\\
&= \frac{7}{(x+3)^2}
\end{aligned}$$

(d) No working needed.

(e) No working needed.

(f)
$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= 2\cos(x)(-\sin x)\\
&= -2\cos x\sin x\\
&= -\sin 2x
\end{aligned}$$

(g) No working needed.

(h) No working needed.

(i) No working needed.

(j)
$$\begin{aligned}
y &= x(\sin^2 x + \cos^2 x)\\
&= x\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= 1
\end{aligned}$$

(k) No working needed.

(l)
$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= e^{\sin x} + xe^{\sin x}(\cos x)\\
&= e^{\sin x}(1 + x\cos x)
\end{aligned}$$

(m)
$$\begin{aligned}
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= 6x\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= 6xy
\end{aligned}$$

(n)
$$\begin{aligned}
4y + 4x\frac{\mathrm{d}y}{\mathrm{d}x} + 5y^4\frac{\mathrm{d}y}{\mathrm{d}x} - 15 &= 8\cos 2x\\
\frac{\mathrm{d}y}{\mathrm{d}x}(4x+5y^4) &= 8\cos(2x) - 4y + 15\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{8\cos(2x) - 4y + 15}{4x+5y^4}
\end{aligned}$$

(o)
$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= 4t^3\\
\frac{\mathrm{d}x}{\mathrm{d}t} &= 2t-3\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}\\
&= \frac{4t^3}{2t-3}
\end{aligned}$$

(p) $\frac{dy}{dt} = 15\cos 5t$
$\frac{dx}{dt} = 2\cos t$
$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$
$= \frac{15\cos 5t}{2\cos t}$

2. $e^{kt} = \frac{N}{N_0}$
$e^{5k} = 2$
$5k = \ln 2$
$k = 0.2\ln 2$
$\approx 0.139$

3. For any matrices X and Y, for XY to be a possible product, we need columns(X)=rows(Y). Thus
   (a) AB is possible $(1 = 1)$
   (b) AC is not possible $(1 \neq 2)$
   (c) BC is possible $(2 = 2)$
   (d) CB is not possible $(2 \neq 1)$
   (e) BD is possible $(2 = 2)$
   (f) CD is possible $(2 = 2)$
   (g) AD is not possible $(1 \neq 2)$
   (h) DA is possible $(3 = 3)$

4. (a) $\mathrm{AB} = \begin{bmatrix} 5+0 & -5+9 \\ -2+0 & 2-3 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ -2 & -1 \end{bmatrix}$

(b) $\det \mathrm{A} = (5)(-1) - (3)(-2) = 1$

(c) $\mathrm{A}^{-1} = \frac{1}{\det \mathrm{A}}\begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix}$

(d) $\mathrm{B}^{-1} = \frac{1}{(1)(3)-(-1)(0)}\begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$

(e) $\mathrm{C} = \mathrm{A}^{-1}\mathrm{B}$
$= \begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$
$= \begin{bmatrix} -1+0 & 1-9 \\ 2+0 & -2+15 \end{bmatrix}$
$= \begin{bmatrix} -1 & -8 \\ 2 & 13 \end{bmatrix}$

(f) $\mathrm{D} = \mathrm{BA}^{-1}$
$= \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix}$
$= \begin{bmatrix} -1-2 & -3-5 \\ 0+6 & 0+15 \end{bmatrix}$
$= \begin{bmatrix} -3 & -8 \\ 6 & 15 \end{bmatrix}$

5. (a) No working required.
   (b) $\mathrm{T}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 0 \end{bmatrix}$

$\mathrm{T} = \begin{bmatrix} 5 & 4 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$
$= \begin{bmatrix} 5 & 4 \\ 3 & 0 \end{bmatrix}\frac{1}{-3}\begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}$
$= \frac{1}{3}\begin{bmatrix} 5 & 4 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$
$= \frac{1}{3}\begin{bmatrix} 5+4 & 5-8 \\ 3+0 & 3+0 \end{bmatrix}$
$= \frac{1}{3}\begin{bmatrix} 9 & -3 \\ 3 & 3 \end{bmatrix}$
$= \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$

6. $\mathrm{PQ} = \mathrm{R}$
$\mathrm{P} = \mathrm{RQ}^{-1}$
$= \begin{bmatrix} 6 & 1 & 4 \\ 7 & 5 & 3 \\ -3 & 3 & -3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}^{-1}$
$= \begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & 3 \\ -2 & -1 & 1 \end{bmatrix}$

7. (a) No working required.
   (b) No working required.

8. (a) $\begin{bmatrix} 4 \\ 1 \end{bmatrix}\begin{bmatrix} -3, 5 \end{bmatrix} = \begin{bmatrix} -12 & 20 \\ -3 & 5 \end{bmatrix}$
   (b) $\begin{bmatrix} -3, 5 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -12+5 \end{bmatrix} = \begin{bmatrix} -7 \end{bmatrix}$

9. 
- AB is not possible (A has 1 column; B has 2 rows)
- AC is possible (columns(A)=rows(C)=1) and has size rows(A)×columns(C)= $(3 \times 4)$
- BA is possible (columns(B)=rows(A)=3 and has size rows(B)×columns(A)= $(2 \times 1)$
- BC is not possible (B has 3 columns; C has 1 row)
- CA is not possible (C has 4 columns; A has 3 rows)
- CB is not possible (C has 4 columns; B has 2 rows)

Thus A can pre-multiply C and B can pre-multiply A so BAC is a possible product and has dimensions rows(B)×columns(C)= $(2 \times 4)$.

$\mathrm{BAC} = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}$
$= \begin{bmatrix} 10 \\ 8 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}$
$= \begin{bmatrix} 10 & 0 & 10 & 10 \\ 8 & 0 & 8 & 8 \end{bmatrix}$

10.
$$AB = BA$$
$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 3x & 3y \\ 0 & z \end{bmatrix} = \begin{bmatrix} 3x & y \\ 0 & z \end{bmatrix}$$

This gives us no restriction on $x$ or $z$ (since $3x = 3x$ is true for all $x$, and $z = z$ for all $z$), but $y$ must be zero (since $3y = y$ is only true for $y = 0$).

11.
$$M^{-1} = \frac{1}{2}\begin{bmatrix} a & 1 \\ -2 & 0 \end{bmatrix}$$
$$M^{-1}M^{-1} = \frac{1}{2}\begin{bmatrix} a & 1 \\ -2 & 0 \end{bmatrix}\frac{1}{2}\begin{bmatrix} a & 1 \\ -2 & 0 \end{bmatrix}$$
$$= \frac{1}{4}\begin{bmatrix} a^2-2 & a \\ -2a & -2 \end{bmatrix}$$
$$\therefore \quad \begin{bmatrix} b & 1 \\ c & d \end{bmatrix} = \frac{1}{4}\begin{bmatrix} a^2-2 & a \\ -2a & -2 \end{bmatrix}$$
$$\begin{bmatrix} 4b & 4 \\ 4c & 4d \end{bmatrix} = \begin{bmatrix} a^2-2 & a \\ -2a & -2 \end{bmatrix}$$
$$\therefore \quad a = 4$$
$$\therefore \quad \begin{bmatrix} 4b & 4 \\ 4c & 4d \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ -8 & -2 \end{bmatrix}$$
$$b = \frac{14}{4} = \frac{7}{2}$$
$$c = -2$$
$$d = -\frac{2}{4} = -\frac{1}{2}$$

12. Without loss of generality, consider just one point:
$$P' = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} P$$
$$P'' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} P'$$
$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} P$$
$$= \begin{bmatrix} 0 & 1 \\ 1 & 5 \end{bmatrix} P$$

Thus the single matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 5 \end{bmatrix}$

13.
$$\begin{bmatrix} 2x & x \\ 4 & y \end{bmatrix}^2 = \begin{bmatrix} 24 & p \\ 0 & q \end{bmatrix}$$
$$\begin{bmatrix} 2x & x \\ 4 & y \end{bmatrix}\begin{bmatrix} 2x & x \\ 4 & y \end{bmatrix} = \begin{bmatrix} 24 & p \\ 0 & q \end{bmatrix}$$
$$\begin{bmatrix} 4x^2+4x & 2x^2+xy \\ 8x+4y & 4x+y^2 \end{bmatrix} = \begin{bmatrix} 24 & p \\ 0 & q \end{bmatrix}$$
$$4x^2+4x = 24$$
$$4x^2+4x-24 = 0$$
$$x^2+x-6 = 0$$
$$(x+3)(x-2) = 0$$
$$x = -3$$
$$\text{or} \quad x = 2$$

for $x = -3$ :
$$\begin{bmatrix} 4(-3)^2+4(-3) & 2(-3)^2+(-3)y \\ 8(-3)+4y & 4(-3)+y^2 \end{bmatrix} = \begin{bmatrix} 24 & p \\ 0 & q \end{bmatrix}$$
$$\begin{bmatrix} 24 & 18-3y \\ 4y-24 & y^2-12 \end{bmatrix} = \begin{bmatrix} 24 & p \\ 0 & q \end{bmatrix}$$
$$4y-24 = 0$$
$$y-6 = 0$$
$$y = 6$$
$$\begin{bmatrix} 24 & 18-3(6) \\ 4(6)-24 & (6)^2-12 \end{bmatrix} = \begin{bmatrix} 24 & p \\ 0 & q \end{bmatrix}$$
$$\begin{bmatrix} 24 & 0 \\ 0 & 24 \end{bmatrix} = \begin{bmatrix} 24 & p \\ 0 & q \end{bmatrix}$$
$$p = 0$$
$$q = 24$$

for $x = 2$ :
$$\begin{bmatrix} 4(2)^2+4(2) & 2(2)^2+(2)y \\ 8(2)+4y & 4(2)+y^2 \end{bmatrix} = \begin{bmatrix} 24 & p \\ 0 & q \end{bmatrix}$$
$$\begin{bmatrix} 24 & 8+2y \\ 16+4y & 8+y^2 \end{bmatrix} = \begin{bmatrix} 24 & p \\ 0 & q \end{bmatrix}$$
$$16+4y = 0$$
$$4+y = 0$$
$$y = -4$$
$$\begin{bmatrix} 24 & 8+2(-4) \\ 16+4(-4) & 8+(-4)^2 \end{bmatrix} = \begin{bmatrix} 24 & p \\ 0 & q \end{bmatrix}$$
$$\begin{bmatrix} 24 & 0 \\ 0 & 24 \end{bmatrix} = \begin{bmatrix} 24 & p \\ 0 & q \end{bmatrix}$$
$$p = 0$$
$$q = 24$$

Thus $p = 0$ and $q = 24$ and $(x, y) \in \{(-3, 6), (2, -4)\}$

14. (a)
$$A^2 = BCB^{-1}BCB^{-1}$$
$$= BCCB^{-1}$$
$$= BC^2B^{-1}$$

(b)
$$A^3 = A^2A$$
$$= BC^2B^{-1}BCB^{-1}$$
$$= BC^2CB^{-1}$$
$$= BC^3B^{-1}$$

(c) $A^n = BC^nB^{-1}$

(You should be able to see how you could use mathematical induction to prove this quite simply.)

15. $$L = \begin{bmatrix} 0 & 1.7 & 2.8 & 0.2 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \end{bmatrix}$$

$$P_0 = \begin{bmatrix} 1020 \\ 1560 \\ 1100 \\ 540 \end{bmatrix}$$

(a) $P_6 = L^3 P_0$

$$= \begin{bmatrix} 4750 \\ 1370 \\ 1402 \\ 122 \end{bmatrix}$$

There will be about 122 4th generation females in 6 years.

(b) $P_{10} = L^5 P_0$

$$= \begin{bmatrix} 5672 \\ 2511 \\ 1140 \\ 411 \end{bmatrix}$$

There will be about 2511 2nd generation females in 10 years.

16. To prove: $2^{n-1} + 3^{2n+1}$ is a multiple of 7 for all integer $n$, $n \geq 1$.

For $n = 1$,

$$\begin{aligned} 2^{n-1} + 3^{2n+1} &= 2^{1-1} + 3^{2(1)+1} \\ &= 2^0 + 3^3 \\ &= 28 \\ &= 7 \times 4 \end{aligned}$$

Assume the proposition is true for $n = k$, i.e.

$$2^{k-1} + 3^{2k+1} = 7a$$

for some integer $a$.

Then for $n = k + 1$ we need to demonstrate that

$$2^{k+1-1} + 3^{2(k+1)+1} = 2^k + 3^{2k+3}$$

is a multiple of 7.

$$\begin{aligned} 2^k + 3^{2k+3} &= 2(2^{k-1}) + 9(3^{2k+1}) \\ &= 2(2^{k-1}) + (2+7)(3^{2k+1}) \\ &= 2(2^{k-1}) + 2(3^{2k+1}) + 7(3^{2k+1}) \\ &= 2(2^{k-1} + 3^{2k+1}) + 7(3^{2k+1}) \\ &= 2(7a) + 7(3^{2k+1}) \\ &= 7(2a + 3^{2k+1}) \end{aligned}$$

which is a multiple of 7 as required.

Therefore, by mathematical induction $2^{n-1} + 3^{2n+1}$ is a multiple of 7 for all integer $n$, $n \geq 1$. □

17. $$\begin{aligned} \frac{dy}{dx} &= 0 \\ 4x - \frac{1}{x} &= 0 \\ 4x^2 - 1 &= 0 \\ x^2 &= \frac{1}{4} \\ x &= \frac{1}{2} \\ y &= 2(\frac{1}{2})^2 - \log_e \frac{1}{2} \\ &= \frac{1}{2} + \log_e 2 \\ \frac{d^2y}{dx^2} &= 4 + \frac{1}{x^2} \\ \text{at } x = \frac{1}{2}, \quad \frac{d^2y}{dx^2} &= 8 \end{aligned}$$

$\frac{d^2y}{dx^2} > 0 \implies \frac{dy}{dx}$ is increasing so the stationary point at $(\frac{1}{2}, \frac{1}{2} + \log_e 2)$ is a minimum.

18. (a) $$\begin{aligned} \frac{P}{P_0} &= e^{0.1t} \\ t &= 10 \ln \frac{P}{P_0} \\ &= 10 \ln 2 \\ &\approx 6.93 \text{ years} \\ &\approx 6 \text{ years 11 months.} \end{aligned}$$

(b) $$\begin{aligned} t &= 10 \ln \frac{40\,000}{10\,000} \\ &= 10 \ln 4 \\ &\approx 13.86 \text{ years} \\ &\approx 13 \text{ years 10 months.} \end{aligned}$$

(c) $$\begin{aligned} t &= 10 \ln \frac{80\,000}{10\,000} \\ &= 10 \ln 8 \\ &\approx 20.79 \text{ years} \\ &\approx 20 \text{ years 10 months.} \end{aligned}$$

Note that the answer to (b) is double the answer to (a) since the principal has to double twice. Similarly, the answer to (c) is three times the answer to (a) since the principle has to double three times ($8 = 2^3$).

19. $$\begin{aligned} \frac{dN}{dt} &= -0.18N \\ N_0 &= 12000 \\ N &= N_0 e^{-0.18t} \\ e^{-0.18t} &= \frac{2\,000}{12\,000} \\ -0.18t &= \ln \frac{1}{6} \\ 0.18t &= \ln 6 \\ t &= \frac{\ln 6}{0.18} \\ &\approx 9.95 \end{aligned}$$

The critical situation will occur in about 10 years time.

20. (a) No working required.

(b) i. $\begin{bmatrix} 5\,465 & 2\,535 \end{bmatrix} \begin{bmatrix} 0.98 & 0.02 \\ 0.03 & 0.97 \end{bmatrix} = \begin{bmatrix} 5\,402 & 2\,568 \end{bmatrix}$

(populations shown in thousands).

ii. $\mathrm{P} \begin{bmatrix} 0.98 & 0.02 \\ 0.03 & 0.97 \end{bmatrix} = \begin{bmatrix} 5\,465 & 2\,535 \end{bmatrix}$

$$\mathrm{P} = \begin{bmatrix} 5\,465 & 2\,535 \end{bmatrix} \begin{bmatrix} 0.98 & 0.02 \\ 0.03 & 0.97 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 5\,469 & 2\,501 \end{bmatrix}$$

(populations shown in thousands).

(c) $$\begin{bmatrix} 0.98 & 0.02 \\ 0.03 & 0.97 \end{bmatrix}^{10} = \begin{bmatrix} 0.84 & 0.16 \\ 0.24 & 0.76 \end{bmatrix}$$

$$\begin{bmatrix} 0.98 & 0.02 \\ 0.03 & 0.97 \end{bmatrix}^{50} = \begin{bmatrix} 0.63 & 0.37 \\ 0.55 & 0.45 \end{bmatrix}$$

$$\begin{bmatrix} 0.98 & 0.02 \\ 0.03 & 0.97 \end{bmatrix}^{100} = \begin{bmatrix} 0.60 & 0.40 \\ 0.60 & 0.40 \end{bmatrix}$$

After about a hundred years, everything else being equal(!), the population would stabilize with 60% of the total in the city and 40% in the country, i.e.

$$\begin{bmatrix} 5\,465 & 2\,535 \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} = \begin{bmatrix} 4\,782 & 3\,188 \end{bmatrix}$$

(Whether or not this makes sense is a different question. Real population modelling for a city would include calculations like this but would be much more complex as many other factors would need to be taken into consideration. Even then, sensibly forecasting 100 years into the future is well beyond current capabilities.)

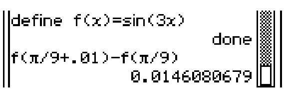
# Chapter 7

## Exercise 7A

1.
$$\begin{aligned}
f'(x) &= 3x^2 - 5\\
f'(5) &= 3(5)^2 - 5\\
&= 70\\
\frac{\delta f(x)}{\delta x} &\approx f'(x)\\
\delta f(5) &\approx f'(5)\delta x\\
&= 70 \times 0.01\\
&= 0.7\\
f(5.01) - f(5) &= (5.01)^3 - 5(5.01) - (5)^3 + 5(5)\\
&= 0.701501
\end{aligned}$$

2.
$$\begin{aligned}
f'(x) &= 10x + 2\\
f'(10) &= 10(10) + 2\\
&= 102\\
\frac{\delta f(x)}{\delta x} &\approx f'(x)\\
\delta f(10) &\approx f'(10)\delta x\\
&= 102 \times 0.1\\
&= 10.2\\
f(10.1) - f(10) &= 5(10.1)^2 + 2(10.1) - 5(10)^2\\
&\quad - 2(10)\\
&= 10.25
\end{aligned}$$

3.
$$\begin{aligned}
f'(x) &= 3\cos 3x\\
f'(\frac{\pi}{9}) &= 3\cos\frac{\pi}{3}\\
&= 1.5\\
\frac{\delta f(x)}{\delta x} &\approx f'(x)\\
\delta f(\frac{\pi}{9}) &\approx f'(\frac{\pi}{9})\delta x\\
&= 1.5 \times 0.01\\
&= 0.015\\
f(\frac{\pi}{9} + 0.01) - f(\frac{\pi}{9}) &= 0.0146
\end{aligned}$$




4.
$$\begin{aligned}
f'(x) &= 30\sin^2 5x\cos 5x\\
f'(\frac{\pi}{5}) &= 30\sin^2\frac{5\pi}{3}\cos\frac{5\pi}{3}\\
&= 30\left(-\frac{\sqrt{3}}{2}\right)^2\left(\frac{1}{2}\right)\\
&= 15\left(\frac{3}{4}\right)\\
&= 11.25\\
\delta f(\frac{\pi}{3}) &\approx f'(\frac{\pi}{3})\delta x\\
&= 11.25 \times 0.001\\
&= 0.01125\\
f(\frac{\pi}{3} + 0.001) - f(\frac{\pi}{3}) &= 0.011266
\end{aligned}$$

5. The marginal cost is $\frac{\mathrm{d}}{\mathrm{d}x}C(x)$. As the expressions given can be easily differentiated, no working is required.

6. Marginal cost $C'(x) = \frac{\mathrm{d}C}{\mathrm{d}x} = \frac{10}{\sqrt{x}}$

   (a) $C'(25) = \frac{10}{\sqrt{25}}$
   $= \$2$ per unit

   (b) $C'(100) = \frac{10}{\sqrt{100}}$
   $= \$1$ per unit

   (c) $C'(400) = \frac{10}{\sqrt{400}}$
   $= \$0.50$ per unit

7. Marginal cost is $C'(x) = 750 - 30x + 0.3x^2$ dollars per tonne.

   (a) $C'(30) = 750 - 30(30) + 0.3(30)^2$
   $= \$120$ per tonne

   (b) $C'(60) = 750 - 30(60) + 0.3(60)^2$
   $= \$30$ per tonne

   (c) $C'(100) = 750 - 30(100) + 0.3(100)^2$
   $= \$750$ per tonne

8. $C'(x) = x$
   $C'(10) = \$10$ per item

   It costs approximately an additional \$10 to produce the next item. (It's only approximate because $\frac{\delta C}{\delta x} \approx \frac{\mathrm{d}C}{\mathrm{d}x}$.)

9. (a) $P(2) = R(2) - C(2)$
   $= 850 - 446$
   $= \$404$

   (b) $\frac{P(2)}{2} = \$202$ profit per item

(c) $$\begin{aligned} P(10) &= R(10) - C(10) \\ &= 4250 - 1150 \\ &= \$3100 \end{aligned}$$

(d) $\frac{P(10)}{10} = \$310$ profit per item

(e) $$\begin{aligned} P'(x) &= R'(x) - C'(x) \\ &= 425 - (200 - 60x + 6x^2) \\ &= 225 + 60x - 6x^2 \end{aligned}$$

At maximum profit,

$$\begin{aligned} P'(x) &= 0 \\ 225 + 60x - 6x^2 &= 0 \\ x &= 12.9 \end{aligned}$$

(ignoring the negative root). Thus, to the nearest integer, the maximum profit will be achieved when 13 items are produced. This profit is

$$\begin{aligned} P(13) &= R(13) - C(13 \\ &= 5525 - 2074 \\ &= \$3451 \end{aligned}$$

10. (a) $$\begin{aligned} C(x) &= \int (4x + 2)\,\mathrm{d}x \\ &= 2x^2 + 2x + c \\ C(0) &= 100 \\ c &= 100 \\ C(x) &= 2x^2 + 2x + 100 \end{aligned}$$

(b) $$\begin{aligned} C(x) &= \int x(3x + 4)\,\mathrm{d}x \\ &= \int (3x^2 + 4x)\,\mathrm{d}x \\ &= x^3 + 2x^2 + c \\ C(0) &= 8000 \\ x &= 8000 \\ C(x) &= x^3 + 2x^2 + 8000 \end{aligned}$$

(c) $$\begin{aligned} C(x) &= \int 20(5x + \sin 2x)\,\mathrm{d}x \\ &= 20\left(\frac{5x^2}{2} - \frac{\cos 2x}{2}\right) + c \\ &= 10\left(5x^2 - \cos 2x\right) + c \\ C(0) &= 30 \\ 10\left(-\cos 0\right) + c &= 30 \\ -10 + c &= 30 \\ c &= 40 \\ C(x) &= 10\left(5x^2 - \cos 2x\right) + 40 \\ &= 10\left(5x^2 - \cos 2x + 4\right) \end{aligned}$$

(d) $$\begin{aligned} C(x) &= \int 4(3x^3 + 5\cos^2 x)\,\mathrm{d}x \\ &= 3x^4 + 10\int (2\cos^2 x - 1 + 1)\,\mathrm{d}x \\ &= 3x^4 + 10\int (\cos 2x + 1)\,\mathrm{d}x \\ &= 3x^4 + 5\sin 2x + 10x + c \\ C(0) &= 100 \\ c &= 100 \\ C(x) &= 3x^4 + 5\sin 2x + 10x + 100 \end{aligned}$$

11. For a revenue function, it is usually safe to assume $R(0) = 0$.

(a) $$\begin{aligned} R(x) &= \int 500\,\mathrm{d}x \\ &= 500x \end{aligned}$$

(b) $$\begin{aligned} R(x) &= \int (60 - 0.1x)\,\mathrm{d}x \\ &= 60x - 0.05x^2 \end{aligned}$$

12. (a) $$\begin{aligned} N(x) &= \int 40\pi \cos\frac{\pi t}{600} dt \\ &= 24\,000 \sin\frac{\pi t}{600} + c \\ N(0) &= 0 \\ c &= 0 \\ N(x) &= 24\,000 \sin\frac{\pi t}{600} \end{aligned}$$

(b) $$\begin{aligned} N(100) &= 24\,000 \sin\frac{100\pi}{600} \\ &= 24\,000 \sin\frac{\pi}{6} \\ &= 12\,000 \end{aligned}$$

(c) $$\begin{aligned} N(100) - N(99) &= 12\,000 - 24\,000 \sin\frac{99\pi}{600} \\ &= 12\,000 - 11\,891 \\ &= 109 \end{aligned}$$

