Chapter 6

Exercise 6A

1. (a) $N = -\log_{10}(6.4 \times 10^{-8}) = 7.19$

   (b)
   $$\begin{aligned} -N &= \log_{10}(2L) \\ 10^{-N} &= 2L \\ L &= \frac{10^{-N}}{2} \\ &= \frac{10^{-9.5}}{2} \\ &= 1.58 \times 10^{-10} \end{aligned}$$
   (Alternatively, use calculator skills to solve this.)

2. (a) $x = \dfrac{1}{\log 2} \times \log \dfrac{50}{20}$
   $= 1.32$ octaves

   (b)
   $$\begin{aligned} 3 &= \frac{1}{\log 2} \times \log \frac{f_2}{f_1} \\ 3\log 2 &= \log \frac{f_2}{f_1} \\ \log 2^3 &= \log \frac{f_2}{f_1} \\ 8 &= \frac{f_2}{f_1} \\ f_2 &= 8f_1 \end{aligned}$$

3. (a)
   $$\begin{aligned} 7 &= -\log(\mathrm{H}^+) \\ \log(\mathrm{H}^+) &= -7 \\ \mathrm{H}^+ &= 10^{-7} \text{ moles per litre} \end{aligned}$$

   (b) $\mathrm{pH} = -\log(0.01)$
   $= 2$

   (c) $\mathrm{pH} = -\log(4 \times 10^{-8})$
   $= 7.40$

4. (a) $\mathrm{logit}(0.2) = \ln\left(\dfrac{0.2}{0.8}\right)$
   $= -1.39$

   (b)
   $$\begin{aligned} 4 &= \ln\left(\frac{p}{1-p}\right) \\ \frac{p}{1-p} &= e^4 \\ p &= e^4 - e^4 p \\ p + e^4 p &= e^4 \\ p(1+e^4) &= e^4 \\ p &= \frac{e^4}{1+e^4} \\ &= 0.98 \end{aligned}$$

   (c) If p is negative, then
   $$\begin{aligned} \frac{p}{1-p} &< 1 \\ p &< 1-p \\ 2p &< 1 \\ p &< 0.5 \end{aligned}$$
   which is to say that the event has a less than even chance of occurring.

   (d)
   $$\begin{aligned} \ln\left(\frac{x}{1-x}\right) &= k \\ \frac{x}{1-x} &= e^k \\ x &= e^k(1-x) \\ &= e^k + e^k x \\ x + e^k x &= e^k \\ x(1+e^k) &= e^k \\ x &= \frac{e^k}{1+e^k} \end{aligned}$$
   For real k, $e^k > 0$. From this we can conclude that the value of x is positive, and that the denominator is greater than the numerator, hence
   $$0 < x < 1 \quad \forall\, k \in \Re$$

5. No working required.

Exercise 6B

1. $A = A_0 e^{1.5t}$
   $= 100e^{1.5t}$

   (a) $A = 100e^{1.5} = 488$

   (b) $A = 100e^{5\times1.5} = 180\,804 \approx 181\,000$

2. $P = P_0 e^{0.25t}$
   $= 5\,000e^{0.25t}$

   (a) $A = 5,000e^{0.25\times5} = 17\,452 \approx 17\,000$

   (b) $A = 5,000e^{0.25\times25} = 2\,590\,064 \approx 2\,600\,000$

3. $A = 500e^{0.02t}$
   (a) $A = 500e^{0.2} = 611$
   (b) $A = 500e^{0.5} = 824$

4. $Q = 100\,000e^{-0.01t}$
   (a) $Q = 100\,000e^{-0.2} = 81\,873 \approx 82\,000$
   (b) $Q = 100\,000e^{-0.5} = 60\,653 \approx 61\,000$

5. $$\begin{aligned} X &= X_0e^{0.25t} \\ 5\times 10^6 &= X_0e^1 \\ X_0 &= 1\,839\,397 \\ X &= 1\,839\,397e^{0.25t} \end{aligned}$$
   or alternatively
   $$\begin{aligned} X &= X_0e^{0.25t} \\ 5\times 10^6 &= X_0e^1 \\ X_0 &= 5\times 10^6e^{-1} \\ X &= 5\times 10^6e^{-1}e^{0.25t} \\ &= 5\times 10^6e^{0.25t-1} \end{aligned}$$
   (a) $X = 1\,839\,397e^{1.25} = 6\,420\,127 \approx 6.4$ million
   (b) $X = 5\times 10^6e^{6.25-1} = 952\,831\,342 \approx 953$ million

6. $$\begin{aligned} Y &= Y_0e^{0.045t} \\ 25\,000 &= X_0e^{0.45} \\ X_0 &= 25\,000e^{-0.45} \\ X &= 25\,000e^{0.045t-0.45} \end{aligned}$$
   (a) $X = 25\,000e^{0.9-0.45} = 39\,208 \approx 39\,000$
   (b) $X = 25\,000e^{0.045\times 25-0.45} = 49\,101 \approx 49\,000$

7. $$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} &= -0.08A \\ A &= A_0e^{-0.08t} \\ &= 5e^{-0.08\times 25} \\ &= 5e^{-2} \\ &= 0.677\text{kg} \end{aligned}$$