Alternatively estimate using

$$\begin{aligned} \frac{\mathrm{d}N}{\mathrm{d}t} &= 40\pi\cos\frac{100\pi}{600} \\ &= 40\pi\cos\frac{\pi}{6} \\ &= 20\pi\sqrt{3} \\ &= 108.8 \\ &\approx 109 \end{aligned}$$

13. $$\begin{aligned} C(x) &= \int \frac{20}{\sqrt{x}}\,\mathrm{d}x \\ &= 40\sqrt{x} + c \\ C(0) &= 500 \\ c &= 500 \\ C(x) &= 40\sqrt{x} + 500 \\ C(100) &= 40\sqrt{100} + 500 \\ &= \$900 \\ C(400) &= 40\sqrt{400} + 500 \\ &= \$1\,300 \end{aligned}$$

Average cost

(a) for $x = 100$ is $\frac{900}{100} = \$9$ per unit
(b) for $x = 400$ is $\frac{1300}{400} = \$3.25$ per unit

14. $$\begin{aligned} &\int_{25}^{30} (x^2 + 10x)\,\mathrm{d}x \\ &= \left[\frac{x^3}{3} + 5x^2\right]_{25}^{30} \\ &= (9000 + 4500) - (5208\frac{1}{3} + 3125) \\ &= 13\,500 - 8\,333\frac{1}{3} \\ &= 5166\frac{2}{3} \\ &\approx \$5200 \end{aligned}$$

15. (a) Note that $\frac{\mathrm{d}V}{\mathrm{d}t} < 0$ for $t < 4$ so

$$\int_0^1 \frac{\mathrm{d}V}{\mathrm{d}t} dt$$

gives the opposite of the amount drained, so the integral we need is

$$\begin{aligned} -\int_0^1 \frac{\mathrm{d}V}{\mathrm{d}t} dt &= -\int_0^1 (5t - 40) dt \\ &= \int_0^1 (40 - 5t) dt \end{aligned}$$

(b) $$\begin{aligned} \int_3^4 (40 - 5t) dt &= \left[40t - \frac{5t^2}{2}\right]_3^4 \\ &= (160 - 40) - (120 - 22.5) \\ &= 22.5 \text{ kL} \end{aligned}$$

(c) $$\begin{aligned} \int_0^4 (40 - 5t) dt &= \left[40t - \frac{5t^2}{2}\right]_0^4 \\ &= (160 - 40) - 0 \\ &= 120 \text{ kL} \end{aligned}$$

16. Let $\theta$ be the size in radians of the nominally $60°$ angle. Let $h$ be the height of the triangle. The area of the triangle is

$$\begin{aligned} A &= \frac{1}{2} 8h \\ &= 4h \\ \tan\theta &= \frac{h}{8} \\ h &= 8\tan\theta \\ A &= 4(8\tan\theta) \\ &= 32\tan\theta \\ \frac{\mathrm{d}A}{\mathrm{d}\theta} &= \frac{32}{\cos^2\theta} \\ \frac{\Delta A}{\Delta\theta} &\approx \frac{32}{\cos^2\theta} \\ \Delta A &\approx \frac{32}{\cos^2\theta}\Delta\theta \\ &= \frac{32}{\cos^2 60°}\left(\frac{\pi \times 0.5}{180}\right) \\ &= 128\left(\frac{\pi}{360}\right) \\ &\approx 1.1\text{cm}^2 \end{aligned}$$

## Exercise 7B

1. No working required.

2. (a) Initial displacement is $x = 5(0)^2 - 2(0) + 8 = 8$m

(b) Initial distance is $|x| = 8$m

(c) Initial velocity:

$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 10t - 2 \\ \frac{\mathrm{d}v}{\mathrm{d}t}_{|t=0} &= -2\text{ms}^{-1} \end{aligned}$$

(d) Speed at $t = 2$ is $|v|_{|t=2} = 10(2) - 2 = 18\text{ms}^{-1}$

(e) Acceleration at $t = 5$ is

$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= 10\text{ms}^{-2} \end{aligned}$$

3. (a) Initial displacement is $x = 0(2(0)+1) = 0$m
   (b) Initial distance is $|x| = 0$m
   (c) Initial velocity:

$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 4t + 1 \\ \frac{\mathrm{d}v}{\mathrm{d}t}_{|t=0} &= 1\text{ms}^{-1} \end{aligned}$$

   (d) Speed at $t = 2$ is $|v|_{|t=2} = 4(2)+1 = 9\text{ms}^{-1}$
   (e) Acceleration at $t = 5$ is

$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= 4\text{ms}^{-2} \end{aligned}$$

4. (a) Initial displacement is $x = 4(0)^4 + 3(0) = 0$m
   (b) Initial distance is $|x| = 0$m
   (c) Initial velocity:

$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 12t^2 + 3 \\ \frac{\mathrm{d}v}{\mathrm{d}t}_{|t=0} &= 3\text{ms}^{-1} \end{aligned}$$

   (d) Speed at $t = 2$ is $|v|_{|t=2} = 12(2)^2 + 3 = 51\text{ms}^{-1}$
   (e) Acceleration at $t = 5$ is

$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= 24t \\ a_{|t=5} &= 120\text{ms}^{-2} \end{aligned}$$

5. (a) Initial displacement is $x = 30 - 6(0) = 30$m
   (b) Initial distance is $|x| = 30$m
   (c) Initial velocity:

$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= -6\text{ms}^{-1} \end{aligned}$$

   (d) Speed at $t = 2$ is $|v|_{|t=2} = |(| - 6) = 6\text{ms}^{-1}$
   (e) Acceleration at $t = 5$ is

$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= 0\text{ms}^{-2} \end{aligned}$$

6. (a) Initial displacement is $x = 2(0)^3 - 30(0) - 1 = -1$m
   (b) Initial distance is $|x| = 1$m
   (c) Initial velocity:

$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 6t^2 - 30 \\ \frac{\mathrm{d}v}{\mathrm{d}t}_{|t=0} &= -30\text{ms}^{-1} \end{aligned}$$

   (d) Speed at $t = 2$ is $|v|_{|t=2} = |6(2)^2 - 30| = 6\text{ms}^{-1}$
   (e) Acceleration at $t = 5$ is

$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= 12t \\ a_{|t=5} &= 60\text{ms}^{-2} \end{aligned}$$

7. (a) Initial displacement is $x = (1-4(0))^3 = 1$m
   (b) Initial distance is $|x| = 1$m
   (c) Initial velocity:

$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 3(1-4t)^2(-4) \\ &= -12(1-4t)^2 \\ \frac{\mathrm{d}v}{\mathrm{d}t}_{|t=0} &= -12\text{ms}^{-1} \end{aligned}$$

   (d) Speed at $t = 2$ is

$$\begin{aligned} |v|_{|t=2} &= |-12(1-4(2))^2| \\ &= 12 \times 49 \\ &= 588\text{ms}^{-1} \end{aligned}$$

   (e) Acceleration at $t = 5$ is

$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= -24(1-4t)(-4) \\ &= 96(1-4t) \\ a_{|t=5} &= 96(1-20) \\ &= -1824\text{ms}^{-2} \end{aligned}$$

8. (a)

$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 6t + 4 \\ v_{|t=3} &= 6(3) + 4 \\ &= 22\,\text{m/s} \end{aligned}$$

   (b)

$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= 6\,\text{m/s}^2 \end{aligned}$$

9. (a)

$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= 12t \\ a_{|t=3} &= 12(3) \\ &= 36\,\text{m/s}^2 \end{aligned}$$

(b)
$$\begin{aligned}
x &= \int v\,\mathrm{d}t \\
&= 2t^3 - t + c \\
51 &= 2(1)^3 - (1) + c \\
c &= 50 \\
x &= 2t^3 - t + 50 \\
x_{t=4} &= 2(4)^3 - 4 + 50 \\
&= 174\,\mathrm{m}
\end{aligned}$$

Another possible approach is to use the known position and a definite integral from the known time:

$$\begin{aligned}
x &= 51 + \int_1^4 (6t^2 - 1)\,\mathrm{d}t \\
&= 51 + \left[2t^3 - t\right]_1^4 \\
&= 51 + \left(2(4)^3 - 4\right) - \left(2(1)^3 - 1\right) \\
&= 51 + 124 - 1 \\
&= 174\,\mathrm{m}
\end{aligned}$$

10. (a)
$$\begin{aligned}
a_{|t=0} &= 6(0) + 4 \\
&= 4\,\mathrm{m/s^2}
\end{aligned}$$

(b)
$$\begin{aligned}
v &= \int (6t + 4)\,\mathrm{d}t \\
&= 3t^2 + 4t + c \\
8 &= 3(1)^2 + 4(1) + c \\
c &= 1 \\
v &= 3t^2 + 4t + 1 \\
v_{t=3} &= 3(3)^2 + 4(3) + 1 \\
&= 40\,\mathrm{m/s}
\end{aligned}$$

(c)
$$\begin{aligned}
x_{t=2} &= 9 + \int_1^2 (3t^2 + 4t + 1)dt \\
&= 9 + \left[t^3 + 2t^2 + t\right]_1^2 \\
&= 9 + \left((2)^3 + 2(2)^2 + 2\right) \\
&\quad - \left((1)^3 + 2(1)^2 + 1\right) \\
&= 9 + 18 - 4 \\
&= 23\,\mathrm{m}
\end{aligned}$$

11. (a)
$$\begin{aligned}
a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\
&= 6\sqrt{16 + t^2} + \frac{6t\left(\frac{1}{2}\right)(2t)}{\sqrt{16 + t^2}} \\
&= 6\sqrt{16 + t^2} + \frac{6t^2}{\sqrt{16 + t^2}} \\
&= \frac{6(16 + t^2) + 6t^2}{\sqrt{16 + t^2}} \\
&= \frac{96 + 6t^2 + 6t^2}{\sqrt{16 + t^2}} \\
&= \frac{96 + 12t^2}{\sqrt{16 + t^2}} \\
&= 12\frac{8 + t^2}{\sqrt{16 + t^2}} \\
a_{|t=0} &= 12\frac{8 + (0)^2}{\sqrt{16 + (0)^2}} \\
&= 24\,\mathrm{m/s^2}
\end{aligned}$$

(b)
$$\begin{aligned}
x &= \int v\,\mathrm{d}t \\
&= \int (6t\sqrt{16 + t^2})\,\mathrm{d}t \\
&= 3\int 2t(16 + t^2)^{0.5}\,\mathrm{d}t \\
&= \frac{3(16 + t^2)^{1.5}}{1.5} + c \\
&= 2(16 + t^2)^{1.5} + c \\
8 &= 2(16 + 0^2)^{1.5} + c \\
8 &= 128 + c \\
c &= -120 \\
x &= 2(16 + t^2)^{1.5} - 120 \\
x_{|t=3} &= 2(16 + (3)^2)^{1.5} - 120 \\
&= 2(25)^{1.5} - 120 \\
&= 2(125) - 120 \\
&= 250 - 120 \\
&= 130\,\mathrm{m}
\end{aligned}$$

12.
$$\begin{aligned}
v &= \int a\,\mathrm{d}t \\
&= \int \frac{6t(t^2 + 2t + 1)}{5}\,\mathrm{d}t \\
&= \int \frac{6t^3 + 12t^2 + 6t}{5}\,\mathrm{d}t \\
&= 0.3t^4 + 0.8t^3 + 0.6t^2 + c \\
2 &= 0.3(1)^4 + 0.8(1)^3 + 0.6(1)^2 + c \\
2 &= 1.7 + c \\
c &= 0.3 \\
v_{|t=0} &= 0.3\,\mathrm{m/s}
\end{aligned}$$

13. (a)
$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= -2\sin t \\ v_{|t=\frac{\pi}{6}} &= -2\sin\frac{\pi}{6} \\ &= -1\,\mathrm{m/s} \end{aligned}$$

(b)
$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= -2\cos t \\ a_{|t=\frac{\pi}{2}} &= -2\cos\frac{\pi}{2} \\ &= 0 \end{aligned}$$

It is correct but not strictly necessary to give units for the acceleration, since zero acceleration means the same regardless of the units being used.

14. (a)
$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= 8\cos 2t \\ a_{|t=\frac{\pi}{6}} &= 8\cos\frac{\pi}{3} \\ &= 4\,\mathrm{m/s^2} \end{aligned}$$

(b)
$$\begin{aligned} x &= \int v\,dt \\ &= -2\cos 2t + c \\ 3 &= -2\cos 0 + c \\ c &= 5 \\ x &= 5 - 2\cos 2t \\ x_{|t=\frac{\pi}{2}} &= 5 - 2\cos\pi \\ &= 7\,\mathrm{m} \end{aligned}$$

15. This question is simplified if you first recognise that $4\sin t\cos t = 2\sin 2t$

(a)
$$\begin{aligned} v &= \int 2\sin 2t\,\mathrm{d}t \\ &= -\cos 2t + c \\ 3 &= -\cos 0 + c \\ c &= 4 \\ v &= 4 - \cos 2t \\ v_{|t=\frac{\pi}{3}} &= 4 - \cos\frac{2\pi}{3} \\ &= 4.5\,\mathrm{m/s} \end{aligned}$$

(b)
$$\begin{aligned} x &= 5 + \int_0^{\frac{\pi}{3}} (4 - \cos 2t)\,\mathrm{d}t \\ &= 5 + [4t - 0.5\sin 2t]_0^{\frac{\pi}{3}} \\ &= 5 + \left(\frac{4\pi}{3} - \frac{\sqrt{3}}{4}\right) - (0 - 0) \\ &= 5 + \frac{4\pi}{3} - \frac{\sqrt{3}}{4} \\ &\approx 8.756\,\mathrm{m} \end{aligned}$$

16.
$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 18(3t+1)^2 \\ a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= 108(3t+1) \end{aligned}$$

At $t = 2$,
$$\begin{aligned} x &= 2(3(2)+1)^3 \\ &= 686\,\mathrm{m} \\ v &= 18(3(2)+1)^2 \\ &= 882\,\mathrm{m/s} \\ a &= 108(3(2)+1) \\ &= 756\,\mathrm{m/s^2} \end{aligned}$$

17. (a)
$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \frac{(2t+3) - 2(t+1)}{(2t+3)^2} \\ &= \frac{1}{(2t+3)^2} \\ a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= \frac{-4}{(2t+3)^3} \end{aligned}$$

(b) At $t = 1$,
$$\begin{aligned} x &= \frac{1+1}{2(1)+3} \\ &= \frac{2}{5} \\ &= 0.4\,\mathrm{m} \\ v &= \frac{1}{(2(1)+3)^2} \\ &= \frac{1}{25} \\ &= 0.04\,\mathrm{m/s} \\ a &= \frac{-4}{(2(1)+3)^3} \\ &= -\frac{4}{125} \\ &= -0.032\,\mathrm{m/s^2} \end{aligned}$$

18. (a)
$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= t^2 - 12t - 45 \\ &= (t-15)(t+3) \end{aligned}$$

when $v = 0$
$$\begin{aligned} t &= 15 \text{ (given that } t \geq 0) \\ x &= \frac{15^3}{3} - 6(15)^2 - 45(15) + 1000 \\ &= 100\,\mathrm{m} \end{aligned}$$

(b)
$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= 2t - 12 \end{aligned}$$
when $a = 0$
$$\begin{aligned} t &= 6 \\ x &= \frac{6^3}{3} - 6(6)^2 - 45(6) + 1000 \\ &= 586\,\text{m} \end{aligned}$$

19. "Hit the ground" means $h = 0$

$$\begin{aligned} 42 + 29t - 5t^2 &= 0 \\ t &= 7\,\text{s (ignoring the negative root)} \\ v &= \frac{\mathrm{d}h}{\mathrm{d}t} \\ &= 29 - 10t \\ v_{|t=7} &= 29 - 70 \\ &= -41\,\text{m/s} \end{aligned}$$

20. (a)
$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 16 - 2t \\ v_{|t=20} &= 16 - 40 \\ &= -24\,\text{m/s} \end{aligned}$$
$\therefore$ speed $= 24\,\text{m/s}$

(b)
$$\begin{aligned} v &= 0 \\ 16 - 2t &= 0 \\ t &= 8\,\text{s} \\ x &= 8(16 - 8) \\ &= 64\,\text{m} \end{aligned}$$

(c) Provided there is no change in direction, distance travelled is equal to the difference between the displacements:

$$\begin{aligned} d &= x_{|t=5} - x_{|t-1} \\ &= 5(16 - 5) - 1(16 - 1) \\ &= 55 - 15 \\ &= 40\,\text{m} \end{aligned}$$

(d) Between 5 and 10 seconds the particle reaches its maximum displacement and returns. We could find the distance travelled by integrating the absolute value of the velocity, but it's simpler to do it in two parts: from 5 to 8 seconds, and from 8 to 10 seconds.

$$\begin{aligned} d &= (64 - 55) + (64 - 10(16 - 10)) \\ &= 9 + (64 - 60) \\ &= 13\,\text{m} \end{aligned}$$

21.
$$\begin{aligned} v &= c - 9.8t \\ \text{initial velocity } v_{|t=0} &= c \\ \text{at max. height } v &= 0 \\ c - 9.8t &= 0 \\ c &= 9.8t \\ 250 &= (9.8t)t - 4.9t^2 \\ 250 &= 9.8t^2 - 4.9t^2 \\ 250 &= 4.9t^2 \\ t &= \sqrt{\frac{250}{4.9}} \\ c &= 9.8\sqrt{\frac{250}{4.9}} \\ &= 70\,\text{m/s} \end{aligned}$$

22.
$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 3t^2 - 24t + 36 \\ d &= \int_1^8 |3t^2 - 24t + 36|\,\mathrm{d}t \\ &= 71\,\text{m} \end{aligned}$$

This could also be done without a calculator. First determine whether there is a change of direction in the interval of interest:

$$\begin{aligned} v &= 0 \\ 3t^2 - 24t + 36 &= 0 \\ t^2 - 8t + 12 &= 0 \\ (t - 2)(t - 6) &= 0 \end{aligned}$$

There are two changes of direction in the interval of interest so we need to find the distance in three sub-intervals: 1–2, 2–6 and 6–8:

$$\begin{aligned} d &= |x(2) - x(1)| \\ &\quad + |x(6) - x(2)| \\ &\quad + |x(8) - x(6)| \\ x(1) &= 1^3 - 12(1)^2 + 36(1) + 15 \\ &= 40 \\ x(2) &= 2^3 - 12(2)^2 + 36(2) + 15 \\ &= 47 \\ x(6) &= 6^3 - 12(6)^2 + 36(6) + 15 \\ &= 15 \\ x(8) &= 8^3 - 12(8)^2 + 36(8) + 15 \\ &= 47 \\ d &= 7 + 32 + 32 \\ &= 71\,\text{m} \end{aligned}$$

(You should know how to do this without a calculator, but your first impulse for a question like this is to use the calculator if it is available to you.)

23.
$$\begin{aligned}
v &= \int a\,\mathrm{d}t \\
&= 3t^2 - 24t + c \\
35 &= 3(0)^2 - 24(0) + c \\
c &= 35 \\
v &= 3t^2 - 24t + 35 \\
x &= \int v\,\mathrm{d}t \\
&= t^3 - 12t^2 + 35t + k \\
0 &= (0)^3 - 12(0)^2 + 35(0) + k \\
k &= 0 \\
x &= t^3 - 12t^2 + 35t
\end{aligned}$$
solve for the first $x = 0, t > 0$
$$\begin{aligned}
t^3 - 12t^2 + 35t &= 0 \\
t(t-5)(t-7) &= 0 \\
t &= 5 \\
v &= 3(5)^2 - 24(5) + 35 \\
&= -10\,\mathrm{m/s}
\end{aligned}$$

24.
$$\begin{aligned}
v &= \int a\,\mathrm{d}t \\
&= 2t - t^2 + c \\
v_{|t=0} &= 24 \\
\therefore\quad v &= 24 + 2t - t^2 \\
x &= \int v\,\mathrm{d}t \\
&= 24t + t^2 - \frac{t^3}{3} + k \\
x_{|t=0} &= 0 \\
\therefore\quad x &= 24t + t^2 - \frac{t^3}{3}
\end{aligned}$$

(a)
$$\begin{aligned}
v &= 0 \\
24 + 2t - t^2 &= 0 \\
(4+t)(6-t) &= 0 \\
t &= 6\,\mathrm{s} \\
x &= 24(6) + (6)^2 - \frac{(6)^3}{3} \\
&= 144 + 36 - 72 \\
&= 108\,\mathrm{m}
\end{aligned}$$

(b)
$$\begin{aligned}
x_{|t=3} &= 24(3) + (3)^2 - \frac{(3)^3}{3} \\
&= 72 + 9 - 9 \\
&= 72\,\mathrm{m} \\
x_{|t=9} &= 24(9) + (9)^2 - \frac{(9)^3}{3} \\
&= 216 + 81 - 243 \\
&= 54\,\mathrm{m}
\end{aligned}$$

(c)
$$\begin{aligned}
d &= (108 - 72) + (108 - 54) \\
&= 90\,\mathrm{m}
\end{aligned}$$

25. (a)
$$\begin{aligned}
v &= \int a\,\mathrm{d}t \\
&= 0.2t \\
x &= \int v\,\mathrm{d}t \\
&= 0.1t^2 \\
x_{|t=180} &= 0.1(180)^2 \\
&= 3240\,\mathrm{m}
\end{aligned}$$
(Both constants of integration are zero to account for the initial position and velocity both being zero.)

(b)
$$\begin{aligned}
v &= 0.2 \times 180 \\
&= 36\,\mathrm{m/s}
\end{aligned}$$

26. "In the fourth second" means from $t = 3$ to $t = 4$:
$$\begin{aligned}
d &= \int_3^4 |7 + 2t|\,\mathrm{d}t \\
&= \int_3^4 (7 + 2t)\,\mathrm{d}t \\
&= \left[7t + t^2\right]_3^4 \\
&= (28 + 16) - (21 + 9) \\
&= 14\,\mathrm{m}
\end{aligned}$$
(The absolute value can safely be dispensed with since $v$ is positive for all positive $t$.)

27. "In the fourth second" means from $t = 3$ to $t = 4$. Here $v$ changes sign at $t = 3.5$ so the absolute value must be retained. If doing the problem without a calculator, the integral should be divided into two parts, as follows:
$$\begin{aligned}
d &= \int_3^4 |7 - 2t|\,\mathrm{d}t \\
&= \int_3^{3.5} (7 - 2t)\,\mathrm{d}t + \int_{3.5}^4 (2t - 7)\,\mathrm{d}t \\
&= \left[7t - t^2\right]_3^{3.5} + \left[t^2 - 7t\right]_{3.5}^4 \\
&= (24.5 - 12.25) - (21 - 9) \\
&\quad + (16 - 28) - (12.25 - 24.5) \\
&= 12.25 - 12 + (-12) - (-12.25) \\
&= 0.5\,\mathrm{m}
\end{aligned}$$

28. (a) The maximum value of $\sin 2t$ is 1, so the maximum velocity is $2 \times 1 = 2\,\mathrm{m/s}$.

(b)
$$\begin{aligned}
a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\
&= 4\cos 2t
\end{aligned}$$

(c) The maximum value of $\cos 2t$ is 1, so the maximum acceleration is $4 \times 1 = 4\,\mathrm{m/s^2}$.

(d)
$$\begin{aligned}
x &= \int v\,\mathrm{d}t \qquad = -\cos 2t + c \\
0 &= -\cos 0 + c \\
c &= 1 \\
x &= 1 - \cos 2t
\end{aligned}$$

(e) The maximum value of $-\cos 2t$ is 1, so the maximum displacement is $1 + 1 = 2\,\mathrm{m}$.

29. (a) The minimum value of $\sin^2 t$ is 0, so the minimum velocity is $3 \times 0 = 0\,\mathrm{m/s}$. (I.e. the particle never moves backward.)

(b)
$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= 6\sin t\cos t \\ &= 3(2\sin t\cos t) \\ &= 3\sin 2t \end{aligned}$$

(c) The maximum value of $\sin 2t$ is 1 which first occurs when $2t = \frac{\pi}{2}$, i.e. $t = \frac{\pi}{4}$ s.

(d)
$$\begin{aligned} x &= \int v\,\mathrm{d}t \\ &= \int 3\sin^2 t\,\mathrm{d}t \\ &= -1.5\int (1 - 2\sin^2 t - 1)\,\mathrm{d}t \\ &= -1.5\int (\cos 2t - 1)\,\mathrm{d}t \\ &= -0.75\sin 2t + 1.5t + c \\ 0 &= -0.75\sin 0 + 1.5(0) + c \\ c &= 0 \\ x &= 1.5t - 0.75\sin 2t \end{aligned}$$

(e)
$$\begin{aligned} x &= 1.5\left(\frac{\pi}{6}\right) - 0.75\sin\left(\frac{\pi}{3}\right) \\ &= \left(\frac{\pi}{4} - \frac{3\sqrt{3}}{8}\right)\,\mathrm{m} \end{aligned}$$

30. (a)
$$\begin{aligned} \Delta x &= \int_0^4 3\sqrt{1+2t}\,\mathrm{d}t \\ &= \left[\frac{3(1+2t)^{1.5}}{1.5 \times 2}\right]_0^4 \\ &= \left[(1+2t)^{1.5}\right]_0^4 \\ &= 9^{1.5} - 1^{1.5} \\ &= 26\,\mathrm{m} \end{aligned}$$

(b) Use the substitution

$$u = 1 + 2t \qquad t = \frac{u-1}{2}$$
$$du = 2dt \qquad dt = \frac{du}{2}$$

$$\begin{aligned} &\int_0^4 3t\sqrt{1+2t}\,\mathrm{d}t \\ &= \int_{1+2(0)}^{1+2(4)} \frac{3(u-1)}{2}\sqrt{u}\frac{du}{2} \\ &= \int_1^9 0.75(u-1)\sqrt{u}\,\mathrm{d}u \\ &= 0.75\int_1^9 u^{1.5} - u^{0.5}\,\mathrm{d}u \\ &= 0.75\left[\frac{u^{2.5}}{2.5} - \frac{u^{1.5}}{1.5}\right]_1^9 \\ &= \left[0.3u^{2.5} - 0.5u^{1.5}\right]_1^9 \\ &= \left(0.3(9)^{\frac{5}{2}} - 0.5(9)^{\frac{3}{2}}\right) \\ &\quad - \left(0.3(1)^{\frac{5}{2}} - 0.5(1)^{\frac{3}{2}}\right) \\ &= \left(0.3(3)^5 - 0.5(3)^3\right) - (0.3 - 0.5) \\ &= (0.3(243) - 0.5(27)) + 0.2 \\ &= 72.9 - 13.5 + 0.2 \\ &= 59.6\,\mathrm{m} \end{aligned}$$

31.
$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= \sqrt{3}\cos t + \sin t \end{aligned}$$

when $a = 0$

$$\begin{aligned} \sqrt{3}\cos t + \sin t &= 0 \\ \tan t &= -\sqrt{3} \\ t &= \frac{2\pi}{3} \\ \text{or } t &= \frac{5\pi}{3} \end{aligned}$$

$$\begin{aligned} x &= \int v\,\mathrm{d}t \\ &= -\sqrt{3}\sin t - \cos t + c \\ 0 &= -\sqrt{3}\sin\frac{\pi}{6} - \cos\frac{\pi}{6} + c \\ 0 &= -\sqrt{3}\left(\frac{1}{2}\right) - \frac{\sqrt{3}}{2} + c \\ 0 &= -\sqrt{3} + c \\ c &= \sqrt{3} \\ x &= \sqrt{3} - \sqrt{3}\sin t - \cos t \end{aligned}$$

when $t = \dfrac{2\pi}{3}$

$$\begin{aligned} x &= \sqrt{3} - \sqrt{3}\sin\frac{2\pi}{3} - \cos\frac{2\pi}{3} \\ &= \sqrt{3} - \frac{\sqrt{3}}{2} + \frac{1}{2} \\ &= \frac{\sqrt{3}+1}{2} \end{aligned}$$

when $t = \frac{5\pi}{3}$

$$\begin{aligned}x &= \sqrt{3} - \sqrt{3}\sin\frac{5\pi}{3} - \cos\frac{5\pi}{3}\\ &= \sqrt{3} + \frac{\sqrt{3}}{2} - \frac{1}{2}\\ &= \frac{3\sqrt{3}-1}{2}\end{aligned}$$

32. (a)
$$\begin{aligned}v &= 3x + 2\\ \frac{\mathrm{d}v}{\mathrm{d}t} &= 3\frac{\mathrm{d}x}{\mathrm{d}t}\\ a &= 3v\\ &= 3(3x+2)\\ &= (9x+6)\,\mathrm{m/s^2}\end{aligned}$$

(b)
$$\begin{aligned}v &= 3(4) + 2\\ &= 14\,\mathrm{m/s}\\ a &= 9(4) + 6\\ &= 42\,\mathrm{m/s^2}\end{aligned}$$

33. (a)
$$\begin{aligned}v &= 3x^2 - 2\\ \frac{\mathrm{d}v}{\mathrm{d}t} &= 6x\frac{\mathrm{d}x}{\mathrm{d}t}\\ a &= 6xv\\ &= 6x(3x^2-2)\\ &= (18x^3 - 12x)\,\mathrm{m/s^2}\end{aligned}$$

(b)
$$\begin{aligned}v &= 3(1)^2 - 2\\ &= 1\,\mathrm{m/s}\\ a &= 18(1)^3 - 12(1)\\ &= 6\,\mathrm{m/s^2}\end{aligned}$$

## Exercise 7C

1.
$$\begin{aligned}\frac{\mathrm{d}X}{\mathrm{d}t} &= \frac{\mathrm{d}X}{\mathrm{d}t}\frac{\mathrm{d}p}{\mathrm{d}t}\\ &= (2\cos 2p)(2)\\ &= 4\cos 2p\\ &= 4\cos\frac{\pi}{3}\\ &= 2\end{aligned}$$

2.
$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{\mathrm{d}A}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}\\ &= 0.6\sin(3x)\cos(3x)\\ &= 0.3\sin 6x\\ &= 0.3\sin\frac{\pi}{6}\\ &= 0.15\end{aligned}$$

3.
$$\begin{aligned}y^2 &= 3x^3 + 1\\ 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 9x^2\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{9x^2}{2y}\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}\\ &= \frac{0.45x^2}{y}\end{aligned}$$

When $y = 5$

$$\begin{aligned}(5^2) &= 3x^3 + 1\\ 3x^3 &= 24\\ x^3 &= 8\\ x &= 2\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{0.45(2)^2}{5}\\ &= 0.36\end{aligned}$$

4.
$$\begin{aligned}x^2 + y^2 &= 400\\ 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x}{y}\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}\\ &= -6\frac{x}{y}\end{aligned}$$

When $y = 12$

$$\begin{aligned}x^2 + (12)^2 &= 400, x \geq 0\\ x &= \sqrt{400 - 144}\\ &= 16\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -6\frac{16}{12}\\ &= -8\end{aligned}$$

5. $$\begin{aligned} A &= \frac{1}{2}(10)(10)\sin x \\ &= 50\sin x \\ \frac{\mathrm{d}A}{\mathrm{d}x} &= 50\cos x \\ \frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{\mathrm{d}A}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 0.5\cos x \\ &= 0.5\cos\frac{\pi}{3} \\ &= 0.25\,\mathrm{cm}^2/\mathrm{s} \end{aligned}$$

6. $$\begin{aligned} A &= \frac{1}{2}x^2\sin 45^\circ \\ &= \frac{\sqrt{2}\,x^2}{4} \\ \frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{\sqrt{2}\,x}{2}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \frac{0.1\sqrt{2}\,x}{2} \\ &= 0.05\sqrt{2}\,x \\ &= 0.05\sqrt{2}\,(10) \\ &= 0.5\sqrt{2} \\ &\approx 0.707\,\mathrm{cm}^2/\mathrm{s} \end{aligned}$$

7. $$\begin{aligned} x^2 + y^2 &= 10^2 \\ 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x}{y} \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= -0.1\frac{x}{y} \end{aligned}$$

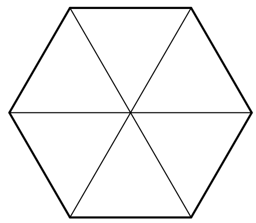
When $t = 20$

$$\begin{aligned} x &= 4 + 0.1t \\ &= 6\,\mathrm{cm} \\ y &= \sqrt{100 - 6^2} \\ &= 8\,\mathrm{cm} \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -0.1\frac{6}{8} \\ &= -0.075\,\mathrm{cm/s} \end{aligned}$$

8. Let $x$ be the side length of the triangle.

$$\begin{aligned} A &= \frac{1}{2}x^2\sin 60^\circ \\ &= \frac{\sqrt{3}\,x^2}{4} \\ \frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{\sqrt{3}\,x}{2}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \frac{\sqrt{3}\,(20)}{2}(0.2) \\ &= 2\sqrt{3}\,\mathrm{cm}^2/\mathrm{s} \end{aligned}$$

9. A regular hexagon can be divided into six equilateral triangles as shown.



$$\begin{aligned} A &= 6 \times \frac{\sqrt{3}\,x^2}{4} \\ &= \frac{3\sqrt{3}\,x^2}{2} \\ \frac{\mathrm{d}A}{\mathrm{d}t} &= 3\sqrt{3}\,x\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 3\sqrt{3}\,(20)(1) \\ &= 60\sqrt{3}\,\mathrm{cm}^2/\mathrm{minute} \\ &= \sqrt{3}\,\mathrm{cm}^2/\mathrm{s} \end{aligned}$$

10. From the cross-section being an equilateral triangle we have the relationship between height and radius:

$$\begin{aligned} \tan 60^\circ &= \frac{h}{r} \\ h &= \sqrt{3}\,r \end{aligned}$$

The volume is

$$\begin{aligned} V &= \frac{\pi r^2 h}{3} \\ &= \frac{\pi\sqrt{3}\,r^3}{3} \end{aligned}$$

Differentiating with respect to $t$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \pi\sqrt{3}\,r^2\frac{\mathrm{d}r}{\mathrm{d}t}$$

and substitute for $r$ and $\frac{\mathrm{d}r}{\mathrm{d}t}$

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= \pi\sqrt{3}\,(20)^2(0.5) \\ &= 200\pi\sqrt{3} \\ &\approx 1090\,\mathrm{cm}^3/\mathrm{s} \end{aligned}$$

11. Let $x$ be the distance between the base of the ladder and the wall. Let $y$ be the height of the top of the ladder above the ground.

$$\begin{aligned} x^2 + y^2 &= 5.2^2 \\ 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x}{y} \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= -0.1\frac{x}{y} \\ x &= \sqrt{5.2^2 - 4.8^2} \\ &= 2.0 \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -0.1\frac{2.0}{4.8} \\ &= -0.0417\,\mathrm{m/s} \end{aligned}$$

The top is moving down at approximately 4.2 cm/s.

12. (a) Let $l$ be the length of the shadow and let $d$ be the person's distance from the lamppost. Using the similar triangles as a starting point,

$$\begin{aligned}\frac{l+d}{l} &= \frac{4.5}{1.5}\\ &= 3\\ l+d &= 3l\\ d &= 2l\\ l &= 0.5d\\ \frac{\mathrm{d}l}{\mathrm{d}t} &= 0.5\frac{\mathrm{d}d}{\mathrm{d}t}\\ &= 1\,\mathrm{m/s}\end{aligned}$$

The shadow grows by 1 m/s.

(b) The tip of the shadow moves with the combined speed of the person and the increasing length of the shadow, i.e. $1+2=3$ m/s.

13.
$$\begin{aligned}r^2+(2-h)^2 &= 2^2\\ 2r\frac{\mathrm{d}r}{\mathrm{d}t}-2(2-h)\frac{\mathrm{d}h}{\mathrm{d}t} &= 0\\ 2r\frac{\mathrm{d}r}{\mathrm{d}t} &= 2(2-h)\frac{\mathrm{d}h}{\mathrm{d}t}\\ \frac{\mathrm{d}r}{\mathrm{d}t} &= \frac{2-h}{r}\frac{\mathrm{d}h}{\mathrm{d}t}\\ \frac{\mathrm{d}h}{\mathrm{d}t} &= -0.005\,\mathrm{m/s}\\ h &= 1\\ r &= \sqrt{2^2-(2-1)^2}\\ &= \sqrt{3}\\ \frac{\mathrm{d}r}{\mathrm{d}t} &= \frac{2-1}{\sqrt{3}}(-0.005)\\ &= -\frac{1}{200\sqrt{3}}\\ &= -\frac{\sqrt{3}}{6}00\,\mathrm{m/s}\\ &= -\frac{\sqrt{3}}{6}\,\mathrm{cm/s}\end{aligned}$$

The radius is decreasing at a rate of $\frac{\sqrt{3}}{6}\approx 0.29$ cm/s

14. Let BC= $x$ and AB= $y$.

$$\begin{aligned}y^2 &= x^2+20^2\\ 2y\frac{\mathrm{d}y}{\mathrm{d}t} &= 2x\frac{\mathrm{d}x}{\mathrm{d}t}\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{x}{y}\frac{\mathrm{d}x}{\mathrm{d}t}\\ &= \frac{48}{\sqrt{48^2+20^2}}(15)\\ &= \frac{48\times 15}{52}\\ &= \frac{180}{13}\\ &\approx 13.85\,\mathrm{ms}^{-1}\end{aligned}$$

15. Let $d$ be the distance from A to the balloon and let $h$ be the height of the balloon.

$$\begin{aligned}d^2 &= h^2+60^2\\ 2d\frac{\mathrm{d}d}{\mathrm{d}t} &= 2h\frac{\mathrm{d}h}{\mathrm{d}t}\\ \frac{\mathrm{d}d}{\mathrm{d}t} &= \frac{h}{d}\frac{\mathrm{d}h}{\mathrm{d}t}\\ &= \frac{80}{\sqrt{80^2+60^2}}(5)\\ &= 4\,\mathrm{m/s}\end{aligned}$$

16.
$$\begin{aligned}x &= 8\tan\theta\\ \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{8}{\cos^2\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}\\ \cos^2\theta &= \left(\frac{8}{\sqrt{8^2+5^2}}\right)^2\\ &= \frac{64}{89}\\ \frac{\mathrm{d}x}{\mathrm{d}t} &= (8)\left(\frac{89}{64}\right)(4\pi)\\ &= 44.5\pi\\ &\approx 139.8\,\mathrm{ms}^{-1}\end{aligned}$$

Note that it is not necessary to determine $\theta$ in order to find $\cos\theta$. Use the definition of cosine as $\frac{\text{adjacent}}{\text{hypotenuse}}$ to determine $\cos\theta$ directly.

17. Rotation of 5 revolutions per minute is $10\pi$ radians per minute or $\frac{\pi}{6}$ radians per second.

Let $y$ be the distance along the coastline from the nearest point at time $t$. Let $x$ be the straight line distance to the lighthouse at time $t$. We want $\frac{\mathrm{d}y}{\mathrm{d}t}$ when $x=4$.

$$\begin{aligned}y^2+3^2 &= x^2\\ 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x}{y}\end{aligned}$$

$$\begin{aligned}\text{When } x=4,\quad y &= \sqrt{4^2-3^2}\\ &= \sqrt{7}\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4}{\sqrt{7}}\end{aligned}$$

$$\cos\theta = \frac{3}{x}$$
$$x = \frac{3}{\cos\theta}$$
$$\frac{dx}{d\theta} = 3\frac{\sin\theta}{\cos^2\theta}$$

When $x = 4$, $\cos\theta = \frac{3}{4}$

$$\sin\theta = \frac{\sqrt{7}}{4}$$
$$\frac{dx}{d\theta} = 3 \times \frac{\frac{\sqrt{7}}{4}}{\frac{9}{16}} = 3 \times \frac{4\sqrt{7}}{9} = \frac{4\sqrt{7}}{3}$$
$$\frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta} = \left(\frac{4}{\sqrt{7}}\right)\left(\frac{4\sqrt{7}}{3}\right) = \frac{16}{3}$$
$$\frac{dy}{dt} = \frac{dy}{d\theta}\frac{d\theta}{dt} = \left(\frac{16}{3}\right)\left(\frac{\pi}{6}\right) = \frac{8\pi}{9} \approx 2.79\,\text{km/s}$$

18.
$$\tan\theta = \frac{h}{600}$$
$$\frac{1}{\cos^2\theta}\frac{d\theta}{dt} = \frac{1}{600}\frac{dh}{dt}$$
$$\frac{d\theta}{dt} = \frac{\cos^2\theta}{600}\frac{dh}{dt}$$
$$\cos\theta = \frac{600}{\sqrt{600^2 + 800^2}} = 0.6$$
$$\frac{d\theta}{dt} = \frac{0.6^2}{600} \times 10\frac{dh}{dt} = 0.006 \text{ rads/sec}$$

19. Let $x$ be the horizontal distance AB at time $t$.

$$\tan\theta = \frac{800}{x}$$
$$\frac{1}{\cos^2\theta}\frac{d\theta}{dt} = -\frac{800}{x^2}\frac{dx}{dt}$$
$$\frac{d\theta}{dt} = -\frac{800\cos^2\theta}{x^2}\frac{dx}{dt}$$
$$\cos^2\theta = \left(\frac{1000}{\sqrt{1000^2 + 800^2}}\right) = \frac{25}{41}$$
$$\frac{d\theta}{dt} = -\frac{800 \times \frac{25}{41}}{1000^2}(-200) = \frac{4\,000\,000}{41\,000\,000} = \frac{4}{41} \text{ rads/s}$$

20.
$$\tan\theta = \frac{h}{200}$$
$$\frac{1}{\cos^2\theta}\frac{d\theta}{dt} = \frac{1}{200}\frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{200}{\cos^2\theta}\left(\frac{1}{20}\right) = \frac{10}{\cos^2\theta}$$
$$\frac{d^2h}{dt^2} = -\frac{20(-\sin\theta)}{\cos^3\theta}\frac{d\theta}{dt} = \frac{20\tan\theta}{\cos^2\theta}\left(\frac{1}{20}\right) = \frac{\tan\theta}{\cos^2\theta}$$

(a) When $\theta = \frac{\pi}{6}$,

$$v = \frac{dh}{dt} = \frac{10}{\cos^2\theta} = \frac{10}{\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{10}{\frac{3}{4}} = \frac{40}{3}\text{ ms}^{-1}$$
$$a = \frac{d^2h}{dt^2} = \frac{\tan\theta}{\cos^2\theta} = \frac{\frac{\sqrt{3}}{3}}{\frac{3}{4}} = \frac{4\sqrt{3}}{9}\text{ ms}^{-2}$$

(b) When $\theta = \frac{\pi}{3}$,

$$\begin{aligned} v &= \frac{\mathrm{d}h}{\mathrm{d}t} \\ &= \frac{10}{\cos^2\theta} \\ &= \frac{10}{\left(\frac{1}{2}\right)^2} \\ &= \frac{10}{\frac{1}{4}} \\ &= 40\,\mathrm{ms}^{-1} \\ a &= \frac{\mathrm{d}^2h}{\mathrm{d}t^2} \\ &= \frac{\tan\theta}{\cos^2\theta} \\ &= \frac{\sqrt{3}}{\frac{1}{4}} \\ &= 4\sqrt{3}\,\mathrm{ms}^{-2} \end{aligned}$$

21. Let $s$ be the length of the shadow CE and $d$ be the distance AC. By similar triangles

$$\begin{aligned} \frac{s+d}{s} &= \frac{9}{1.8} \\ s+d &= 5s \\ d &= 4s \\ s &= 0.25d \\ \frac{\mathrm{d}s}{\mathrm{d}d} &= 0.25 \end{aligned}$$

$$\begin{aligned} d^2 &= 12^2 + y^2 \\ 2d\frac{\mathrm{d}d}{\mathrm{d}y} &= 2y \\ \frac{\mathrm{d}d}{\mathrm{d}y} &= \frac{y}{d} \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}s}{\mathrm{d}t} &= \frac{\mathrm{d}s}{\mathrm{d}d}\frac{\mathrm{d}d}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= (0.25)\left(\frac{y}{d}\right)(2) \\ &= \frac{y}{2d} \end{aligned}$$

When AC = 20, $y = 20$

$$\begin{aligned} d &= \sqrt{12^2 + 20^2} \\ &= 4\sqrt{34} \\ \frac{\mathrm{d}s}{\mathrm{d}t} &= \frac{20}{8\sqrt{34}} \\ &= \frac{5}{2\sqrt{34}} \\ &\approx 0.43\,\mathrm{m/s} \end{aligned}$$

The shadow is growing 0.43 metres per second.

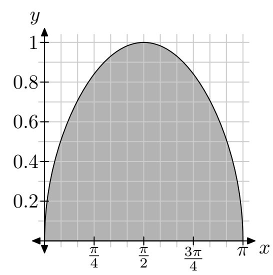
## Exercise 7D

1–3 No working required.

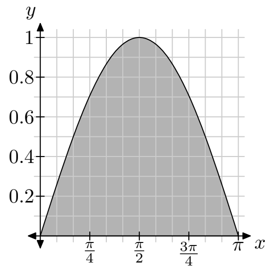
4.
$$\begin{aligned} u &= 2x \\ \frac{\mathrm{d}u}{\mathrm{d}x} &= 2 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_1^{2x} 5^t dt\right) &= \frac{\mathrm{d}}{\mathrm{d}u}\left(\int_1^{u} 5^t dt\right)\frac{\mathrm{d}u}{\mathrm{d}x} \\ &= (5^u)(2) \\ &= 2(5)^{2x} \end{aligned}$$

5.
$$\begin{aligned} u &= 5x \\ \frac{\mathrm{d}u}{\mathrm{d}x} &= 5 \\ \frac{\mathrm{d}}{\mathrm{d}x}&\left(\int_1^{5x} (3+4t+\sin t)dt\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}u}\left(\int_1^{u} (3+4t+\sin t)dt\right)\frac{\mathrm{d}u}{\mathrm{d}x} \\ &= (3+4u+\sin u)(5) \\ &= 5\,(3+20x+\sin(5x)) \\ &= 15+100x+5\sin(5x) \end{aligned}$$

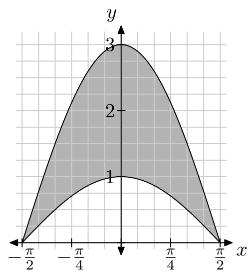
6.



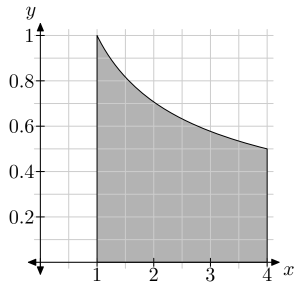

$$\begin{aligned} V &= \int_0^{\pi} \pi y^2 dx \\ &= \int_0^{\pi} \pi(\sqrt{\sin x})^2 dx \\ &= \int_0^{\pi} \pi\sin x\,\mathrm{d}x \\ &= [\pi(-\cos x)]_0^{\pi} \\ &= (-\pi\cos\pi) - (-\pi\cos 0) \\ &= \pi + \pi \\ &= 2\pi \text{ units}^3 \end{aligned}$$

7. 

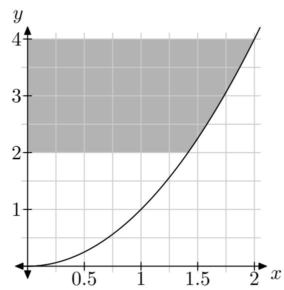

$$\begin{aligned}
V &= \int_0^{\pi} \pi(\sin x)^2 dx \\
&= -\frac{\pi}{2}\int_0^{\pi} -2\sin^2 x \, \mathrm{d}x \\
&= -\frac{\pi}{2}\int_0^{\pi} (1 - 2\sin^2 x - 1) \, \mathrm{d}x \\
&= -\frac{\pi}{2}\int_0^{\pi} (\cos(2x) - 1) \, \mathrm{d}x \\
&= -\frac{\pi}{2}\left[\frac{\sin 2x}{2} - x\right]_0^{\pi} \\
&= -\frac{\pi}{2}\left(\left(\frac{\sin 2\pi}{2} - \pi\right) - \left(\frac{\sin 0}{2} - 0\right)\right) \\
&= -\frac{\pi}{2}(-\pi - 0) \\
&= \frac{\pi^2}{2} \text{ units}^3
\end{aligned}$$

8. 

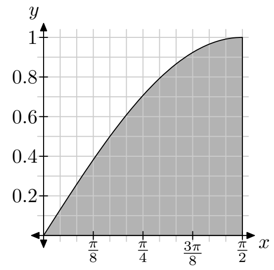

$$\begin{aligned}
V &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi y_1^2 dx - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi y_2^2 dx \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi(y_1^2 - y_2^2) dx \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi\left((3\cos x)^2 - (\cos x)^2\right) dx \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 8\pi\cos^2 x \, \mathrm{d}x \\
&= 4\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos^2 x - 1 + 1) dx \\
&= 4\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos 2x + 1) dx \\
&= 4\pi\left[\frac{\sin 2x}{2} + x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= 4\pi\left(\left(\sin\frac{\pi}{2} + \frac{\pi}{2}\right) - \left(\sin\left(-\frac{\pi}{2}\right) - \frac{\pi}{2}\right)\right) \\
&= 4\pi\left(\frac{\pi}{2} + \frac{\pi}{2}\right) \\
&= 4\pi^2 \text{ units}^3
\end{aligned}$$

9. 


$$\begin{aligned}
V &= \int_1^4 \pi\left(\frac{1}{\sqrt{x}}\right)^2 dx \\
&= \int_1^4 \frac{\pi}{x} dx \\
&= \pi\left[\ln x\right]_1^4 \\
&= \pi(\ln 4 - \ln 1) \\
&= \pi \ln 4 \text{ units}^3
\end{aligned}$$

10. 

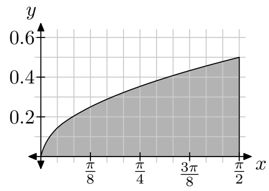

$$\begin{aligned}
V &= \int_2^4 \pi x^2 dy \\
&= \int_2^4 \pi y \, dx \\
&= \left[\frac{\pi y^2}{2}\right]_2^4 \\
&= \frac{\pi}{2}(4^2 - 2^2) \\
&= 6\pi \text{ units}^3
\end{aligned}$$




11. First possibility:

$$\begin{aligned}V &= \int_0^{\frac{\pi}{2}} \pi \left(\frac{\sin x}{2}\right)^2 dy\\ &= \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \sin^2 x \, \mathrm{d}y\\ &= -\frac{\pi}{8}\int_0^{\frac{\pi}{2}} (1 - 2\sin^2 x - 1)\, \mathrm{d}y\\ &= -\frac{\pi}{8}\int_0^{\frac{\pi}{2}} (\cos 2x - 1)\, \mathrm{d}y\\ &= -\frac{\pi}{8}\left[\frac{\sin 2x}{2} - x\right]_0^{\frac{\pi}{2}}\\ &= -\frac{\pi}{8}\left(-\frac{\pi}{2} - 0\right)\\ &= \frac{\pi^2}{16}\,\mathrm{m}^3\end{aligned}$$

Second possibility:




$$\begin{aligned}V &= \int_0^{\frac{\pi}{2}} \pi \left(\sqrt{\frac{x}{2\pi}}\right)^2 dy\\ &= \int_0^{\frac{\pi}{2}} \pi\frac{x}{2\pi} dy\\ &= \int_0^{\frac{\pi}{2}} \frac{x}{2} dy\\ &= \left[\frac{x^2}{4}\right]_0^{\frac{\pi}{2}}\\ &= (\frac{\pi^2}{16} - 0)\\ &= \frac{\pi^2}{16}\,\mathrm{m}^3\end{aligned}$$

## Chapter 7 Extension Activity

1. The $y$-coordinate of the centre of gravity is 0 from the symmetry of the shape.

   Sum of moments is

$$\begin{aligned}\int_0^r mg2yx\,\mathrm{d}x &= -mg\int_0^r -2x\sqrt{r^2 - x^2}\,\mathrm{d}x\\ &= -mg\left[\frac{2}{3}(r^2 - x^2)^{\frac{3}{2}}\right]_0^r\\ &= -\frac{2}{3}mg\left((0)^{\frac{3}{2}} - (r^2)^{\frac{3}{2}}\right)\\ &= \frac{2}{3}mgr^3\end{aligned}$$

   The moment of the sum is

$$mg\frac{\pi r^2}{2}(\bar{x})$$

   Thus

$$\begin{aligned}mg\frac{\pi r^2}{2}(\bar{x}) &= \frac{2}{3}mgr^3\\ \frac{\pi}{2}(\bar{x}) &= \frac{2}{3}r\\ \bar{x} &= \frac{4r}{3\pi}\end{aligned}$$

   $\therefore$ the centre of gravity is at the point $\left(\frac{4r}{3\pi}, 0\right)$.

2. The $y$-coordinate of the centre of gravity is 0 from the symmetry of the shape.

   The line segment AB is on the line passing through the origin and $(a, b)$, i.e. $y = \frac{b}{a}x$.

   Sum of moments is

$$\begin{aligned}\int_0^a mg2yx\,\mathrm{d}x &= 2mg\int_0^a \left(\frac{b}{a}x\right)x\,\mathrm{d}x\\ &= \frac{2mgb}{a}\int_0^a x^2\,\mathrm{d}x\\ &= \frac{2mgb}{a}\left[\frac{x^3}{3}\right]_0^a\\ &= \frac{2mgb}{3a}\left[x^3\right]_0^a\\ &= \frac{2mgb}{3a}\left(a^3 - 0^3\right)\\ &= \frac{2mga^2b}{3}\end{aligned}$$

   The moment of the sum is

$$mgab\bar{x}$$

Thus

$$\begin{aligned} mgab\bar{x} &= \frac{2mga^2b}{3} \\ \bar{x} &= \frac{2a}{3} \end{aligned}$$

$\therefore$ the centre of gravity is at the point $\left(\frac{2a}{3}, 0\right)$.

3. Area of one strip:

$$\begin{aligned} A_x &\approx ((0.5x+2) - (-0.5x-2))\,\delta x \\ &= (x+4)\delta x \end{aligned}$$

Moment of one strip:

$$I_x \approx mgx(x+4)\delta x$$

Sum of moments:

$$\begin{aligned} I &= \int_2^6 mgx(x+4)\,\mathrm{d}x \\ &= mg\left[\frac{x^3}{3} + 2x^2\right]_2^6 \\ &= mg\left(\left(\frac{6^3}{3} + 2(6)^2\right)\right. \\ &\qquad \left.-\left(\frac{2^3}{3} + 2(2)^2\right)\right) \\ &= mg\left((72+72) - \left(\frac{8}{3} + 8\right)\right) \\ &= \frac{400}{3}mg \end{aligned}$$

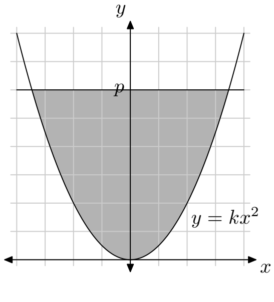
Total area:

$$\begin{aligned} A &= \frac{1}{2}\left((2+4) + (6+4)\right)(6-2) \\ &= 2\,(6+10) \\ &= 32 \end{aligned}$$

Moment of sum:

$$\begin{aligned} I &= 32mg\bar{x} \\ \therefore \quad 32mg\bar{x} &= \frac{400}{3}mg \\ \bar{x} &= \frac{25}{6} \end{aligned}$$

The symmetry of the figure gives us $\bar{y} = 0$ so the centre of gravity is at $\left(\frac{25}{6}, 0\right)$.

4. 


From the symmetry of the figure, $\bar{x} = 0$.

Taking horizontal strips of height $\delta y$ and the moment of inertia about the $x-$axis,

$$\begin{aligned} y &= kx^2 \\ x &= \pm\sqrt{\frac{y}{k}} \end{aligned}$$

Area of one strip:

$$A_y \approx 2\sqrt{\frac{y}{k}}\delta y$$

Moment of one strip:

$$\begin{aligned} I_y &\approx mgy\left(2\sqrt{\frac{y}{k}}\right)\delta y \\ &= \frac{2mg}{\sqrt{k}}y^{\frac{3}{2}}\delta y \end{aligned}$$

Sum of moments:

$$\begin{aligned} I &= \int_0^p \frac{2mg}{\sqrt{k}}y^{\frac{3}{2}}\,\mathrm{d}y \\ &= \frac{2mg}{\sqrt{k}}\left[\frac{2}{5}y^{\frac{5}{2}}\right]_0^p \\ &= \frac{4mg}{5\sqrt{k}}\left[y^{\frac{5}{2}}\right]_0^p \\ &= \frac{4mgp^{\frac{5}{2}}}{5\sqrt{k}} \end{aligned}$$

Total area:

$$\begin{aligned} A &= \int_0^p 2\sqrt{\frac{y}{k}}\,\mathrm{d}y \\ &= \frac{2}{\sqrt{k}}\left[\frac{2}{3}y^{\frac{3}{2}}\right]_0^p \\ &= \frac{4p^{\frac{3}{2}}}{3\sqrt{k}} \end{aligned}$$

Moment of sum:

$$\begin{aligned} I &= \frac{4p^{\frac{3}{2}}}{3\sqrt{k}}mg\bar{y} \\ &= \frac{4mgp^{\frac{3}{2}}}{3\sqrt{k}}\bar{y} \\ \therefore \quad \frac{4mgp^{\frac{3}{2}}}{3\sqrt{k}}\bar{y} &= \frac{4mgp^{\frac{5}{2}}}{5\sqrt{k}} \\ \bar{y} &= \frac{3p}{5} \end{aligned}$$

Hence the centre of gravity is at $\left(0, \frac{3p}{5}\right)$.

5. From the symmetry of the figure, $\bar{y} = 0$.

The radius of one disc is

$$y = \sqrt{r^2 - x^2}$$

Volume of one disc:

$$\begin{aligned}V_x &\approx \pi y^2 \delta x\\ &= \pi(r^2 - x^2)\delta x\end{aligned}$$

Moment of one strip (where $m$ is mass per unit volume):

$$\begin{aligned}I_x &\approx mgx\pi(r^2 - x^2)\delta x\\ &= \pi mg(r^2 x - x^3)\delta x\end{aligned}$$

Sum of moments:

$$\begin{aligned}I &= \int_0^r \pi mg(r^2 x - x^3)\,\mathrm{d}x\\ &= \pi mg\left[\frac{r^2x^2}{2} - \frac{x^4}{4}\right]_0^r\\ &= \frac{\pi mg}{4}\left[2r^2x^2 - x^4\right]_0^r\\ &= \frac{\pi mgr^4}{4}\end{aligned}$$

Total volume:

$$V = \frac{2\pi r^3}{3}$$

Moment of sum:

$$\begin{aligned}I &= \frac{2\pi r^3}{3}mg\bar{x}\\ &= \frac{2\pi mgr^3}{3}\bar{x}\\ \therefore \quad \frac{2\pi mgr^3}{3}\bar{x} &= \frac{\pi mgr^4}{4}\\ \bar{x} &= \frac{3r}{8}\end{aligned}$$

Hence the centre of gravity is at $\left(\frac{3r}{8}, 0\right)$.

6. From the symmetry of the figure, $\bar{y} = 0$.

   Slice the figure into vertical discs of thickness $\delta x$. The radius of one disc is

$$y = \frac{xr}{h}$$

   Volume of one disc:

$$\begin{aligned}V_x &\approx \pi y^2 \delta x\\ &= \pi\left(\frac{xr}{h}\right)^2 \delta x\\ &= \frac{\pi r^2 x^2}{h^2}\delta x\end{aligned}$$

   Moment of one strip (where $m$ is mass per unit volume):

$$\begin{aligned}I_x &\approx mgx\left(\frac{\pi r^2 x^2}{h^2}\right)\delta x\\ &= \left(\frac{\pi mgr^2x^3}{h^2}\right)\delta x\end{aligned}$$

   Sum of moments:

$$\begin{aligned}I &= \int_0^h \frac{\pi mgr^2x^3}{h^2}\,\mathrm{d}x\\ &= \frac{\pi mgr^2}{h^2}\left[\frac{x^4}{4}\right]_0^h\\ &= \frac{\pi mgr^2h^2}{4}\end{aligned}$$

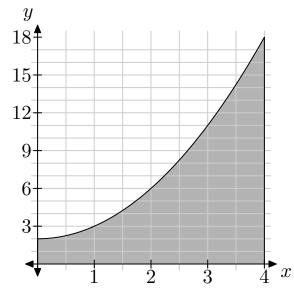
   Total volume:

$$V = \frac{\pi r^2 h}{3}$$

   Moment of sum:

$$\begin{aligned}I &= \frac{\pi r^2 h}{3}mg\bar{x}\\ &= \frac{\pi mgr^2h}{3}\bar{x}\\ \therefore \quad \frac{\pi mgr^2h}{3}\bar{y} &= \frac{\pi mgr^2h^2}{4}\\ \bar{y} &= \frac{3h}{4}\end{aligned}$$

   Hence the centre of gravity is at $\left(\frac{3h}{4}, 0\right)$.

7. 


   Divide the shape into vertical strips of width $\delta x$. Area of one strip:

$$A_x \approx \left(x^2 + 2\right)\delta x$$

   Moment of one strip about the $y$-axis:

$$I_x \approx mgx(x^2 + 2)\delta x$$

   Sum of moments:

$$\begin{aligned}I &= \int_0^4 mgx(x^2+2)\,\mathrm{d}x\\ &= mg\int_0^4 (x^3 + 2x)\,\mathrm{d}x\\ &= mg\left[\frac{x^4}{4} + x^2\right]_0^4\\ &= mg(64 + 16)\\ &= 80mg\end{aligned}$$

Total area:

$$\begin{aligned} A &= \int_0^4 (x^2+2)\,\mathrm{d}x \\ &= \left[\frac{x^3}{3}+2x\right]_0^4 \\ &= \frac{64}{3}+8 \\ &= \frac{88}{3} \end{aligned}$$

Moment of sum:

$$\begin{aligned} I &= \frac{88}{3}mg\bar{x} \\ \therefore \quad \frac{88}{3}mg\bar{x} &= 80mg \\ \bar{x} &= \frac{240}{88} \\ &= \frac{30}{11} \end{aligned}$$

Now consider the moment of each strip about the $x$-axis. The centre of gravity of each strip is $0.5y$ from the $x$-axis (as the hint states) so the moment of each strip is

$$\begin{aligned} I_x &\approx mg\frac{y}{2}(x^2+2)\delta x \\ &= \frac{mg}{2}(x^2+2)(x^2+2)\delta x \\ &= \frac{mg}{2}(x^4+4x^2+4)\delta x \end{aligned}$$

Sum of moments:

$$\begin{aligned} I &= \int_0^4 \frac{mg}{2}(x^4+4x^2+4)\,\mathrm{d}x \\ &= \frac{mg}{2}\left[\frac{x^5}{5}+\frac{4x^3}{3}+4x\right]_0^4 \\ &= \frac{mg}{2}\left(\frac{1024}{5}+\frac{256}{3}+16\right) \\ &= \frac{2296mg}{15} \end{aligned}$$

Moment of sum:

$$\begin{aligned} I &= mgA\bar{y} \\ &= \frac{88mg}{3}\bar{y} \therefore \quad \frac{88mg}{3}\bar{y} = \frac{2296mg}{15} \\ \bar{y} &= \frac{287}{55} \end{aligned}$$

The centre of gravity is $(\bar{x},\bar{y}) = \left(\frac{30}{11}, \frac{287}{55}\right)$.

## Miscellaneous Exercise 7

1-2 What working there is for this question is trivial enough (I hope) to not need further elucidation here.

3. (a)

$$\begin{aligned} 2|x|-3 &< 0 \\ 2|x| &< 3 \\ |x| &< \frac{3}{2} \\ x &< \frac{3}{2} \\ \text{and } -x &< -\frac{3}{2} \\ \therefore \quad -\frac{3}{2} < x &< \frac{3}{2} \end{aligned}$$

(b) First consider the case $2x-3 \geq 0$ so $|2x-3| = 2x-3$. It also follows that $x \geq \frac{3}{2}$ so $|x| = x$. The inequality then gives

$$\begin{aligned} 2|x|-3 &< |2x-3| \\ 2x-3 &< 2x-3 \end{aligned}$$

This is clearly a contradiction and so we must exclude $x \geq \frac{3}{2}$ from the solution.

Next consider the case $2x-3 < 0$ and $x \geq 0$. This gives $|2x-3| = -(2x-3)$ and $|x| = x$. Hence

$$\begin{aligned} 2|x|-3 &< |2x-3| \\ 2x-3 &< -(2x-3) \\ 2x-3 &< -2x+3 \\ 4x &< 6 \\ x &< \frac{3}{2} \end{aligned}$$

Finally consider the case $2x-3 < 0$ and $x < 0$. This gives $|2x-3| = -(2x-3)$ and $|x| = -x$. Hence

$$\begin{aligned} 2|x|-3 &< |2x-3| \\ -2x-3 &< -(2x-3) \\ -2x-3 &< -2x+3 \\ -3 &< 3 \end{aligned}$$

This is true regardless of the value of $x$. Combining these results gives us the solution $x < \frac{3}{2}$.

4. (a) Consider the three possible cases: $x > 3$, $1 < x \leq 3$ and $x \leq 1$.
For $x > 3$, $x - 3 > 0$ so $|x-3| = x - 3$ and $x - 1 > 0$ so $|x - 1| = x - 1$;

$$\begin{aligned} |x-3| + |x-1| &= 4 \\ x - 3 + x - 1 &= 4 \\ 2x - 4 &= 4 \\ x &= 4 \end{aligned}$$

For $1 < x \leq 3$, $x-3 \leq 0$ so $|x-3| = -x+3$ and $x - 1 > 0$ so $|x-1| = x - 1$;

$$\begin{aligned} |x-3| + |x-1| &= 4 \\ -x + 3 + x - 1 &= 4 \\ 2 &= 4 \end{aligned}$$

No solution.
For $x \leq 1$, $x - 3 < 0$ so $|x - 3| = -x + 3$ and $x - 1 \leq 0$ so $|x - 1| = -x + 1$;

$$\begin{aligned} |x-3| + |x-1| &= 4 \\ -x + 3 - x + 1 &= 4 \\ -2x + 4 &= 4 \\ x &= 0 \end{aligned}$$

The two solutions are $x = 0$ and $x = 4$.

(b) Again, consider the three possible cases: $x > 3$, $1 < x \leq 3$ and $x \leq 1$.
For $x > 3$, $x - 3 > 0$ so $|x - 3| = x - 3$ and $x - 1 > 0$ so $|x - 1| = x - 1$;

$$\begin{aligned} |x-3| + |x-1| &= 2 \\ x - 3 + x - 1 &= 2 \\ 2x - 4 &= 2 \\ x &= 3 \end{aligned}$$

Strictly speaking this is not in the part of the domain we are considering (but $x = 3$ *is* a solution as shown by the next part.)
For $1 < x \leq 3$, $x-3 \leq 0$ so $|x-3| = -x+3$ and $x - 1 > 0$ so $|x - 1| = x - 1$;

$$\begin{aligned} |x-3| + |x-1| &= 2 \\ -x + 3 + x - 1 &= 2 \\ 2 &= 2 \end{aligned}$$

This is true for all $1 < x \leq 3$
For $x \leq 1$, $x - 3 < 0$ so $|x - 3| = -x + 3$ and $x - 1 \leq 0$ so $|x - 1| = -x + 1$;

$$\begin{aligned} |x-3| + |x-1| &= 2 \\ -x + 3 - x + 1 &= 2 \\ -2x + 4 &= 2 \\ -2x &= -2 \\ x &= 1 \end{aligned}$$

The two solution is $1 \leq x \leq 3$.

5.
$$\begin{aligned} \frac{\delta y}{\delta x} &\approx \frac{\mathrm{d}y}{\mathrm{d}x} \\ &= \sin x + x\cos x \\ \delta y &\approx \delta x(\sin x + x\cos x) \\ &= 0.05(\sin 2.5 + 2.5\cos 2.5) \\ &= -0.07 \end{aligned}$$

6.
$$\begin{aligned} A &= ACC^{-1} \\ &= \begin{bmatrix} 2 & -4 \\ 11 & 3 \end{bmatrix}\begin{bmatrix} 0.2 & 0.1 \\ -0.4 & 0.3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \\ B &= C^{-1}CB \\ &= \begin{bmatrix} 0.2 & 0.1 \\ -0.4 & 0.3 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 6 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

7.
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 1 & 1 \end{bmatrix}$$
$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}^{-1} &= \frac{1}{-3+2}\begin{bmatrix} -3 & -2 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix} \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 4 & 9 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \end{aligned}$$

$a = 3, b = -1, c = 2, d = 1$

8.
$$\begin{aligned} AB &= \begin{bmatrix} 5 & 2 \\ x & y \end{bmatrix}\begin{bmatrix} 1 & 6 \\ -3 & z \end{bmatrix} \\ &= \begin{bmatrix} -1 & 30+2z \\ x-3y & 6x+yz \end{bmatrix} \\ BA &= \begin{bmatrix} 1 & 6 \\ -3 & z \end{bmatrix}\begin{bmatrix} 5 & 2 \\ x & y \end{bmatrix} \\ &= \begin{bmatrix} 5+6x & 2+6y \\ -15+xz & -6+yz \end{bmatrix} \\ AB &= BA \end{aligned}$$
$$\begin{bmatrix} -1 & 30+2z \\ x-3y & 6x+yz \end{bmatrix} = \begin{bmatrix} 5+6x & 2+6y \\ -15+xz & -6+yz \end{bmatrix}$$
$$\begin{aligned} 5 + 6x &= -1 \\ x &= -1 \end{aligned}$$
$$\begin{bmatrix} -1 & 30+2z \\ -1-3y & -6+yz \end{bmatrix} = \begin{bmatrix} -1 & 2+6y \\ -15-z & -6+yz \end{bmatrix}$$
$$\begin{aligned} 30 + 2z &= 2 + 6y \\ 2z &= -28 + 6y \\ z &= 3y - 14 \\ \text{check:}\ -1 - 3y &= -15 - z \\ 14 - 3y &= -z \\ z &= 3y - 14 \end{aligned}$$

9. No working required.

10.
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x\ln x + x^2\frac{1}{x}\\ &= 2x\ln x + x\\ &= x(2\ln x + 1)\end{aligned}$$

11-12 No working required.

13. (a)
$$\begin{aligned}\mathrm{e}^{\frac{\pi \mathrm{i}}{2}} &= \cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\\ &= 0 + \mathrm{i}\\ &= (0, 1)\end{aligned}$$

(b)
$$\begin{aligned}\mathrm{e}^{(2-0.5\pi \mathrm{i})} &= (\mathrm{e}^2)(\mathrm{e}^{0.5\pi\mathrm{i}})\\ &= \mathrm{e}^2\left(\cos(-0.5\pi) + \mathrm{i}\sin(-0.5\pi)\right)\\ &= \mathrm{e}^2(0 - \mathrm{i})\\ &= (0, -\mathrm{e}^2)\end{aligned}$$

(c)
$$\begin{aligned}\mathrm{e}^2 + 4\mathrm{e}^{\frac{\pi\mathrm{i}}{3}} &= \mathrm{e}^2 + 4\cos\frac{\pi}{3} + 4\mathrm{i}\sin\frac{\pi}{3}\\ &= \mathrm{e}^2 + 2 + 2\sqrt{3}\mathrm{i}\\ &= (\mathrm{e}^2 + 2, 2\sqrt{3})\end{aligned}$$

14.
$$\begin{aligned}\int\frac{dA}{A} &= \int -0.02\,\mathrm{d}t\\ \ln A &= -0.02t + c\\ A &= A_0\mathrm{e}^{-0.02t}\\ \frac{A}{A_0} &= \mathrm{e}^{-0.02t}\\ \mathrm{e}^{-0.02t_h} &= 0.5\\ -0.02t_h &= \ln(0.5)\\ t_h &= \frac{\ln(0.5)}{-0.02}\\ &\approx 34.66 \text{ years}\end{aligned}$$

15. (a)
$$\begin{aligned}u &= 3x^2 - 5\\ du &= 6x\,\mathrm{d}x\\ \int x(3x^2-5)^7\,\mathrm{d}x &= \int u^7\,\frac{du}{6}\\ &= \frac{u^8}{8\times 6} + c\\ &= \frac{(3x^2-5)^8}{48} + c\end{aligned}$$

(b)
$$\begin{aligned}u &= x - 5\\ x &= u + 5\\ du &= dx\\ \int x(x-5)^7\,\mathrm{d}u &= \int (u+5)(u)^7\,\mathrm{d}u\\ &= \int (u^8 + 5u^7)\,\mathrm{d}u\\ &= \frac{u^9}{9} + \frac{5u^8}{8} + c\\ &= \frac{u^8}{72}(8u + 45) + c\\ &= \frac{(x-5)^8}{72}(8(x-5) + 45) + c\\ &= \frac{(x-5)^8}{72}(8x - 40 + 45) + c\\ &= \frac{(x-5)^8}{72}(8x - 5) + c\end{aligned}$$

(c)
$$\begin{aligned}u &= x^2 - 3\\ du &= 2\,\mathrm{d}x\\ \int\frac{8x}{\sqrt{x^2-3}}dx &= \int\frac{4}{\sqrt{u}}du\\ &= \frac{4\sqrt{u}}{0.5} + c\\ &= 8\sqrt{x^2-3} + c\end{aligned}$$

(d)
$$\begin{aligned}u &= 5x - 2\\ x &= \frac{u+2}{5}\\ du &= 5\,\mathrm{d}x\end{aligned}$$
$$\begin{aligned}&\int 10x\sqrt{5x-2}\,\mathrm{d}x\\ &= \int 2(u+2)\sqrt{u}\frac{du}{5}\\ &= \frac{2}{5}\int (u^{\frac{3}{2}} + 2u^{\frac{1}{2}})\,\mathrm{d}u\\ &= \frac{2}{5}\left(\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}(2u^{\frac{3}{2}})\right) + c\\ &= \frac{4}{75}u^{\frac{3}{2}}(3u + 10) + c\\ &= \frac{4}{75}(5x-2)^{\frac{3}{2}}(3(5x-2) + 10) + c\\ &= \frac{4}{75}(5x-2)^{\frac{3}{2}}(15x - 6 + 10) + c\\ &= \frac{4}{75}(5x-2)^{\frac{3}{2}}(15x + 4) + c\end{aligned}$$

(e)
$$\begin{aligned}u &= x^2 - 5\\ du &= 2x\,\mathrm{d}x\\ \int 8x\sin(x^2-5)\,\mathrm{d}x &= \int 4\sin u\,\mathrm{d}u\\ &= -4\cos u + c\\ &= -4\cos(x^2-5) + c\end{aligned}$$

(f)
$$\begin{aligned}u &= 1 + \mathrm{e}^x\\ \mathrm{d}u &= \mathrm{e}^x\,\mathrm{d}x\\ \int \mathrm{e}^x(1+\mathrm{e}^x)^4\,\mathrm{d}x &= \int u^4\,\mathrm{d}u\\ &= \frac{u^5}{5} + c\\ &= \frac{1+\mathrm{e}^x}{5} + c\end{aligned}$$

(g)
$$\begin{aligned}
u &= x-3\\
x &= u+3\\
du &= dx\\
\int \frac{4x}{\sqrt{x-3}}dx &= \int \frac{4(u+3)}{\sqrt{u}}du\\
&= \int (4u^{0.5}+12u^{-0.5})du\\
&= \frac{4u^{1.5}}{1.5}+\frac{12u^{0.5}}{0.5}+c\\
&= \frac{8u^{1.5}}{3}+24u^{0.5}+c\\
&= \frac{8u^{1.5}+72u^{0.5}}{3}+c\\
&= \frac{8}{3}\sqrt{u}(u+9)+c\\
&= \frac{8}{3}\sqrt{x-3}(x-3+9)+c\\
&= \frac{8}{3}\sqrt{x-3}(x+6)+c
\end{aligned}$$

(h)
$$\begin{aligned}
u &= x+2\\
x &= u-2\\
2x &= 2u-4\\
2x+1 &= 2u-3\\
du &= dx\\
\int \frac{2x+1}{(x+2)^3}dx &= \int \frac{2u-3}{u^3}dx\\
&= \int (2u^{-2}-3u^{-3})dx\\
&= \frac{2u^{-1}}{-1}-\frac{3u^{-2}}{-2}+c\\
&= -2u^{-1}+1.5u^{-2}+c\\
&= \frac{1.5-2u}{u^2}+c\\
&= \frac{1.5-2(x+2)}{(x+2)^2}+c\\
&= \frac{1.5-2x-4}{(x+2)^2}+c\\
&= \frac{-2x-2.5}{(x+2)^2}+c\\
&= -\frac{4x-5}{2(x+2)^2}+c
\end{aligned}$$

(i)
$$\begin{aligned}
u &= 2^x\\
&= e^{\ln 2^x}\\
&= e^{x\ln 2}\\
du &= \ln(2)e^{x\ln 2}\,\mathrm{d}x\\
&= \ln(2)\,2^x\,\mathrm{d}x\\
\int 2^x\,\mathrm{d}x &= \int \frac{du}{\ln 2}\\
&= \frac{u}{\ln 2}+c\\
&= \frac{2^x}{\ln 2}+c
\end{aligned}$$

(j)
$$\begin{aligned}
u &= 5^{x+1}\\
du &= \ln(5)\,5^{x+1}\,\mathrm{d}x\\
\int 5^{x+1}\,\mathrm{d}x &= \int \frac{du}{\ln 5}\\
&= \frac{u}{\ln 5}+c\\
&= \frac{5^{x+1}}{\ln 5}+c
\end{aligned}$$

16.
$$\begin{aligned}
\frac{\mathrm{d}M}{\mathrm{d}t} &= -kM\\
\int \frac{dM}{M} &= \int k\,\mathrm{d}t\\
\ln M &= kt+c\\
M &= M_0\mathrm{e}^{kt}\\
0.5 &= \mathrm{e}^{1600k}\\
1600k &= -\ln 2\\
k &= \frac{-\ln 2}{1600}\\
&\approx -0.000433\\
M_0 &= 5000\,\mathrm{g}\\
M &= 5000\mathrm{e}^{-0.000433t}\ \mathrm{g}
\end{aligned}$$

After 100 years,

$$\begin{aligned}
M &= 5000\mathrm{e}^{-0.000433\times 100}\\
&= 5000\mathrm{e}^{-0.0433}\\
&\approx 4\,788\,\mathrm{g}
\end{aligned}$$

17. (a)
$$\begin{aligned}
\int_0^{\frac{2\pi}{3}} \sin x\,\mathrm{d}x &= [-\cos x]_0^{\frac{2\pi}{3}}\\
&= \frac{1}{2}+1\\
&= 1.5
\end{aligned}$$

(b)
$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{0} \sin 2x\,\mathrm{d}x &= \left[\frac{-\cos 2x}{2}\right]_{-\frac{\pi}{2}}^{0}\\
&= -\frac{1}{2}-\frac{1}{2}\\
&= -1
\end{aligned}$$

(c)
$$\begin{aligned}
&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\,\mathrm{d}x\\
&= -\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-2\sin^2 x-1)\,\mathrm{d}x\\
&= -\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos 2x-1)\,\mathrm{d}x\\
&= -\frac{1}{2}\left[\frac{\sin 2x}{2}-x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\\
&= -\frac{1}{4}\left[\sin(2x)-2x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\\
&= -\frac{1}{4}\left((\sin(\pi)-\pi)-(\sin(-\pi)+\pi)\right)\\
&= -\frac{1}{4}(-\pi-\pi)\\
&= \frac{\pi}{2}
\end{aligned}$$

18. (a)
$$\begin{aligned}\int_1^3 x^a \,\mathrm{d}x &= \left[\frac{x^{a+1}}{a+1}\right]_1^3\\ &= \frac{3^{a+1}}{a+1} - \frac{1^{a+1}}{a+1}\\ &= \frac{(3)3^a - 1}{a+1}\end{aligned}$$

(b)
$$\begin{aligned}\int_2^a 6x^2 \,\mathrm{d}x &= \left[2x^3\right]_2^a\\ &= 2a^3 - 16\end{aligned}$$

(c)
$$\begin{aligned}\text{Let } u &= \sin 2x\\ \mathrm{d}u &= 2\cos 2x \,\mathrm{d}x\\ \int_0^{\frac{\pi}{12}} \sin^a 2x \cos 2x \,\mathrm{d}x &= \int_0^{\sin\frac{\pi}{6}} u^a \frac{\mathrm{d}u}{2}\\ &= \int_0^{\frac{1}{2}} \frac{u^a}{2}\,\mathrm{d}u\\ &= \left[\frac{u^{a+1}}{2(a+1)}\right]_0^{\frac{1}{2}}\\ &= \left(\frac{\left(\frac{1}{2}\right)^{a+1}}{2(a+1)}\right)\\ &= \frac{1}{2(a+1)2^{a+1}}\\ &= \frac{1}{(a+1)2^{a+2}}\end{aligned}$$

19. (a)
$$\begin{aligned}d &= \int_0^{0.5} |v| \,\mathrm{d}t\\ &= \int_0^{0.5} |5\cos 2t| \,\mathrm{d}t\\ 5\cos 2t &> 0 \;\forall 0 \le t \le 0.5\\ \therefore \quad d &= \int_0^{0.5} 5\cos(2t)\,\mathrm{d}t\\ &= \left[\frac{5}{2}\sin 2t\right]_0^{0.5}\\ &= 2.5\sin 1\\ &\approx 2.10\text{m}\end{aligned}$$

(b)
$$\begin{aligned}d &= \int_0^1 |5\cos 2t| \,\mathrm{d}t\\ &= \int_0^{\frac{\pi}{4}} 5\cos(2t)\,\mathrm{d}t - \int_{\frac{\pi}{4}}^1 5\cos(2t)\,\mathrm{d}t\\ &= \left[\frac{5}{2}\sin 2t\right]_0^{\frac{\pi}{4}} - \left[\frac{5}{2}\sin 2t\right]_{\frac{\pi}{4}}^1\\ &= (2.5 - 0) - (2.5\sin 2 - 2.5)\\ &= 5 - 2.5\sin 2\\ &\approx 2.73\text{m}\end{aligned}$$

20. To prove:

$$\sin^3\theta = \frac{3\sin\theta - \sin(3\theta)}{4}$$

Proof:

$$\begin{aligned}\sin(3\theta) &= \mathrm{Im}(\mathrm{cis}(3\theta))\\ &= \mathrm{Im}(\mathrm{cis}^3\theta)\\ &= \mathrm{Im}(\cos^3\theta + 3\mathrm{i}\sin\theta\cos^2\theta\\ &\qquad - 3\sin^2\theta\cos\theta - \mathrm{i}\sin^3\theta)\\ &= 3\sin\theta\cos^2\theta - \sin^3\theta\\ &= 3\sin\theta(1-\sin^2\theta) - \sin^3\theta\\ &= 3\sin\theta - 3\sin^3\theta - \sin^3\theta\\ &= 3\sin\theta - 4\sin^3\theta\\ \text{RHS} &= \frac{3\sin\theta - \sin(3\theta)}{4}\\ &= \frac{3\sin\theta - (3\sin\theta - 4\sin^3\theta)}{4}\\ &= \frac{4\sin^3\theta}{4}\\ &= \sin^3\theta\\ &= \text{LHS}\end{aligned}$$

□

$$\begin{aligned}A &= \int_0^{\frac{\pi}{2}} |\sin^3\theta| \,\mathrm{d}\theta\\ &= \int_0^{\frac{\pi}{2}} \sin^3\theta \,\mathrm{d}\theta\\ &= \int_0^{\frac{\pi}{2}} \frac{3\sin\theta - \sin(3\theta)}{4}\,\mathrm{d}\theta\\ &= \left[\frac{-3\cos\theta + \frac{1}{3}\cos(3\theta)}{4}\right]_0^{\frac{\pi}{2}}\\ &= \left[\frac{\cos(3\theta) - 9\cos\theta}{12}\right]_0^{\frac{\pi}{2}}\\ &= \left(\frac{\cos\frac{3\pi}{2} - 9\cos\frac{\pi}{2}}{12}\right)\\ &\qquad - \left(\frac{\cos 0 - 9\cos 0}{12}\right)\\ &= 0 - \left(\frac{-8}{12}\right)\\ &= \frac{2}{3}\text{ units}^2\end{aligned}$$

$$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} \sin^3\theta\,\mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{2}} \sin\theta(1-\cos^2\theta)\,\mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{2}} (\sin\theta - \cos^2\theta\sin\theta)\,\mathrm{d}\theta \\ &= \left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^{\frac{\pi}{2}} \\ &= \left(-\cos\frac{\pi}{2} + \frac{\cos^3\frac{\pi}{2}}{3}\right) \\ &\quad - \left(-\cos 0 + \frac{\cos^3 0}{3}\right) \\ &= 0 - \left(-1 + \frac{1}{3}\right) \\ &= \frac{2}{3}\text{ units}^2 \end{aligned}$$

21. (a)

$$\begin{array}{lc|cccc} & & \multicolumn{4}{c}{\text{To (next location)}} \\ & & A & B & C & D \\ \text{From} & A & 0 & 1/2 & 0 & 1/2 \\ \text{(present} & B & 1/3 & 0 & 1/3 & 1/3 \\ \text{loca-} & C & 0 & 1/2 & 0 & 1/2 \\ \text{tion)} & D & 1/3 & 1/3 & 1/3 & 0 \end{array}$$

(b) 17 seconds is after 3 transitions, so

$$\begin{bmatrix} 0 & 0 & 200 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}^3 = \begin{bmatrix} 22 & 78 & 22 & 78 \end{bmatrix}$$

(c) $$\begin{bmatrix} 0 & 0 & 0 & 200 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}^3 = \begin{bmatrix} 52 & 52 & 52 & 44 \end{bmatrix}$$

(d) Intuitively I would expect 50% more people at B and D than at A and C, so about 40 at each of A and C and 60 at each of B and D. This would be my intuitive guess regardless of initial positions.

$$\begin{bmatrix} 200 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}^{15} = \begin{bmatrix} 40 & 60 & 40 & 60 \end{bmatrix}$$

check that this has in fact reached the long term average:

$$\begin{bmatrix} 200 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}^{16} = \begin{bmatrix} 40 & 60 & 40 & 60 \end{bmatrix}$$

(e) $$\begin{bmatrix} 0 & 200 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}^{15} = \begin{bmatrix} 40 & 60 & 40 & 60 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 200 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}^{16} = \begin{bmatrix} 40 & 60 & 40 & 60 \end{bmatrix}$$

(f) $$\begin{bmatrix} 50 & 50 & 50 & 50 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}^{15} = \begin{bmatrix} 40 & 60 & 40 & 60 \end{bmatrix}$$

$$\begin{bmatrix} 50 & 50 & 50 & 50 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}^{16} = \begin{bmatrix} 40 & 60 & 40 & 60 \end{bmatrix}$$

22.

$$\begin{aligned} \frac{\mathrm{d}T}{\mathrm{d}t} &= kT \\ \int \frac{\mathrm{d}T}{T} &= \int k\,\mathrm{d}t \\ \ln T &= kt + c \\ T &= T_0\mathrm{e}^{kt} \\ 27.7 - 22 &= (28.6 - 22)\mathrm{e}^{1k} \\ k &= \ln\frac{27.7-22}{28.6-22} \\ &\approx -0.147 \\ 28.6 - 22 &= (37-22)\mathrm{e}^{-0.147t} \\ -0.147t &= \ln\frac{28.6-22}{37-22} \\ t &= \frac{\ln\frac{28.6-22}{37-22}}{-0.147} \\ &= 5.6 \end{aligned}$$

5.6 hours before 1.30pm is 7:54am (although it's unlikely that such precision is realistic—8am is a more sensible estimate.)

# Chapter 8

## Exercise 8A

1. No working required.

2. (a) $k^2 = 4$ so $k = 2$ and $T = \frac{2\pi}{2} = \pi$.
   (b) $k^2 = 1$ so $k = 1$ and $T = \frac{2\pi}{1} = 2\pi$.
   (c) $k^2 = 25$ so $k = 5$ and $T = \frac{2\pi}{5} = 0.4\pi$.

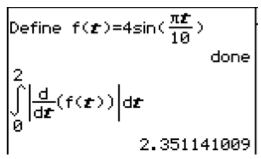
3. (a) $T = \frac{2\pi}{k} = 4\pi$ so $k = 0.5$ and $x = \sin 0.5t$.
   (b) $x = -\sin 0.5t$
   (c) $T = \frac{2\pi}{k} = \pi$ so $k = 2$ and $x = 3\sin 2t$.
   (d) $T = \frac{2\pi}{k} = 2$ so $k = \pi$ and $x = -0.5\sin \pi t$.

4. (a) $T = \frac{2\pi}{k} = \pi$ so $k = 2$ and $x = 2\cos 2t$.
   (b) $T = \frac{2\pi}{k} = 0.5\pi$ so $k = 4$ and $x = 1.5\cos 4t$.
   (c) $T = \frac{2\pi}{k} = 0.5$ so $k = 4\pi$ and $x = 0.5\cos 4\pi t$.

5. (a) $T = \frac{2\pi}{k} = \pi$ so $k = 2$ and $x = 2.5\sin 2t$ or $x = -2.5\sin 2t$ (depending on the direction of the motion at time $t = 0$).
   (b)
$$\begin{aligned} v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 5\cos 2t \\ &= 5\cos\frac{\pi}{3} \\ &= 2.5\text{ms}^{-1} \end{aligned}$$

6. (a)
$$\begin{aligned} x &= 5\cos 5t + 3\sin 5t \\ &= r\sin(5t + \alpha) \end{aligned}$$
where
$$\begin{aligned} r\sin\alpha &= 5 \\ r\cos\alpha &= 3 \\ \therefore\quad r^2 &= 5^2 + 3^2 \end{aligned}$$
Hence the amplitude is $\sqrt{34}$m. (We could proceed to find the phase angle $\alpha$ but this is not requested by the question.)
Period $T = \frac{2\pi}{5} = 0.4\pi$s.
   (b) Amplitude is $\sqrt{3^2 + 7^2} = \sqrt{58}$m. (You can use the approach in part (a) above, but you should probably remember the general result for questions like this.)
Period $T = \frac{2\pi}{2} = \pi$s.

7. (a) To prove: $\ddot{x} = -k^2 x$
Proof:
$$\begin{aligned} \dot{x} &= \frac{\mathrm{d}}{\mathrm{d}t} 4\sin\frac{\pi t}{10} \\ &= \frac{2\pi}{5}\cos\frac{\pi t}{10} \\ \ddot{x} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{2\pi}{5}\cos\frac{\pi t}{10} \\ &= -\frac{\pi^2}{25}\sin\frac{\pi t}{10} \\ &= -\frac{\pi^2}{10^2}\left(4\sin\frac{\pi t}{10}\right) \\ &= -\left(\frac{\pi}{10}\right)^2 x \end{aligned}$$
Taking $k = \frac{\pi}{10}$ this gives $\ddot{x} = -k^2 x$. □
(In my opinion the wording of this question is a little unclear. Since it might well be reasonable to define simple harmonic motion as $x = a\sin(kt + \phi)$, the proof could be so trivial as to be non-existant. In order to proceed, I have taken the question to mean that we are required to prove that the motion satisfies the differential equation definition of SHM.)
   (b) The period of the motion is $T = 2\pi \times \frac{10}{\pi} = 20$s.
Amplitude is 4m.
   (c) In the first two seconds the movement is all in the same direction and the distance moved is
$$\begin{aligned} d &= x(2) - x(0) \\ &= 4\sin\frac{\pi}{5} \\ &\approx 2.35\text{m} \end{aligned}$$
If using technology, it's probably simpler to take a definite integral of the absolute value of the velocity over the given interval:




With this approach there is no need to first analyse whether the object changes direction during the interval under consideration: the absolute value takes care of that. Be warned, however, that handheld technology may take longer than a few seconds to evaluate this.

8. (a) To prove: $\ddot{x} = -k^2 x$
Proof:
$$\begin{aligned} \dot{x} &= \frac{\mathrm{d}}{\mathrm{d}t} 2\sin\frac{\pi t}{3} \\ &= \frac{2\pi}{3}\cos\frac{\pi t}{3} \\ \ddot{x} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{2\pi}{3}\cos\frac{\pi t}{3} \\ &= -\frac{2\pi^2}{9}\sin\frac{\pi t}{3} \\ &= -\frac{\pi^2}{3^2}\left(2\sin\frac{\pi t}{3}\right) \\ &= -\left(\frac{\pi}{3}\right)^2 x \end{aligned}$$
Taking $k = \frac{\pi}{3}$ this gives $\ddot{x} = -k^2 x$. □

(b) The period of the motion is $T = 2\pi \times \frac{3}{\pi} = 6$s.
Amplitude is 2m.

(c) In the first two seconds the movement is not all in the same direction so the distance moved must be determined in two parts.
From $t = 0$ to $t = \frac{T}{4} = 1.5$ seconds the body moves through its amplitude: 2m.
From $t = 1.5$ to $t = 2$ seconds the body moves back to $x = 2\sin\frac{2\pi}{3} = \sqrt{3}$, thus moving through a further distance of $2 - \sqrt{3}$m.
Hence the total distance moved is

$$\begin{aligned} d &= (2) + (2 - \sqrt{3}) \\ &= (4 - \sqrt{3})\,\text{m} \end{aligned}$$

(See the note to the previous question about using technology.)

9. (a) To prove: $\ddot{x} = -k^2 x$
Proof:

$$\begin{aligned} \dot{x} &= \frac{\mathrm{d}}{\mathrm{d}t} 3\sin\left(2t + \frac{\pi}{6}\right) \\ &= 6\cos\left(2t + \frac{\pi}{6}\right) \\ \ddot{x} &= \frac{\mathrm{d}}{\mathrm{d}t} 6\cos\left(2t + \frac{\pi}{6}\right) \\ &= -12\sin\left(2t + \frac{\pi}{6}\right) \\ &= -4\left(3\sin\left(2t + \frac{\pi}{6}\right)\right) \\ &= -2^2 x \end{aligned}$$

Taking $k = 2$ this gives $\ddot{x} = -k^2 x$. □

(b) The period of the motion is $T = \frac{2\pi}{2} = \pi$s.
Amplitude is 3m.

(c) The body first reaches maximum displacement when

$$\begin{aligned} 3\sin\left(2t + \frac{\pi}{6}\right) &= 3 \\ 2t + \frac{\pi}{6} &= \frac{\pi}{2} \\ 2t &= \frac{\pi}{3} \\ t &= \frac{\pi}{6} \end{aligned}$$

From $t = 0$ to $t = \frac{\pi}{6}$ seconds the body moves

$$\begin{aligned} d &= 3 - 3\sin\left(2(0) + \frac{\pi}{6}\right) \\ &= 1.5\text{m} \end{aligned}$$

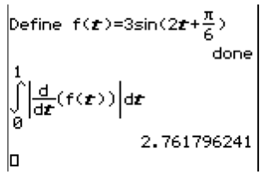
From $t = \frac{\pi}{6}$ to $t = 1$ seconds the body moves a further

$$\begin{aligned} d &= 3 - 3\sin\left(2(1) + \frac{\pi}{6}\right) \\ &\approx 1.26\text{m (2d.p.)} \end{aligned}$$

Hence the total distance moved is

$$\begin{aligned} d &= 1.5 + 1.26 \\ &= 2.76\text{m (2d.p.)} \end{aligned}$$

If using technology:




10. (a)

$$\begin{aligned} x &= a\sin(kt + \alpha) \\ a &= 4 \\ \frac{2\pi}{k} &= 2 \\ k &= \pi \\ 2 &= 4\sin(\pi(0) + \alpha) \\ \sin\alpha &= \frac{1}{2} \\ \alpha &= \frac{\pi}{6} \\ \text{or} \quad \alpha &= \frac{5\pi}{6} \\ v &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 4\pi\cos(\pi t + \alpha) \end{aligned}$$

When $t = 0$,

$$\begin{aligned} v &= 4\pi\cos\alpha \\ \therefore \quad \alpha &= \frac{5\pi}{6} \text{ (to make } v \text{ negative)} \\ x &= 4\sin\left(\pi t + \frac{5\pi}{6}\right) \end{aligned}$$

(b)

$$\begin{aligned} v &= 4\pi\cos\left(\pi t + \frac{5\pi}{6}\right) \\ &= 4\pi\cos\left(\frac{\pi}{6} + \frac{5\pi}{6}\right) \\ &= 4\pi\cos\pi \\ &= -4\pi\,\text{ms}^{-1} \\ \therefore \text{speed} &= 4\pi\,\text{ms}^{-1} \end{aligned}$$

11. (a)
$$\begin{aligned}
x &= a\sin(kt+\alpha)\\
a &= 2\\
\frac{2\pi}{k} &= \frac{2\pi}{5}\\
k &= 5\\
\sqrt{2} &= 2\sin(5(0)+\alpha)\\
\sin\alpha &= \frac{\sqrt{2}}{2}\\
\alpha &= \frac{\pi}{4}\\
\text{or}\quad \alpha &= \frac{3\pi}{4}\\
v &= \frac{\mathrm{d}x}{\mathrm{d}t}\\
&= 10\cos(5t+\alpha)
\end{aligned}$$
When $t=0$,
$$\begin{aligned}
v &= 10\cos\alpha\\
\therefore\quad \alpha &= \frac{\pi}{4}\ \text{(to make } v \text{ positive)}\\
x &= 2\sin\left(5t+\frac{\pi}{4}\right)
\end{aligned}$$

(b) $v = 10\cos\left(5t+\frac{\pi}{4}\right)$ which has an amplitude of $10\mathrm{ms}^{-1}$ so the greatest speed is $10\mathrm{ms}^{-1}$.

(c)
$$\begin{aligned}
a &= \frac{\mathrm{d}v}{\mathrm{d}t}\\
&= -50\sin\left(5t+\frac{\pi}{4}\right)
\end{aligned}$$
This has an amplitude of $50\mathrm{ms}^{-2}$ so the maximum accelleration is $50\mathrm{ms}^{-2}$.

12.
$$\begin{aligned}
\ddot{x} &= -k^2x\\
&= -4x\\
\therefore\quad k &= 2\\
x &= 0.6\sin 2t
\end{aligned}$$

(a)
$$\begin{aligned}
x &= 0.6\sin\frac{2\pi}{6}\\
&= 0.6\sin\frac{\pi}{3}\\
&= 0.3\sqrt{3}\,\mathrm{m}
\end{aligned}$$

(b)
$$\begin{aligned}
x &= 0.6\sin\frac{2\pi}{3}\\
&= -0.3\sqrt{3}\,\mathrm{m}
\end{aligned}$$

(c)
$$\begin{aligned}
|x| &= 0.3\\
x &= \pm 0.3\\
0.6\sin 2t &= \pm 0.3\\
\sin 2t &= \pm 0.5\\
2t &\in \left\{\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6},\dots\right\}\\
t &\in \left\{\frac{\pi}{12},\frac{5\pi}{12},\frac{7\pi}{12},\frac{11\pi}{12},\dots\right\}
\end{aligned}$$

i. $t=\frac{\pi}{12}\mathrm{s}$

ii. $t=\frac{5\pi}{12}\mathrm{s}$

iii. $t=\frac{7\pi}{12}\mathrm{s}$

13.
$$\begin{aligned}
\ddot{x} &= -k^2x\\
&= -\pi^2x\\
\therefore\quad k &= \pi\\
x &= -3\sin\pi t
\end{aligned}$$

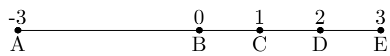
(a)
$$\begin{aligned}
x &= -3\sin\frac{\pi}{3}\\
&= -\frac{3\sqrt{3}}{2}\,\mathrm{m}
\end{aligned}$$

(b)
$$\begin{aligned}
v &= \frac{\mathrm{d}x}{\mathrm{d}t}\\
&= -3\pi\cos\pi t
\end{aligned}$$
when $t=\frac{1}{3}$,
$$\begin{aligned}
v &= -3\pi\cos\frac{\pi}{3}\\
&= -\frac{3\pi}{2}\,\mathrm{ms}^{-1}
\end{aligned}$$

(c) speed$=|v|=\frac{3\pi}{2}\,\mathrm{ms}^{-1}$

(d)
$$\begin{aligned}
|v| &= \frac{3\pi}{2}\\
v &= \pm\frac{3\pi}{2}\\
-3\pi\cos\pi t &= \pm\frac{3\pi}{2}\\
\cos\pi t &= \pm 0.5\\
\pi t &\in \left\{\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3},\dots\right\}\\
t &\in \left\{\frac{1}{3},\frac{2}{3},\frac{4}{3},\frac{5}{3},\dots\right\}
\end{aligned}$$
The body next has the same speed it had at $t=\frac{1}{3}\mathrm{s}$ when $t=\frac{2}{3}\mathrm{s}$.

14.




Without loss of generality, suppose that the particle is at point A when $t=0$. Then
$$\begin{aligned}
x &= -3\cos kt\\
\frac{2\pi}{k} &= \pi\\
\therefore\quad k &= 2\\
\therefore\quad x &= -3\cos 2t
\end{aligned}$$

(a)
$$\begin{aligned}
x &= 1\\
-3\cos 2t &= 1\\
\cos 2t &= -\frac{1}{3}\\
t &= \frac{\cos^{-1}-\frac{1}{3}}{2}\\
&= 0.9553\dots\\
&\approx 0.96\mathrm{s}
\end{aligned}$$

(b)
$$\begin{aligned} x &= 2 \\ -3\cos 2t &= 2 \\ \cos 2t &= -\frac{2}{3} \\ t &= \frac{\cos^{-1} -\frac{2}{3}}{2} \\ &= 1.1503\ldots \\ 1.1503 - 0.9553 &= 0.1949 \\ &\approx 0.19\text{s} \end{aligned}$$

(c)
$$\begin{aligned} x &= 3 \\ -3\cos 2t &= 3 \\ \cos 2t &= -1 \\ t &= \frac{\pi}{2} \\ &= 1.5708\ldots \\ 1.5708 - 1.1503 &= 0.4205 \\ &\approx 0.42\text{s} \end{aligned}$$

Check: the total times from A to E should be half the period: $\frac{\pi}{2} \approx 1.57$s–

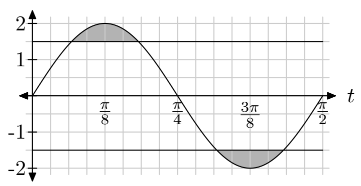
$$0.96 + 0.19 + 0.42 = 1.57$$

(d) If the particle is moving left-to-right when it passes D, the time to get back to D is double the time needed to go from D to E (since it moves from D to E and back again, and the symmetry makes these times DE and ED equal):

$$\begin{aligned} t &= 2 \times 0.4205 \\ &= 0.84\text{s} \end{aligned}$$

If the particle is moving right-to-left when it passes D, the time to get back to D again is a whole period less the D-E-D time, i.e.

$$\begin{aligned} t &= \pi - 0.84 \\ &\approx 2.30\text{s} \end{aligned}$$

15. 


Find the time the body first reaches 1.5m away from the mean position O and determine the length of time between that point and when it reaches maximum displacement. Then use the symmetry of the sine curve to determine the total time.

$$\begin{aligned} 2\sin 4t &= 1.5 \\ \sin 4t &= 0.75 \\ 4t &= \sin^{-1} 0.75 \\ t &= \frac{\sin^{-1} 0.75}{4} \\ &\approx 0.212 \\ \frac{\pi}{8} - t &\approx 0.181 \\ 0.181 \times 4 &\approx 0.72\text{s} \end{aligned}$$

16.
$$\begin{aligned} \ddot{x} &= -k^2 x \\ \ddot{x} &= -4x \\ k &= 2 \\ x &= a\sin(2t + \alpha) \\ v &= \dot{x} \\ &= 2a\cos(2t + \alpha) \end{aligned}$$

(a)
$$\begin{aligned} x &= a\sin(2t + \alpha) \\ 0 &= a\sin\alpha \\ \alpha &= 0 \\ v &= 2a\cos(2t + \alpha) \\ 4 &= 2a\cos 0 \\ a &= 2 \\ \therefore \quad x &= 2\sin 2t \end{aligned}$$

(b)
$$\begin{aligned} v &= 2a\cos(2t + \alpha) \\ 0 &= 2a\cos\alpha \\ \alpha &= \frac{\pi}{2} \\ x &= a\sin(2t + \alpha) \\ 4 &= a\sin\frac{\pi}{2} \\ a &= 4 \\ \therefore \quad x &= 4\sin\left(2t + \frac{\pi}{2}\right) \\ &= 4\cos 2t \end{aligned}$$

17. (a) Since the mass is at rest 2cm below equilibrium the amplitude of its motion is 2cm.

(b)
$$\begin{aligned} k^2 &= 64 \\ k &= 8 \\ \text{period} &= \frac{2\pi}{k} \\ &= \frac{\pi}{4}\text{ s} \end{aligned}$$

(c) This represents a quarter of a full cycle and takes a quarter of the period, i.e. $\frac{\pi}{16}$ s.

(d) The mass is at maximum speed when passing through the equilibrium point, so its speed is

$$\begin{aligned} s &= |ka| \\ &= 8 \times 2 \\ &= 16\text{cm/s} \end{aligned}$$

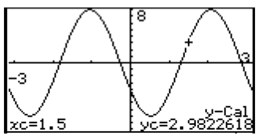
(e)
$$\begin{aligned} x &= -2\cos 8t \\ v &= 16\sin 8t \\ 16\sin 8t &= 8 \\ \sin 8t &= \frac{1}{2} \\ 8t &= \frac{\pi}{6} \\ t &= \frac{\pi}{48}\text{ s} \end{aligned}$$

18. (a)
$$\begin{aligned} x &= -4\sqrt{3}\sin 0 - 4\cos 0 \\ &= -4 \end{aligned}$$
The object is 4m from O.

(b) To prove:
$$\ddot{x} = -k^2 x$$
Proof:
$$\begin{aligned} x &= -4\sqrt{3}\sin 2t - 4\cos 2t \\ \dot{x} &= -8\sqrt{3}\cos 2t + 8\sin 2t \\ \ddot{x} &= 16\sqrt{3}\sin 2t + 16\cos 2t \\ &= -4(-4\sqrt{3}\sin 2t - 4\cos 2t) \\ &= -2^2 x \end{aligned}$$
as required, for $k = 2$. □

(c) This question is trivial to do using technology, as illustrated in previous questions. This gives an answer of
$$\int_0^{1.5} \left|\frac{\mathrm{d}x}{\mathrm{d}t}\right| dt = 14.98\text{m}$$
To work this question without technology is not within the scope of the course since it requires (amongst other things) calculating sin 3 and cos 3 which you are not expected to do without a calculator.

Another approach using technology is to graph the motion.




The object starts at $x = -4$, moves to the maximum negative displacement of $-8$ then comes back to $x = 2.98$ at $t = 1.5$. Thus the distance is $|-8 - -4| + |2.98 - -8| = 14.98$.

19. (a)
$$\begin{aligned} p &= x - 3 \\ &= 4\sin \pi t \\ \dot{p} &= 4\pi\cos \pi t \\ \ddot{p} &= -4\pi^2 \sin \pi t \\ &= -\pi^2 p \end{aligned}$$

(b) Period$= \frac{2\pi}{\pi} = 2$s.
Amplitude$= 4$

(c) The mean value of sine is zero, so the mean position of $3 + 4\sin \pi t$ is 3m.

(d) The maximum value of $x$ occurs when $\sin \pi t = 1$:
$$x = 3 + 4 \times 1 = 7\text{m}$$

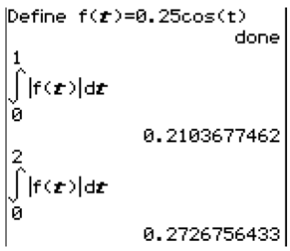
20. (a)
$$\begin{aligned} s &= x - 5 \\ &= -3\cos 2t \\ \dot{s} &= 6\sin 2t \\ \ddot{s} &= 12\cos 2t \\ &= -2^2 s \end{aligned}$$

(b) Period$= \frac{2\pi}{2} = \pi$s.
Amplitude$= 3$

(c) The mean value of cosine is zero, so the mean position of $5 - 3\cos 2t$ is 5m.

(d) The minimum value of $x$ occurs when $\sin 2t = 1$:
$$x = 5 - 3 \times 1 = 2\text{m}$$

21. Determine distance by integrating speed—the absolute value of velocity.




22.
$$\begin{aligned} x &= a\sin(kt + \alpha) \\ v &= ka\cos(kt + \alpha) \end{aligned}$$
When $x = 20$, $v = 30$:
$$\begin{aligned} a\sin(kt+\alpha) &= 20 \\ \sin(kt+\alpha) &= \frac{20}{a} \\ ka\cos(kt+\alpha) &= 30 \\ \cos(kt+\alpha) &= \frac{30}{ka} \\ \sin^2(kt+\alpha) + \cos^2(kt+\alpha) &= 1 \\ \frac{400}{a^2} + \frac{900}{k^2a^2} &= 1 \\ 400k^2 + 900 &= k^2a^2 \end{aligned}$$

When $x = 24$, $v = 14$:

$$\begin{aligned}
a\sin(kt+\alpha) &= 24\\
\sin(kt+\alpha) &= \frac{24}{a}\\
ka\cos(kt+\alpha) &= 14\\
\cos(kt+\alpha) &= \frac{14}{ka}\\
\sin^2(kt+\alpha)+\cos^2(kt+\alpha) &= 1\\
\frac{576}{a^2}+\frac{196}{k^2a^2} &= 1\\
576k^2+196 &= k^2a^2\\
\text{and}\quad 400k^2+900 &= k^2a^2\\
\therefore\quad 576k^2+196 &= 400k^2+900\\
176k^2 &= 704\\
k^2 &= 4\\
k &= \pm 2\\
\text{Period} &= \frac{2\pi}{2}\\
&= \pi\,\text{s}\\
\text{Now}\quad 400k^2+900 &= k^2a^2\\
1600+900 &= 4a^2\\
a^2 &= 625\\
a &= \pm 25\\
\text{Amplitude} &= 25\,\text{m}
\end{aligned}$$

23.
$$\begin{aligned}
x &= a\sin(kt+\alpha)\\
v &= ka\cos(kt+\alpha)
\end{aligned}$$

When $x = 0.6$, $v = 0.75$:

$$\begin{aligned}
a\sin(kt+\alpha) &= 0.6\\
\sin(kt+\alpha) &= \frac{0.6}{a}\\
ka\cos(kt+\alpha) &= 0.76\\
\cos(kt+\alpha) &= \frac{0.75}{ka}\\
\sin^2(kt+\alpha)+\cos^2(kt+\alpha) &= 1\\
\frac{0.36}{a^2}+\frac{0.5625}{k^2a^2} &= 1\\
0.36k^2+0.5625 &= k^2a^2
\end{aligned}$$

When $x = 0.39$, $v = 1.56$:

$$\begin{aligned}
a\sin(kt+\alpha) &= 0.39\\
\sin(kt+\alpha) &= \frac{0.39}{a}\\
ka\cos(kt+\alpha) &= 1.56\\
\cos(kt+\alpha) &= \frac{1.56}{ka}
\end{aligned}$$

$$\begin{aligned}
\sin^2(kt+\alpha)+\cos^2(kt+\alpha) &= 1\\
\frac{0.1521}{a^2}+\frac{2.4336}{k^2a^2} &= 1\\
0.1521k^2+2.4336 &= k^2a^2\\
\text{and} 0.36k^2+0.5625 &= k^2a^2\\
\therefore\quad 0.1521k^2+2.4336 &= 0.36k^2+0.5625\\
0.2079k^2 &= 1.8711\\
k^2 &= 9\\
k &= \pm 3\\
\text{Period} &= \frac{2\pi}{3}\,\text{s}\\
\text{Now}\quad 0.36k^2+0.5625 &= k^2a^2\\
3.24+0.5625 &= 9a^2\\
a^2 &= 0.4225\\
a &= \pm 0.65\\
\text{Amplitude} &= 65\,\text{cm}
\end{aligned}$$

24. (a) To prove: $\ddot{x} = -k^2x$
Proof:

$$\begin{aligned}
x &= A\cos kt\\
\dot{x} &= -kA\sin kt\\
\ddot{x} &= -k^2A\cos kt\\
&= -k^2x
\end{aligned}$$

as required. □

To prove: $|x| \le |A\cos 0|$
Proof:

$$\begin{aligned}
\text{RHS} &= |A|\\
\text{LHS} &= |A\cos kt|\\
&= |A||\cos kt|\\
|\cos kt| &\le 1\\
\therefore\quad |A||\cos kt| &\le |A|\\
\text{LHS} &\le \text{RHS}
\end{aligned}$$

as required. □

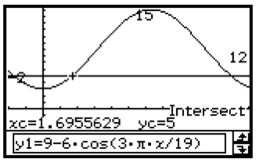
(b) Modelling the tide movement with SHM gives a mean depth of $\frac{3+15}{2} = 9$m and an amplitude of $\frac{15-3}{2} = 6$m. The period is double the time between low and high tides, i.e. $6\frac{1}{3}\times 2 = 12\frac{2}{3}$ hours. Hence

$$\begin{aligned}
\frac{2\pi}{k} &= 12\frac{2}{3}\\
&= \frac{38}{3}\\
38k &= 6\pi\\
k &= \frac{3\pi}{19}
\end{aligned}$$

If we take 7am as our starting time (i.e. $t = 0$ at 7am) the water depth is

$$d = 9 - 6\cos\frac{3\pi t}{19}$$

To determine the times when the water depth is at least 5m, plot a graph of this function and determine when it exceeds 5:




This gives $1.70 \le t \le 10.97$ representing times (to the nearest 5 minutes) between 8:40am and 6:00pm.

## Integration By Parts Extension Exercise

1.
$$u = x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x \qquad v = -\cos x$$
$$\begin{aligned}\int x\sin x\,\mathrm{d}x &= x(-\cos x) - \int -\cos x\,\mathrm{d}x\\ &= -x\cos x + \sin x + c\\ &= \sin x - x\cos x + c\end{aligned}$$

2.
$$u = x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x \qquad v = \sin x$$
$$\begin{aligned}\int x\cos x\,\mathrm{d}x &= x\sin x - \int \sin x\,\mathrm{d}x\\ &= x\sin x - \cos x + c\end{aligned}$$

3.
$$u = 3x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 3$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin 2x \qquad v = -0.5\cos 2x$$
$$\begin{aligned}&\int 3x\sin 2x\,\mathrm{d}x\\ &= 3x(-0.5\cos 2x) - \int (-0.5\cos 2x)(3)\,\mathrm{d}x\\ &= -1.5x\cos 2x + 0.75\sin 2x + c\\ &= \frac{3\sin x - 6x\cos 2x}{4} + c\end{aligned}$$

4.
$$u = x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x} \qquad v = 0.5\mathrm{e}^{2x}$$
$$\begin{aligned}\int x\mathrm{e}^{2x}\,\mathrm{d}x &= x(0.5\mathrm{e}^{2x}) - \int 0.5\mathrm{e}^{2x}\,\mathrm{d}x\\ &= 0.5x\mathrm{e}^{2x} - 0.25\mathrm{e}^{2x}\\ &= \frac{(2x-1)\mathrm{e}^{2x}}{4} + c\end{aligned}$$

5.
$$u = \ln x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = x^2 \qquad v = \frac{x^3}{3}$$
$$\begin{aligned}\int x^2\ln x\,\mathrm{d}x &= \frac{x^3\ln x}{3} - \int \frac{x^3}{3x}\,\mathrm{d}x\\ &= \frac{x^3\ln x}{3} - \int \frac{x^2}{3}\,\mathrm{d}x\\ &= \frac{x^3\ln x}{3} - \frac{x^3}{9} + c\\ &= \frac{x^3(3\ln x - 1)}{9} + c\end{aligned}$$

6.
$$u = x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = (x+2)^5 \qquad v = \frac{(x+2)^6}{6}$$
$$\begin{aligned}\int x(x+2)^5\,\mathrm{d}x &= \frac{x(x+2)^6}{6} - \int \frac{(x+2)^6}{6}\,\mathrm{d}x\\ &= \frac{x(x+2)^6}{6} - \frac{(x+2)^7}{42} + c\\ &= \frac{(x+2)^6(7x-(x+2))}{42} + c\\ &= \frac{(x+2)^6(6x-2)}{42} + c\\ &= \frac{(x+2)^6(3x-1)}{21} + c\end{aligned}$$

7. $$u = x \qquad \frac{du}{dx} = 1$$
$$\frac{dv}{dx} = \sqrt{2x+1} \qquad v = \frac{2(2x+1)^{\frac{3}{2}}}{3 \times 2} = \frac{(2x+1)^{\frac{3}{2}}}{3}$$

$$\begin{aligned}\int x\sqrt{2x+1}\,dx &= \frac{x(2x+1)^{\frac{3}{2}}}{3} - \int \frac{(2x+1)^{\frac{3}{2}}}{3}\,dx \\ &= \frac{x(2x+1)^{\frac{3}{2}}}{3} - \frac{2(2x+1)^{\frac{5}{2}}}{15 \times 2} + c \\ &= \frac{5x(2x+1)^{\frac{3}{2}} - (2x+1)^{\frac{5}{2}}}{15} + c \\ &= \frac{(2x+1)^{\frac{3}{2}}(5x - (2x+1)}{15} + c \\ &= \frac{(2x+1)^{\frac{3}{2}}(5x - 2x - 1)}{15} + c \\ &= \frac{(2x+1)^{\frac{3}{2}}(3x-1)}{15} + c\end{aligned}$$

8. $$u = x^2 \qquad \frac{du}{dx} = 2x$$
$$\frac{dv}{dx} = e^x \qquad v = e^x$$

$$\int x^2 e^x\,dx = x^2 e^x - \int 2xe^x\,dx$$

The integral on the right hand side requires integration by parts again.

$$u = 2x \qquad \frac{du}{dx} = 2$$
$$\frac{dv}{dx} = e^x \qquad v = e^x$$

$$\begin{aligned}\int 2xe^x\,dx &= 2xe^x - \int 2e^x\,dx \\ &= 2xe^x - 2e^x + c \\ &= 2e^x(x-1) + c\end{aligned}$$
$$\begin{aligned}\therefore \int x^2e^x\,dx &= x^2e^x - 2e^x(x-1) + c \\ &= e^x(x^2 - 2x + 2) + c\end{aligned}$$

9. $$u = x^2 \qquad \frac{du}{dx} = 2x$$
$$\frac{dv}{dx} = \sin x \qquad v = -\cos x$$

$$\begin{aligned}\int x^2 \sin x\,dx &= -x^2\cos x - \int -2x\cos x\,dx \\ &= \int 2x\cos x\,dx - x^2\cos x\end{aligned}$$

The integral on the right hand side requires integration by parts again.

$$u = 2x \qquad \frac{du}{dx} = 2$$
$$\frac{dv}{dx} = \cos x \qquad v = \sin x$$

$$\begin{aligned}\int 2x\cos x\,dx &= 2x\sin x - \int 2\sin x\,dx \\ &= 2x\sin x + 2\cos x + c\end{aligned}$$
$$\therefore \int x^2\sin x\,dx = 2x\sin x + 2\cos x - x^2\cos x + c$$

10. The key to this problem is appropriate selection of $u$ and $v$ so that differentiating one and integrating the other leaves an expression that is more amenable to integration. Often this means that we need to end up with a lower power of $x$. Differentiation of $e^{x^2}$ will not achieve this, so we need to look to integrate this part of the expression.

$$u = x^2 \qquad \frac{du}{dx} = 2x$$
$$\frac{dv}{dx} = 2xe^{x^2} \qquad v = e^{x^2}$$

$$\begin{aligned}\int x^3e^{x^2}\,dx &= \int (x^2)(2xe^{x^2})\,dx \\ &= x^2e^{x^2} - \int 2xe^{x^2}\,dx \\ &= x^2e^{x^2} - e^{x^2} + c \\ &= e^{x^2}(x^2 - 1) + c\end{aligned}$$

11. $$u = \ln x \qquad \frac{du}{dx} = \frac{1}{x}$$
$$\frac{dv}{dx} = 1 \qquad v = x$$

$$\begin{aligned}\int \ln x\,dx &= x\ln x - \int \frac{x}{x}\,dx \\ &= x\ln x - x + c \\ &= x(\ln x - 1) + c\end{aligned}$$

12. $$u = \sin x \qquad \frac{du}{dx} = \cos x$$
$$\frac{dv}{dx} = e^x \qquad v = e^x$$

$$\int e^x\sin x\,dx = e^x\sin x - \int e^x\cos x\,dx$$

The integral on the right needs to be done by parts again. Are we going around in circles? (You were warned these are sneaky!)

$$u = \cos x \qquad \frac{du}{dx} = -\sin x$$
$$\frac{dv}{dx} = e^x \qquad v = e^x$$

$$\int e^x \cos x = e^x \cos x + \int e^x \sin x \, dx$$
$$\therefore \int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$
$$\therefore 2\int e^x \sin x \, dx = e^x \sin x - e^x \cos x$$
$$\int e^x \sin x \, dx = \frac{e^x(\sin x - \cos x)}{2} + c$$

13. $$u = \cos 2x \quad \frac{du}{dx} = -2\sin 2x$$
$$\frac{dv}{dx} = e^x \quad v = e^x$$

$$\int e^x \cos 2x \, dx = e^x \cos 2x - \int -2e^x \sin 2x \, dx$$
$$= e^x \cos 2x + 2\int e^x \sin 2x \, dx$$

The integral on the right needs to be done by parts again.

$$u = \sin 2x \quad \frac{du}{dx} = 2\cos 2x$$
$$\frac{dv}{dx} = e^x \quad v = e^x$$

$$\int e^x \sin 2x = e^x \sin 2x - \int 2e^x \cos 2x \, dx$$
$$= e^x \sin 2x - 2\int e^x \cos 2x \, dx$$
$$\therefore \int e^x \cos 2x \, dx = e^x \cos 2x + 2\Big(e^x \sin 2x - 2\int e^x \cos 2x \, dx\Big)$$
$$= e^x \cos 2x + 2e^x \sin 2x - 4\int e^x \cos 2x \, dx$$
$$\therefore 5\int e^x \cos 2x \, dx = e^x \cos 2x + 2e^x \sin 2x$$
$$\int e^x \cos 2x \, dx = \frac{e^x(\cos 2x + 2\sin 2x)}{5} + c$$

## Miscellaneous Exercise 8

1. (c)
$$BA = \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 & -1 \\ 3 & 2 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0-6 & -2-4 & 1+0 \\ 0+0 & 2+0 & -1+0 \end{bmatrix}$$
$$= \begin{bmatrix} -6 & -6 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$
$$BA + C = \begin{bmatrix} -5 & -6 & 1 \\ -2 & 3 & 2 \end{bmatrix}$$

(f)
$$BD = \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} -1-4 \\ 1+0 \end{bmatrix}$$
$$= \begin{bmatrix} -5 \\ 1 \end{bmatrix}$$
$$BD + D = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

2. (a) For $x < -6$, $|x+6| = -(x+6)$ and $|x-2| = -(x-2)$:

$$-(x+6) = 2 - (x-2)$$
$$-x - 6 = 2 - x + 2$$
$$-6 = 4 \implies \text{no solution}$$

For $-6 \le x < 2$, $|x+6| = x+6$ and $|x-2| = -(x-2)$:

$$x + 6 = 2 - (x-2)$$
$$= 2 - x + 2$$
$$2x + 6 = 4$$
$$x = -1$$

For $x \ge 2$, $|x+6| = x+6$ and $|x-2| = x-2$:

$$x + 6 = 2 + x - 2$$
$$6 = 0 \implies \text{no solution}$$

Single solution: $x = -1$.

(b) For $x < -6$, $|x+6| = -(x+6)$ and $|x-2| = -(x-2)$:

$$-(x+6) = 10 - (x-2)$$
$$-x - 6 = 10 - x + 2$$
$$-6 = 12 \implies \text{no solution}$$

For $-6 \le x < 2$, $|x+6| = x+6$ and $|x-2| = -(x-2)$:

$$\begin{aligned} x+6 &= 10-(x-2) \\ &= 10-x+2 \\ 2x+6 &= 12 \\ x &= 3 \implies \text{outside the domain} \end{aligned}$$

For $x \ge 2$, $|x+6| = x+6$ and $|x-2| = x-2$:

$$\begin{aligned} x+6 &= 10+x-2 \\ 6 &= 8 \implies \text{no solution} \end{aligned}$$

No solution.

(c) For $x < -6$, $|x+6| = -(x+6)$ and $|x-2| = -(x-2)$:

$$\begin{aligned} -(x+6) &= 8-(x-2) \\ -x-6 &= 8-x+2 \\ -6 &= 10 \implies \text{no solution} \end{aligned}$$

For $-6 \le x < 2$, $|x+6| = x+6$ and $|x-2| = -(x-2)$:

$$\begin{aligned} x+6 &= 8-(x-2) \\ &= 8-x+2 \\ 2x+6 &= 10 \\ x &= 2 \implies \text{outside the domain} \end{aligned}$$

For $x \ge 2$, $|x+6| = x+6$ and $|x-2| = x-2$:

$$\begin{aligned} x+6 &= 8+x-2 \\ 6 &= 6 \implies \text{all solution} \end{aligned}$$

The solution is $x \ge 2$.

3.
$$\begin{bmatrix} 2x & 6 \\ 4 & 2y \end{bmatrix} + \begin{bmatrix} 3y & 3 \\ -3 & -3x \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 1 & 22 \end{bmatrix}$$
$$\begin{bmatrix} 2x+3y & 9 \\ 1 & 2y-3x \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 1 & 22 \end{bmatrix}$$
$$\begin{aligned} \therefore \quad & 2x+3y = 7 \\ \text{and} \quad & -3x+2y = 22 \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 7 \\ 22 \end{bmatrix} \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \frac{1}{13}\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 22 \end{bmatrix} \\ &= \frac{1}{13}\begin{bmatrix} -52 \\ 65 \end{bmatrix} \\ &= \begin{bmatrix} -4 \\ 5 \end{bmatrix} \end{aligned}$$
$$\therefore \quad x = -4, \quad y = 5$$

(You could, of course, use non-matrix techniques to solve the simultaneous equations if you so choose.)

4.
$$\begin{aligned} PA &= P + 2A \\ PA - P &= 2A \\ P(A-I) &= 2A \\ P &= 2A(A-I)^{-1} \\ &= 2\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \\ &= 2\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} \end{aligned}$$

5.
$$\begin{aligned} A^2 &= \begin{bmatrix} p & -p \\ 0 & q \end{bmatrix}\begin{bmatrix} p & -p \\ 0 & q \end{bmatrix} \\ &= \begin{bmatrix} p^2 & -p^2-pq \\ 0 & q^2 \end{bmatrix} \\ \det A &= pq \\ pqA^{-1} &= \begin{bmatrix} q & p \\ 0 & p \end{bmatrix} \\ p^2qA^{-1} &= \begin{bmatrix} pq & p^2 \\ 0 & p^2 \end{bmatrix} \\ A^2 + p^2qA^{-1} &= \begin{bmatrix} p^2 & -p^2-pq \\ 0 & q^2 \end{bmatrix} + \begin{bmatrix} pq & p^2 \\ 0 & p^2 \end{bmatrix} \\ &= \begin{bmatrix} p^2+pq & -pq \\ 0 & p^2+q^2 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} p^2+pq & -pq \\ 0 & p^2+q^2 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 0 & 10 \end{bmatrix}$$
$$\begin{aligned} pq &= -3 \\ p^2+pq &= 6 \\ p^2-3 &= 6 \\ p^2 &= 9 \\ p &= \pm 3 \\ \therefore \quad q &= \mp 1 \\ \text{check:} \quad p^2+q^2 &= 10 \\ 9+1 &= 10 \quad \text{ok.} \\ \therefore \quad (p,q) &\in \{(3,-1),(-3,1)\} \end{aligned}$$

6. Let $A = (x,y)$ be some arbitrary point on the $x-y$ plane. The matrix $\begin{bmatrix} a & b \\ ka & kb \end{bmatrix}$ transforms this point to $A' = (x', y')$, thus:

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} a & b \\ ka & kb \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} ax+by \\ kax+kby \end{bmatrix} \\ &= \begin{bmatrix} ax+by \\ k(ax+by) \end{bmatrix} \end{aligned}$$
$$\begin{aligned} \text{thus} \quad x' &= ax+by \\ \text{and} \quad y' &= k(ax+by) \\ &= kx' \end{aligned}$$

Thus the transformed point satisfies the equation $y = kx$ and hence lies on the line as required. □

7. (a)
$$\begin{aligned}\dot{x} &= 8\cos 4t\\ \ddot{x} &= -32\sin 4t\\ &= -4^2 x\end{aligned}$$
Period$=\frac{2\pi}{4}=\frac{\pi}{2}$s.
When $t=0$, $x=2\sin 0=0$.
The mean position is at O (distance=0).

(b)
$$\begin{aligned}\dot{x} &= -15\sin 3t\\ \ddot{x} &= -75\cos 3t\\ &= -3^2 x\end{aligned}$$
Period$=\frac{2\pi}{3}$s.
When $t=0$, $x=5\cos 0=5$m.
The mean position is at O (distance=0).

(c)
$$\begin{aligned}\dot{x} &= 4\cos 2t - 8\sin 2t\\ \ddot{x} &= -8\sin 2t - 16\cos 2t\\ &= -2^2 x\end{aligned}$$
Period$=\frac{2\pi}{2}=\pi$s.
When $t=0$, $x=2\cos 0+4\sin 0=2$m.
The mean position is at O (distance=0).

(d)
$$\begin{aligned}\dot{x} &= 15\cos 5t\\ \ddot{x} &= -75\sin 5t\\ &= -5^2(x-1)\end{aligned}$$
Period$=\frac{2\pi}{5}$s.
When $t=0$, $x=1+3\sin 0=1$m.
The mean position of $3\sin 5t$ is 0, so the mean position of $1+3\sin 5t$ is 1m from O.

8. The volume of any prism-like solid is equal to the area of the base times the height. The height here is 5m and the area of the base is determined by
$$\begin{aligned}A &= -\int_0^{\pi} -\sin x\,\mathrm{d}x\\ &= -\left[\cos x\right]_0^{\pi}\\ &= -(-1-1)\\ &= 2\text{m}^2\end{aligned}$$
Thus the volume of sand required is 10m$^3$.

9. First, rewrite each relation with $x$ the dependent variable:
$$\begin{aligned}x &= y+3\\ x &= y^2+1\end{aligned}$$
Now find the points of intersection to determine the bounds for our integrals:
$$\begin{aligned}y+3 &= y^2+1\\ y^2-y-2 &= 0\\ (y-2)(y+1) &= 0\\ y &= -1\\ \text{and}\quad y &= 2\end{aligned}$$
The region we want is right of the parabola and left of the line, i.e. $y^2+1 \le x \le y+3$, so the area is
$$\begin{aligned}A &= \int_{-1}^{2} (y+3)-(y^2+1)\,\mathrm{d}y\\ &= \int_{-1}^{2} (y-y^2+2)\,\mathrm{d}y\\ &= \left[\frac{y^2}{2}-\frac{y^3}{3}+2y\right]_{-1}^{2}\\ &= \left(\frac{4}{2}-\frac{8}{3}+4\right)-\left(\frac{1}{2}+\frac{1}{3}-2\right)\\ &= \frac{10}{3}-\left(-\frac{7}{6}\right)\\ &= \frac{27}{6}\\ &= 4.5\text{ units}^2\end{aligned}$$

10. (c) Let $P_0$ represent the initial population:
$$P_0 = \begin{bmatrix}340\\720\\840\\220\\80\end{bmatrix}$$
i.
$$LP_0 = \begin{bmatrix}1922\\204\\504\\672\\198\end{bmatrix} \approx \begin{bmatrix}1920\\200\\500\\670\\200\end{bmatrix}$$
ii.
$$L^{10}P_0 = \begin{bmatrix}3836\\2213\\1404\\875\\738\end{bmatrix} \approx \begin{bmatrix}3800\\2200\\1400\\900\\700\end{bmatrix}$$

(d) i. $\begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix} L^{19}P_0 = [25\,775]$

ii. $\begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix} L^{20}P_0 = [28\,839]$

iii. $\begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix} L^{29}P_0 = [80\,618]$

iv. $\begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix} L^{30}P_0 = [90\,367]$

v. $\begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix} L^{49}P_0 = [791\,220]$

vi. $\begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix} L^{50}P_0 = [886\,936]$

(e)
$$\frac{P_{20}}{P_{19}} = 1.119$$
$$\frac{P_{30}}{P_{29}} = 1.121$$
$$\frac{P_{50}}{P_{49}} = 1.121$$
This suggests an annual growth rate of 12.1%.

(f)
$$\frac{1}{1.121} = 0.892$$
$$1 - 0.892 = 0.108$$
The harvesting rate should be 10.8%.

(g)
$$(0.95L)^5 P_0 = \begin{bmatrix} 2067 \\ 873 \\ 442 \\ 515 \\ 450 \end{bmatrix} \approx \begin{bmatrix} 2050 \\ 850 \\ 450 \\ 500 \\ 450 \end{bmatrix}$$

11. One approach is to use Euler's formula:
$$\begin{aligned} \text{L.H.S.} &= (\cos\theta + \mathrm{i}\sin\theta)^n \\ &= \left(\mathrm{e}^{\mathrm{i}\theta}\right)^n \\ &= \mathrm{e}^{\mathrm{i}n\theta} \\ &= \cos n\theta + \mathrm{i}\sin n\theta \\ &= \text{R.H.S.} \end{aligned}$$
□

12. This is a pretty standard kind of exam question.
$$\begin{aligned} \cos 4\theta + \mathrm{i}\sin 4\theta &= (\cos\theta + \mathrm{i}\sin\theta)^4 \\ &= \cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta \\ &\quad + 6\mathrm{i}^2\cos^2\theta\sin^2\theta + 4\mathrm{i}^3\cos\theta\sin^3\theta \\ &\quad + \mathrm{i}^4\sin^4\theta \\ &= \cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta \\ &\quad - 6\cos^2\theta\sin^2\theta - 4\mathrm{i}\cos\theta\sin^3\theta \\ &\quad + \sin^4\theta \\ &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\ &\quad + \mathrm{i}(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta) \end{aligned}$$
Equating real parts,
$$\begin{aligned} \cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\ &= \cos^4\theta - 6\cos^2\theta(1 - \cos^2\theta) \\ &\quad + (1 - \cos^2\theta)^2 \\ &= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta \\ &\quad + 1 - 2\cos^2\theta + \cos^4\theta \\ &= 8\cos^4\theta - 8\cos^2\theta + 1 \end{aligned}$$

13. Since the particle has positive acceleration for all $t \geq 0$ and positive initial velocity, its velocity is always positive and the distance travelled in the third second is the difference between its position at $t = 2$ and at $t = 3$.
$$\begin{aligned} v(t) &= \int a(t)\,\mathrm{d}t \\ &= \int (6t + 4)\,\mathrm{d}t \\ &= 3t^2 + 4t + c \\ x(t) &= \int v(t)\,\mathrm{d}t \\ &= \int (3t^2 + 4t + c)\,\mathrm{d}t \\ &= t^3 + 2t^2 + ct + k \end{aligned}$$
$$\begin{aligned} x(3) - x(2) &= 32 \\ (45 + 3c + k) - (16 + 2c + k) &= 32 \\ 29 + c &= 32 \\ c &= 3 \\ \therefore \quad v(1) &= 10\,\mathrm{ms}^{-1} \end{aligned}$$

14. (a) Let $a$ be the surface area and $s$ the side length.
$$\begin{aligned} a &= 6s^2 \\ \frac{\mathrm{d}a}{\mathrm{d}s} &= 12s \\ \frac{\delta a}{\delta s} &\approx 12s \\ \delta a &\approx 12s\delta s \\ &= 12 \times 5 \times 0.2 \\ &= 12\,\mathrm{cm}^2 \end{aligned}$$
(b) Let $v$ be the volume and $s$ the side length.
$$\begin{aligned} v &= s^3 \\ \frac{\mathrm{d}v}{\mathrm{d}s} &= 3s^2 \\ \frac{\delta v}{\delta s} &\approx 3s^2 \\ \delta v &\approx 3s^2\delta s \\ &= 3 \times 25 \times 0.2 \\ &= 15\,\mathrm{cm}^3 \end{aligned}$$

15. (a)
$$\frac{\mathrm{d}C}{\mathrm{d}x} = \frac{200}{1 + x}$$
(b)
$$\begin{aligned} \frac{200}{1 + x} &= 2 \\ 1 + x &= 100 \\ x &= 99 \\ C &= 600 + 200\ln(1 + 99) \\ &= 1521.03 \\ \frac{C}{x} &= \frac{1521.03}{99} \\ &= \$15.36 \text{ per unit} \end{aligned}$$

16.
$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} &= -0.005A \\ A &= A_0\mathrm{e}^{-0.005t} \end{aligned}$$

The half-life represents the time when $A = 0.5A_0$:

$$\begin{aligned}
0.5A_0 &= A_0\mathrm{e}^{-0.005t}\\
\mathrm{e}^{-0.005t} &= 0.5\\
-0.005t &= \ln 0.5\\
&= -\ln 2\\
t &= \frac{\ln 2}{0.005}\\
&\approx 139 \text{ years}
\end{aligned}$$

17.
$$\begin{aligned}
C &= C_0\mathrm{e}^{-kt}\\
0.5C_0 &= C_0\mathrm{e}^{-5700k}\\
-5700k &= \ln 0.5\\
&= -\ln 2\\
k &= \frac{\ln 2}{5700}\\
\therefore \quad C &= C_0\mathrm{e}^{-\frac{t\ln 2}{5700}}\\
&= C_0(\mathrm{e}^{\ln 2})^{-\frac{t}{5700}}\\
&= C_0 2^{-\frac{t}{5700}}
\end{aligned}$$

Given that 65% has decayed, $C = 0.35C_0$,

$$\begin{aligned}
0.35C_0 &= C_0 2^{-\frac{t}{5700}}\\
2^{-\frac{t}{5700}} &= 0.35\\
-\frac{t}{5700} &= \log_2 0.35\\
t &= -5700\log_2 0.35\\
&\approx 8\,600 \text{ years}
\end{aligned}$$

18.
$$\begin{aligned}
C &= C_0\mathrm{e}^{-kt}\\
0.5C_0 &= C_0\mathrm{e}^{-12k}\\
-12k &= \ln 0.5\\
&= -\ln 2\\
k &= \frac{\ln 2}{12}\\
\therefore \quad C &= C_0\mathrm{e}^{-\frac{t\ln 2}{12}}\\
&= C_0(\mathrm{e}^{\ln 2})^{-\frac{t}{12}}\\
&= C_0 2^{-\frac{t}{12}}
\end{aligned}$$

Given $C = 0.05C_0$,

$$\begin{aligned}
0.05C_0 &= C_0 2^{-\frac{t}{12}}\\
2^{-\frac{t}{12}} &= 0.05\\
-\frac{t}{12} &= \log_2 0.05\\
t &= -12\log_2 0.05\\
&\approx 52 \text{ days}
\end{aligned}$$

19. The proposition to prove is:

$$5^n + 3\times 9^n = 4a, \quad a, n \in \mathbb{I}, n \geq 0$$

Proof:

For $n = 0$:

$$\begin{aligned}
5^0 + 3\times 9^0 &= 1 + 4\\
&= 4
\end{aligned}$$

Assume the proposition is true for $n = k$, i.e.:

$$5^k + 3\times 9^k = 4a$$

for some integer $a$.

Then for $n = k+1$,

$$\begin{aligned}
5^{k+1} &+ 3{\times}9^{k+1}\\
&= 5\times 5^k + 3\times 9\times 9^k\\
&= (4+1)\times 5^k + (8+1)\times 3\times 9^k\\
&= 4\times 5^k + 8\times 3\times 9^k + 5^k + 3\times 9^k\\
&= 4(\times 5^k + 2\times 3\times 9^k) + 4a\\
&= 4(\times 5^k + 2\times 3\times 9^k + a)
\end{aligned}$$

Hence if the proposition is true for $n = k$ then it is also true for $n = k+1$, and since it is true for $n = 0$ it is true for all integer $n \geq 0$ by mathematical induction. □

20. (a)
$$\begin{aligned}
v &= \frac{\mathrm{d}x}{\mathrm{d}t}\\
&= \frac{9}{1+t} - 4\\
v &= 0\\
\frac{9}{1+t} - 4 &= 0\\
\frac{9}{1+t} &= 4\\
9 &= 4(1+t)\\
4 + 4t &= 9\\
4t &= 5\\
t &= 1.25\text{s}
\end{aligned}$$

(b)
$$\begin{aligned}
a &= \frac{\mathrm{d}v}{\mathrm{d}t}\\
&= -\frac{9}{(1+t)^2}\\
v &= a\\
\frac{9}{1+t} - 4 &= -\frac{9}{(1+t)^2}\\
-9(1+t) + 4(1+t)^2 &= 9\\
4t^2 + 8t + 4 - 9t - 9 &= 9\\
4t^2 - t - 14 &= 0\\
(4t+7)(t-2) &= 0\\
t &= 2
\end{aligned}$$

(discarding the negative solution for $t$ because we are given $t \geq 0$).

21. Repeatedly rotate 90° anti-clockwise to give $z_2 = -b + a\mathrm{i}$, $z_3 = -a - b\mathrm{i}$, $z_4 = b - a\mathrm{i}$.

22. (a)
$$\begin{aligned} A &= \int 6\mathrm{e}^{2t}\,\mathrm{d}t \\ &= 3\mathrm{e}^{2t} + c \\ 4 &= 3\mathrm{e}^{0} + c \\ c &= 1 \\ A &= 3\mathrm{e}^{2t} + 1 \end{aligned}$$

(b)
$$\begin{aligned} A &= 3\mathrm{e}^{1} + 1 \\ &= 3\mathrm{e} + 1 \end{aligned}$$

(c)
$$\begin{aligned} \delta A &\approx \frac{\mathrm{d}A}{\mathrm{d}t}\delta t \\ &= 6\mathrm{e}^{0} \times 0.01 \\ &= 0.06 \end{aligned}$$

23. Starting from De Moivre's Theorem,

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$$

Let $n = 2$

$$\begin{aligned} (\cos\theta + \mathrm{i}\sin\theta)^2 &= \cos 2\theta + \mathrm{i}\sin 2\theta \\ \cos^2\theta + 2\mathrm{i}\cos\theta\sin\theta + \mathrm{i}^2\sin^2\theta &= \cos 2\theta + \mathrm{i}\sin 2\theta \\ \cos^2\theta - \sin^2\theta + 2\mathrm{i}\cos\theta\sin\theta &= \cos 2\theta + \mathrm{i}\sin 2\theta \end{aligned}$$

Equating real parts gives

$$\cos^2\theta - \sin^2\theta = \cos 2\theta$$

Equating imaginary parts gives

$$2\cos\theta\sin\theta = \sin 2\theta$$

as required. □

24. (a) If you recognise this as being of the form $\int (f(x))^n f'(x)\,\mathrm{d}x$ where $f(x) = \ln x$ then this can be done by inspection: no working required.

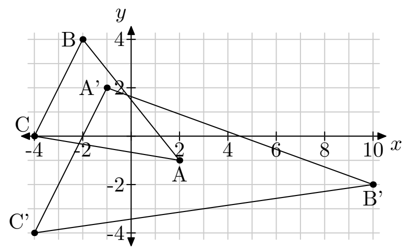
(b) This can also be done by inspection.

(c)
$$\begin{aligned} \int \frac{(\ln x^4)}{x}\,\mathrm{d}x &= \int \frac{4\ln x}{x}\,\mathrm{d}x \\ &= 2(\ln x)^2 + c \end{aligned}$$

25. (a)
$$\begin{aligned} T^{-1} &= \frac{1}{3}\begin{bmatrix} 0 & 3 \\ 1 & -1 \end{bmatrix} \\ A &= T^{-1}A' \\ &= \frac{1}{3}\begin{bmatrix} 0 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ B &= T^{-1}B' \\ &= \frac{1}{3}\begin{bmatrix} 0 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 10 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 4 \end{bmatrix} \\ C &= T^{-1}C' \\ &= \frac{1}{3}\begin{bmatrix} 0 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ -4 \end{bmatrix} \\ &= \begin{bmatrix} -4 \\ 0 \end{bmatrix} \end{aligned}$$

The coordinates of A, B and C are $(2, -1)$, $(-2, 4)$ and $(-4, 0)$ respectively.

(b) $|\det T| = 3$




Let $|\triangle ABC|$ represent the area of triangle ABC. We can determine the area of each triangle by considering its enclosing rectangle and subtracting the right-triangular regions outside the triangle.

$$\begin{aligned} |\triangle ABC| &= 6 \times 5 \\ &\quad - \frac{2\times 4}{2} - \frac{4\times 5}{2} - \frac{6\times 1}{2} \\ &= 13\text{units}^2 \\ |\triangle A'B'C'| &= 14 \times 6 \\ &\quad - \frac{3\times 6}{2} - \frac{11\times 4}{2} - \frac{14\times 2}{2} \\ &= 39\text{units}^2 \\ |\triangle A'B'C'| &= 3|\triangle ABC| \end{aligned}$$

26. (a)
$$\begin{aligned} v(t) &= \int 0.1\mathrm{e}^{0.1t}\,\mathrm{d}t \\ &= \mathrm{e}^{0.1t} + c \\ v(0) &= 0 \\ \mathrm{e}^{0} + c &= 0 \\ c &= -1 \\ v(t) &= \mathrm{e}^{0.1t} - 1 \\ v(10) &= (\mathrm{e} - 1)\,\mathrm{ms}^{-1} \end{aligned}$$

(b)
$$\begin{aligned} x(t) &= \int (e^{0.1t} - 1)\,dt \\ &= 10e^{0.1t} - t + c \\ x(0) &= 0 \\ 10e^0 - 0 + c &= 0 \\ c &= -10 \\ x(t) &= 10e^{0.1t} - t - 10 \\ x(10) &= (10e - 20)\,\text{m} \end{aligned}$$

(c) Distance travelled is equal to the difference in displacement, provided there is no change in sign in velocity. $v = e^{0.1t} - 1$ is positive for all $t > 0$ so

$$\begin{aligned} d(T) &= x(T+1) - x(T) \\ &= 10e^{0.1(T+1)} - (T+1) - 10 \\ &\quad - (10e^{0.1T} - T - 10) \\ &= 10e^{0.1(T+1)} - 1 - 10e^{0.1T} \\ &= 10e^{0.1}e^{0.1T} - 10e^{0.1T} - 1 \\ &= (10e^{0.1T}(e^{0.1} - 1) - 1)\,\text{m} \end{aligned}$$

(d) The third second means from $t = 2$ to $t = 3$, so we want $d(2)$:

$$\begin{aligned} d(2) &= 10e^{0.2}(e^{0.1} - 1) - 1 \\ &= 0.285\,\text{m} \end{aligned}$$

(e)
$$\begin{aligned} d(9) &= 10e^{0.9}(e^{0.1} - 1) - 1 \\ &= 1.587\,\text{m} \end{aligned}$$

27. Although this presents itself as a transition matrix question, it can be answered more intuitively. The long-term distribution will be that distribution that results in a steady state, i.e. when the 6% of the birds at A who switch to B are balanced by the 4% of the birds at B who switch to B.

Let $a$ be the number of birds at A.

Let $b$ be the number of birds at B.

$$\begin{aligned} 0.06a &= 0.04b \\ 1.5a &= b \\ \frac{a}{a+b} &= \frac{a}{a+1.5a} \\ &= \frac{1}{2.5} \\ &= 0.4 \end{aligned}$$

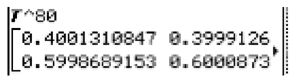
Forty percent of the birds will be at A in the long term.

Here is the matrix approach:

$$T = \begin{array}{cc} & \textit{From} \\ & \begin{array}{cc} A & B \end{array} \\ \text{To} \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.94 & 0.04 \\ 0.06 & 0.96 \end{bmatrix} \end{array}$$

$$\begin{aligned} T\begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} a \\ b \end{bmatrix} \\ 0.94a + 0.04b &= a \\ 0.06a + 0.96b &= b \\ -0.06a + 0.04b &= 0 \\ 0.06a - 0.04b &= 0 \\ b &= 1.5a \\ \frac{a}{a+b} &= \frac{a}{a+1.5a} \\ &= \frac{1}{2.5} \\ &= 0.4 \end{aligned}$$

Alternatively, if using technology, once you've formed $T$, simply raise it to increasingly high powers until the two columns are sufficiently identical and interpret the results.




28. (a)
$$\begin{aligned} \int \sin^3 x\,dx &= \int \sin^2 x \sin x\,dx \\ &= \int (1 - \cos^2 x)\sin x\,dx \\ &= \int (\sin x - \cos^2 x \sin x)\,dx \\ &= -\cos x + \frac{\cos^3 x}{3} + c \end{aligned}$$

(b)
$$\begin{aligned} \int 4\sin^2 x\,dx &= \int -2(-2\sin^2 x)\,dx \\ &= -2\int (1 - 2\sin^2 x - 1)\,dx \\ &= -2\int (\cos 2x - 1)\,dx \\ &= -2\left(\frac{\sin 2x}{2} - x\right) + c \\ &= 2x - \sin 2x + c \end{aligned}$$

29. (a) No working required.

(b)
$$\begin{aligned} u &= 4x \\ \frac{d}{dx}\int_1^{4x} e^{t^2}\,dt &= \frac{du}{dx}\frac{d}{du}\int_1^{u} e^{t^2}\,dt \\ &= 4e^{u^2} \\ &= 4e^{(4x)^2} \\ &= 4e^{16x^2} \end{aligned}$$

30.
$$\begin{aligned}
M &= M_0\mathrm{e}^{-kt}\\
0.5M_0 &= M_0\mathrm{e}^{-30k}\\
-30k &= \ln 0.5\\
&= -\ln 2\\
k &= \frac{\ln 2}{30}\\
\therefore\quad M &= M_0\mathrm{e}^{-\frac{t\ln 2}{30}}\\
&= M_0(\mathrm{e}^{\ln 2})^{-\frac{t}{30}}\\
&= M_0 2^{-\frac{t}{30}}
\end{aligned}$$

Given $M_0$ is 20 times the safe level, we need $M = 0.05M_0$,

$$\begin{aligned}
0.05M_0 &= M_0 2^{-\frac{t}{30}}\\
2^{-\frac{t}{30}} &= 0.05\\
-\frac{t}{30} &= \log_2 0.05\\
t &= -30\log_2 0.05\\
&\approx 130 \text{ years}
\end{aligned}$$

31. (a) There are two paths from A: to D with probability 0.7 and to B with probability $p$. Since the probabilities must add to 1, $p = 0.3$. Similarly there are two paths from C: to D with probability 0.4 and to B with probability $q$, giving $q = 0.6$.

(b) Let $T$ be the transition matrix as follows:

$$T = \text{To}\ \begin{matrix} A\\ B\\ C\\ D\end{matrix}\overset{\textit{From}}{\overset{\begin{matrix}A & B & C & D\end{matrix}}{\begin{bmatrix} 0 & 0.4 & 0 & 0.4\\ 0.3 & 0 & 0.6 & 0.5\\ 0 & 0.1 & 0 & 0.1\\ 0.7 & 0.5 & 0.4 & 0\end{bmatrix}}}$$

Let $S_0$ be the initial state matrix:

$$S_0 = \begin{bmatrix} 1000\\ 0\\ 0\\ 0\end{bmatrix}$$

After one period,

$$\begin{aligned}
S_1 &= TS_0\\
&= \begin{bmatrix} 0\\ 300\\ 0\\ 700\end{bmatrix}
\end{aligned}$$

That is, 300 people at B and 700 at D.

(c)
$$\begin{aligned}
S_2 &= T^2S_0\\
&= \begin{bmatrix} 400\\ 350\\ 100\\ 150\end{bmatrix}
\end{aligned}$$

That is, 400 people at A, 350 at B, 100 at C and 150 at D.

(d)
$$\begin{aligned}
S_3 &= T^3S_0\\
&= \begin{bmatrix} 200\\ 255\\ 50\\ 495\end{bmatrix}
\end{aligned}$$

That is, 200 people at A, 255 at B, 50 at C and 495 at D.

(e)
$$T^{20}S_0 = \begin{bmatrix} 267\\ 302\\ 67\\ 364\end{bmatrix}$$

$$T^{21}S_0 = \begin{bmatrix} 267\\ 302\\ 67\\ 364\end{bmatrix}$$

In the long term there are expected to be 267 people at A, 302 at B, 67 at C and 364 at D.

Alternatively, solve

$$\begin{aligned}
T\begin{bmatrix} a\\ b\\ c\\ d\end{bmatrix} &= \begin{bmatrix} a\\ b\\ c\\ d\end{bmatrix}\\
T\begin{bmatrix} a\\ b\\ c\\ d\end{bmatrix} - \begin{bmatrix} a\\ b\\ c\\ d\end{bmatrix} &= \begin{bmatrix} 0\\ 0\\ 0\\ 0\end{bmatrix}\\
(T-I)\begin{bmatrix} a\\ b\\ c\\ d\end{bmatrix} &= \begin{bmatrix} 0\\ 0\\ 0\\ 0\end{bmatrix}\\
-a + 0.4b + 0.4d &= 0\\
0.3a - b + 0.6c + 0.5d &= 0\\
0.1b - c + 0.1d &= 0\\
0.7a + 0.5b + 0.4c - d &= 0\\
\text{and } a + b + c + d &= 1000
\end{aligned}$$

(Actually one of the first four of these five equations is redundant and can be left out when solving.)

This gives us

$$\begin{aligned}
a &= \frac{800}{3}\\
b &= \frac{2720}{9}\\
c &= \frac{200}{3}\\
d &= \frac{3280}{9}
\end{aligned}$$

Although you should be able to solve four equations in four unknowns, it's difficult to envisage a situation where you would need to do that without the assistance of appropriate technology. With that in mind, the simpler first approach is probably more appropriate.

32. There are several conjectures that could be made here. This solution addresses only the most obvious.

The initial conjecture might deal with only two, three and four digit numbers:

Conjecture: The difference between a 2-, 3- or 4-digit natural number and its reflection is a multiple of 9 (where a number's reflection is another number with the same digits in reverse order).

The obvious extension is to consider numbers with more digits.

Conjecture: The difference between any natural number and its reflection is a multiple of 9.

Test with some five digit numbers:

$$\begin{aligned}|12345 - 54321| &= 41976 = 9 \times 4664\\ |10209 - 90201| &= 79992 = 9 \times 8888\\ |92654 - 45629| &= 47025 = 9 \times 5225\end{aligned}$$

These support the conjecture.

Proof: For two digit numbers, let $a$ and $b$ be single-digit natural numbers. Any two digit natural number $p$ can be represented as

$$p = 10a + b$$

and the reflection of $p$ as

$$p^R = 10b + a$$

This gives the difference as

$$\begin{aligned}|p - p^R| &= |10a + b - (10b + a)|\\ &= |9a - 9b|\\ &= 9|a - b|\end{aligned}$$

Thus the difference is 9 times the difference between the two digits: a multiple of 9 as required.

For three digit numbers, let $a$, $b$ and $c$ be single digit natural numbers. Any three digit natural number $p$ can be represented as

$$p = 100a + 10b + c$$

and the reflection of $p$ as

$$p^R = 100c + 10b + a$$

This gives the difference as

$$\begin{aligned}|p - p^R| &= |100a + 10b + c - (100c + 10b + a)|\\ &= |99a - 99c|\\ &= 99|a - c|\end{aligned}$$

Thus the difference is 99 times the difference between the first and last digits: a multiple of 9 as required.

(A four-digit proof could be given next, but let's be more ambitious.)

Now consider the conjecture for $n$-digit natural numbers. Assume the conjecture to be true for any number $p$ with $k$ digits, that is

$$p - p^R = 9d$$

for some integer d.

Let $a$ and $b$ be single digit natural numbers. If we put $a$ before the digits of $p$ and put $b$ after, we create a new natural number $q$ having $k + 2$ digits:

$$q = 10^{k+1}a + 10p + b$$

and the reflection of $q$ is

$$q^R = 10^{k+1}b + 10p^R + a$$

Then

$$\begin{aligned}&|q - q^R|\\ &= |10^{k+1}a + 10p + b - (10^{k+1}b + 10p^R + a)|\\ &= |(10^{k+1} - 1)a + 10(p - p^R) - (10^{k+1} - 1)b|\\ &= |(10^{k+1} - 1)(a - b) + 90d|\end{aligned}$$

One less than any positive power of 10 is a multiple of 9 (which we could also prove by induction, but we take as self-evident here) so we can conclude that if the conjecture is true for numbers having $k$ digits then it is also true for numbers having $k+2$ digits. Since we have established the conjecture for numbers having 2 and 3 digits, it is proven for all numbers of 2 or more digits by mathematical induction. □

33.
$$\begin{aligned}\left(\frac{z_1}{z_2 z_3}\right)^{-3} &= \left(\frac{\sqrt{6}\operatorname{cis}\frac{5\pi}{6}}{(2\operatorname{cis}\frac{\pi}{2})(3\operatorname{cis}\frac{2\pi}{3})}\right)^{-3}\\ &= \left(\frac{\sqrt{6}\operatorname{cis}\frac{5\pi}{6}}{6\operatorname{cis}\frac{7\pi}{6}}\right)^{-3}\\ &= \left(\frac{\operatorname{cis}-\frac{\pi}{3}}{\sqrt{6}}\right)^{-3}\\ &= 6\sqrt{6}\operatorname{cis}\pi\\ &= -6\sqrt{6}\end{aligned}$$

34. (a) No working required.

(b)
$$\begin{aligned}z^{14} &= (\sqrt{2}\operatorname{cis}-\frac{\pi}{4})^{14}\\ &= 2^{\frac{14}{2}}\operatorname{cis}-\frac{14\pi}{4}\\ &= 2^7\operatorname{cis}-\frac{7\pi}{2}\\ &= 128\operatorname{cis}\frac{\pi}{2}\\ &= 128\text{i}\end{aligned}$$

35.
$$
\begin{aligned}
z &= -1+\sqrt{3}\mathrm{i}\\
&= \sqrt{(-1)^2+(\sqrt{3})^2}\operatorname{cis}\tan^{-1}\frac{\sqrt{3}}{-1}\\
&\quad (2^{\text{nd}} \text{ quadrant})\\
&= 2\operatorname{cis}\frac{2\pi}{3}\\
\bar{z} &= 2\operatorname{cis}-\frac{2\pi}{3}\\
\frac{1}{\bar{z}} &= \frac{1}{2}\operatorname{cis}\frac{2\pi}{3}\\
\left(z+\frac{1}{\bar{z}}\right)^4 &= \left(2\operatorname{cis}\frac{2\pi}{3}+\frac{1}{2}\operatorname{cis}\frac{2\pi}{3}\right)^4\\
&= \left(\frac{5}{2}\operatorname{cis}\frac{2\pi}{3}\right)^4\\
&= \frac{625}{16}\operatorname{cis}\frac{8\pi}{3}\\
&= \frac{625}{16}\operatorname{cis}\frac{2\pi}{3}\\
\left(z-\frac{1}{\bar{z}}\right)^4 &= \left(2\operatorname{cis}\frac{2\pi}{3}-\frac{1}{2}\operatorname{cis}\frac{2\pi}{3}\right)^4\\
&= \left(\frac{3}{2}\operatorname{cis}\frac{2\pi}{3}\right)^4\\
&= \frac{81}{16}\operatorname{cis}\frac{8\pi}{3}\\
&= \frac{81}{16}\operatorname{cis}\frac{2\pi}{3}
\end{aligned}
$$

36. Let $d$ be the distance AB.

$$
\begin{aligned}
\sin\theta &= \frac{h}{50}\\
\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{1}{50}\frac{\mathrm{d}h}{\mathrm{d}t}\\
\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{1.2}{50\cos\theta}\\
d^2+h^2 &= 2500\\
2d\frac{\mathrm{d}d}{\mathrm{d}t}+2h\frac{\mathrm{d}h}{\mathrm{d}t} &= 0\\
d\frac{\mathrm{d}d}{\mathrm{d}t}+h\frac{\mathrm{d}h}{\mathrm{d}t} &= 0\\
d\frac{\mathrm{d}d}{\mathrm{d}t}+1.2h &= 0\\
\frac{\mathrm{d}d}{\mathrm{d}t} &= -\frac{1.2h}{d}
\end{aligned}
$$

When $h = 40$,

$$
\begin{aligned}
d &= \sqrt{2500-1600}\\
&= 30\,\mathrm{m}\\
\cos\theta &= \frac{30}{50}\\
&= 0.6\\
\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{1.2}{50\cos\theta}\\
&= \frac{1.2}{30}\\
&= 0.04 \text{ radians per second}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}d}{\mathrm{d}t} &= -\frac{1.2h}{d}\\
&= -\frac{1.2\times 40}{30}\\
&= -1.6\,\mathrm{ms}^{-1}
\end{aligned}
$$

B approaches A at 1.6 metres per second.

37. (a) $12\times 5\,000+5\times 8\,000 = \$100\,000$

(b)
$$
\begin{aligned}
AD &= 12-DC\\
&= 12-\frac{5}{\tan\theta}\\
DB &= \frac{5}{\sin\theta}\\
C &= 5\,000AD+8\,000DB\\
&= 5\,000\left(12-\frac{5}{\tan\theta}\right)+8\,000\left(\frac{5}{\sin\theta}\right)\\
&= 60\,000-\frac{25\,000}{\tan\theta}+\frac{40\,000}{\sin\theta}
\end{aligned}
$$

as required.

(c) Minimum cost will be at one or other extremes of the domain, or where $\frac{\mathrm{d}C}{\mathrm{d}\theta}=0$. Extremes are where D is coincident with point C—with cost of \$100 000 as seen in part (a)—or where D is coincident with point A, in which case the cost is

$$8\,000\sqrt{5^2+12^2} = \$104\,000$$

$$
\begin{aligned}
\frac{\mathrm{d}C}{\mathrm{d}\theta} &= \frac{25\,000}{\tan^2\theta\cos^2\theta}-\frac{40\,000\cos\theta}{\sin^2\theta}\\
&= \frac{25\,000}{\sin^2\theta}-\frac{40\,000\cos\theta}{\sin^2\theta}
\end{aligned}
$$

Setting $\frac{\mathrm{d}C}{\mathrm{d}\theta}=0$ gives

$$
\begin{aligned}
\frac{25\,000}{\sin^2\theta}-\frac{40\,000\cos\theta}{\sin^2\theta} &= 0\\
25\,000-40\,000\cos\theta &= 0\\
\cos\theta &= \frac{25\,000}{40\,000}\\
&= 0.625\\
\theta &= 51^\circ
\end{aligned}
$$

$$\begin{aligned} C &= 60\,000 - \frac{25\,000}{\tan 51^\circ} + \frac{40\,000}{\sin 51^\circ} \\ &= \$91\,000 \end{aligned}$$

38. For particle A, the amplitude of the displacement gives $c = 5$.

The velocity for particle A is

$$v = k_1 c \cos k_1 t$$

and from the graph,

$$\begin{aligned} k_1 c &= 10 \\ k_1 &= 2 \\ \text{Period} &= \frac{2\pi}{k_1} \\ &= \pi \text{ s} \end{aligned}$$

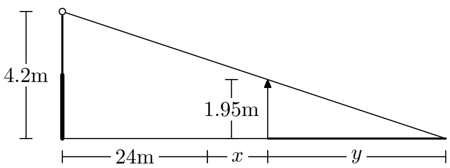
For particle B,

$$\begin{aligned} v &= k_2 d \cos k_2 t \\ a &= -k_2^2 d \sin k_2 t \\ k_2 d &= 3 \\ k_2^2 d &= 1.5 \\ k_2 &= 0.5 \\ d &= 6 \\ \text{Period} &= \frac{2\pi}{k_2} \\ &= 4\pi \text{ s} \end{aligned}$$

Note: the answer of $k_1 = 1$ in Sadler is an error.

39. Let $y$ be the length of the shadow and let $x$ be the distance that has been run.

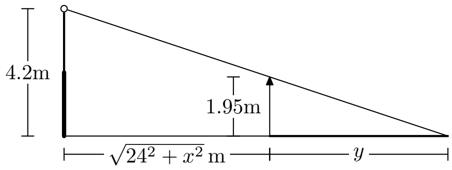
(a)




$$\begin{aligned} \frac{y}{24 + x + y} &= \frac{1.95}{4.2} \\ 4.2y &= 1.95(24 + x + y) \\ 2.25y &= 1.95x + 46.8 \\ y &= \frac{13}{15}x + \frac{937}{45} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{13}{15} \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \frac{13}{15} \times 5 \\ &= \frac{13}{3} \text{ m/s} \end{aligned}$$

(b) The geometry is exactly the same as in (a) except that the sign of $x$ is reversed. The length of the shadow changes at the same speed, but now it is getting shorter at $\frac{13}{3}$ m/s instead of getting longer.

(c) After the runner has travelled $x$ metres, the distance from the lamppost is given by Pythagoras' Theorem:




$$\begin{aligned} \frac{y}{\sqrt{24^2 + x^2} + y} &= \frac{1.95}{4.2} \\ 4.2y &= 1.95(\sqrt{24^2 + x^2} + y) \\ 2.25y &= 1.95\sqrt{24^2 + x^2} \\ y &= \frac{13\sqrt{24^2 + x^2}}{15} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{13x}{15\sqrt{24^2 + x^2}} \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \frac{13x}{15\sqrt{24^2 + x^2}} \times 5 \\ &= \frac{13x}{3\sqrt{24^2 + x^2}} \end{aligned}$$

When $t = 2$, $x = 10$

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{130}{3\sqrt{24^2 + 10^2}} \\ &= \frac{5}{3} \text{ m/s} \end{aligned}$$

40. (a) Let $U$ be the amount used in millions of tonnes, then

$$\begin{aligned} R &= 5\mathrm{e}^{0.08t} \\ U &= \int_0^{10} R \, \mathrm{d}t \\ &= \int_0^{10} 5\mathrm{e}^{0.08t} \, \mathrm{d}t \\ &= \left[\frac{5\mathrm{e}^{0.08t}}{0.08}\right]_0^{10} \\ &= \left[62.5\mathrm{e}^{0.08t}\right]_0^{10} \\ &= 62.5(\mathrm{e}^{0.8} - \mathrm{e}^0) \\ &= 76.60 \text{ million tonnes} \end{aligned}$$

(b)

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} &= -5\mathrm{e}^{0.08t} \\ A(t) &= \int -5\mathrm{e}^{0.08t} \, \mathrm{d}t \\ &= -62.5\mathrm{e}^{0.08t} + c \\ A(0) &= 200 \\ 200 &= -62.5\mathrm{e}^0 + c \\ c &= 262.5 \\ A(t) &= 262.5 - 62.5\mathrm{e}^{0.08t} \end{aligned}$$

Solving for $t$ when $A(t) = 0$,

$$\begin{aligned} 262.5 - 62.5e^{0.08t} &= 0 \\ 62.5e^{0.08t} &= 262.5 \\ e^{0.08t} &= 4.2 \\ 0.08t &= \ln(4.2) \\ t &= \frac{\ln(4.2)}{0.08} \\ &\approx 17.9 \end{aligned}$$

The resource will be exhausted before the end of the eighteenth year.

41. (a) $x = \sin u \qquad \mathrm{d}x = \cos u \,\mathrm{d}u$

$u = \sin^{-1} x$

$$\begin{aligned} \int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x &= \int \frac{\cos u}{\sqrt{1-\sin^2 u}}\,\mathrm{d}u \\ &= \int \frac{\cos u}{\sqrt{\cos^2 u}}\,\mathrm{d}u \\ &= \int \frac{\cos u}{|\cos u|}\,\mathrm{d}u \\ &= u + c \\ &= \sin^{-1} x + c \end{aligned}$$

Note that it is safe to disregard the absolute value since we can cover all permissable values of $x$ by restricting $u$ to first or fourth quadrants where $\cos u \geq 0$.

(b) $x = 5\sin u \qquad \mathrm{d}x = 5\cos u \,\mathrm{d}u$

$u = \sin^{-1} \frac{x}{5}$

$$\begin{aligned} \int \frac{1}{\sqrt{25-x^2}}\,\mathrm{d}x &= \int \frac{5\cos u}{\sqrt{25-5^2\sin^2 u}}\,\mathrm{d}u \\ &= \int \frac{5\cos u}{\sqrt{25\cos^2 u}}\,\mathrm{d}u \\ &= \int \frac{5\cos u}{|5\cos u|}\,\mathrm{d}u \\ &= u + c \\ &= \sin^{-1} \frac{x}{5} + c \end{aligned}$$

(c) $x = \frac{3}{2}\sin u \qquad \mathrm{d}x = \frac{3}{2}\cos u \,\mathrm{d}u$

$u = \sin^{-1} \frac{2x}{3}$

$$\begin{aligned} \int \frac{1}{\sqrt{9-4x^2}}\,\mathrm{d}x &= \int \frac{\frac{3}{2}\cos u}{\sqrt{9-4\left(\frac{3}{2}\right)^2\sin^2 u}}\,\mathrm{d}u \\ &= \int \frac{\frac{3}{2}\cos u}{\sqrt{9\cos^2 u}}\,\mathrm{d}u \\ &= \int \frac{\frac{3}{2}\cos u}{|3\cos u|}\,\mathrm{d}u \\ &= \frac{u}{2} + c \\ &= \frac{1}{2}\sin^{-1} \frac{2x}{3} + c \end{aligned}$$

(d) $x = \sin u \qquad \mathrm{d}x = \cos u \,\mathrm{d}u$

$u = \sin^{-1} x$

$$\begin{aligned} \int \sqrt{1-x^2}\,\mathrm{d}x &= \int \sqrt{1-\sin^2 u}\cos u \,\mathrm{d}u \\ &= \int \sqrt{\cos^2 u}\cos u \,\mathrm{d}u \\ &= \int \cos^2 u \,\mathrm{d}u \\ &= \frac{1}{2}\int (2\cos^2 u - 1 + 1)\,\mathrm{d}u \\ &= \frac{1}{2}\int (\cos 2u + 1)\,\mathrm{d}u \\ &= \frac{\sin 2u}{4} + \frac{u}{2} + c \\ &= \frac{2\sin u\cos u}{4} + \frac{u}{2} + c \\ &= \frac{\sin u\sqrt{1-\sin^2 u}}{2} + \frac{u}{2} + c \\ &= \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1} x}{2} + c \end{aligned}$$

(e) $x = 2\sin u \qquad \mathrm{d}x = 2\cos u \,\mathrm{d}u$

$u = \sin^{-1} \frac{x}{2}$

$$\begin{aligned} &\int \sqrt{4-x^2}\,\mathrm{d}x \\ &= \int \sqrt{4-4\sin^2 u}(2\cos u)\,\mathrm{d}u \\ &= \int 2\sqrt{4\cos^2 u}\cos u \,\mathrm{d}u \\ &= \int 4\cos^2 u \,\mathrm{d}u \\ &= 2\int (2\cos^2 u - 1 + 1)\,\mathrm{d}u \\ &= 2\int (\cos 2u + 1)\,\mathrm{d}u \\ &= \sin 2u + 2u + c \\ &= 2\sin u\cos u + 2u + c \\ &= 2\sin u\sqrt{1-\sin^2 u} + 2u + c \\ &= x\sqrt{1-\frac{x^2}{4}} + 2\sin^{-1}\frac{x}{2} + c \\ &= \frac{x\sqrt{4-x^2}}{2} + 2\sin^{-1}\frac{x}{2} + c \end{aligned}$$

(f) $x = 2\cos u \qquad \mathrm{d}x = -2\sin u\,\mathrm{d}u$

$u = \cos^{-1}\frac{x}{2}$

$$\begin{aligned}
\int \sqrt{4-x^2}\,\mathrm{d}x \\
&= \int \sqrt{4-4\cos^2 u}(-2\sin u)\,\mathrm{d}u \\
&= \int -2\sqrt{4\sin^2 u}\sin u\,\mathrm{d}u \\
&= \int -4\sin^2 u\,\mathrm{d}u \\
&= 2\int (1-2\sin^2 u - 1)\,\mathrm{d}u \\
&= 2\int (\cos 2u - 1)\,\mathrm{d}u \\
&= \sin 2u - 2u + c \\
&= 2\sin u\cos u - 2u + c \\
&= 2\sqrt{1-\cos^2 u}\cos -2u + c \\
&= x\sqrt{1-\frac{x^2}{4}} - 2\cos^{-1}\frac{x}{2} + c \\
&= \frac{x\sqrt{4-x^2}}{2} - 2\cos^{-1}\frac{x}{2} + c
\end{aligned}$$

(Comparing (e) and (f) might suggest that

$$\sin^{-1} x = -\cos^{-1} x$$

but this is not the case because the constants of integration in these two answers are different.)