8. $$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} &= -0.02A \\ A &= A_0e^{-0.02t} \\ &= 20e^{-0.02\times 50} \\ &= 20e^{-1} \\ &= 7.36\text{kg} \end{aligned}$$

9. $$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}t} &= 0.025P \\ P &= P_0e^{0.025t} \end{aligned}$$
   (a) $P = 25e^{0.25} \approx 32$ million.
   (b) i. $t = 2030 - 1995 = 35$
   $P = 25e^{0.025\times 35} \approx 60$ million
   ii. $t = 2060 - 1995 = 65$
   $P = 25e^{0.025\times 65} \approx 127$ million

10. (a) $P = 100e^{-0.005} = 99.5\%$
    (b) $P = 100e^{-0.05} = 95\%$
    (c) $P = 100e^{-0.5} = 61\%$
    (d) $$\begin{aligned} 50 &= 100e^{-0.005t} \\ e^{-0.005t} &= 0.5 \\ -0.005t &= \ln 0.5 \\ t &= \frac{\ln 0.5}{-0.005} \\ &= 138.6 \end{aligned}$$
    The element has a half life of about 140 years.

11. $$\begin{aligned} 0.5A_0 &= A_0e^{-0.001t} \\ e^{-0.001t} &= 0.5 \\ e^{0.001t} &= 2 \\ 0.001t &= \ln 2 \\ t &= 1000\ln 2 \\ &= 693 \end{aligned}$$
    The element has a half life of about 690 years.

12. $$\begin{aligned} 0.0008t &= \ln 2 \\ t &= \frac{\ln 2}{0.0008} \\ &= 866 \end{aligned}$$
    The element has a half life of about 870 years.
    (After a few of these half-life problems, the pattern becomes clear and we can take some shortcuts.)

13. $$\begin{aligned} 200 &= 75e^{0.035t} \\ e^{0.035t} &= \frac{200}{75} \\ 0.035t &= \ln\frac{200}{75} \\ t &= \frac{\ln\frac{200}{75}}{0.035} \\ &= 28.02 \end{aligned}$$
    Population will reach 200 million in approximately 28 years.

14. $$\begin{aligned} t &= \frac{\ln 2}{0.0004} \\ &= 1733 \text{ years} \end{aligned}$$

15. $$\begin{aligned} t &= \frac{\ln 2}{0.009} \\ &= 77 \text{ years} \end{aligned}$$

16. (a) No calculations are needed (based on the definition of half-life, half the 20kg must be left after one half-life.)
    (b) This is similarly straightforward. 100 years is twice the half-life, so the amount has halved twice to 5kg.

(c) From the definition of half life,

$$\begin{aligned} e^{-50k} &= 0.5 \\ \therefore \quad 20e^{-75k} &= 20e^{-50k\times 1.5} \\ &= 20\times 0.5^{1.5} \\ &= 7.07 \text{ kg} \end{aligned}$$

17. $P = 500e^{1.5t}$ or $t = \frac{\ln\left(\frac{P}{500}\right)}{1.5}$

(a) $$\begin{aligned} t &= \frac{\ln\left(\frac{1\,000\,000}{500}\right)}{1.5} \\ &= \frac{\ln(2\,000)}{1.5} \\ &= 5.07 \text{ hours} \\ &= 5 \text{ hours } 4 \text{ minutes} \end{aligned}$$

(b) $$\begin{aligned} t &= \frac{\ln\left(\frac{2\,000\,000}{500}\right)}{1.5} \\ &= \frac{\ln(4\,000)}{1.5} \\ &= 5.53 \text{ hours} \\ &= 5 \text{ hours } 32 \text{ minutes} \end{aligned}$$

The doubling time is the difference between the answers to (a) and (b), i.e. 28 minutes.

18. (a) Based on the half-life, 500g will remain after 30 years.

(b) This is two half lives, so the amount remaining will be

$$1000\left(\tfrac{1}{2}\right)^2 = 250 \text{ g}$$

(c) This is $\frac{4}{3}$ half lives, so the amount remaining will be

$$1000\left(\frac{1}{2}\right)^{\frac{4}{3}} = 397 \text{ g}$$

19. $$\begin{aligned} M &= M_0e^{-kt} \\ \frac{M}{M_0} &= e^{-kt} \\ e^{-250\,000k} &= 0.5 \\ \frac{M}{M_0} &= e^{-250\,000k\times\frac{t}{250\,000}} \\ &= \left(e^{-250\,000k}\right)^{\frac{t}{250\,000}} \\ &= 0.5^{\frac{t}{250\,000}} \\ &= 0.5^{\frac{5\,000}{250\,000}} \\ &= 0.986 \end{aligned}$$

98.6% remains after 5 000 years.

20. $$\begin{aligned} P &= P_0e^{kt} \\ 31\,250\,000 &= 18\,500\,000e^{15k} \\ k &= \frac{\ln\frac{31\,250\,000}{18\,500\,000}}{15} \\ &= 0.0349 \end{aligned}$$

The growth rate about is 3.5% per annum.

21. $$\begin{aligned} P &= P_0e^{kt} \\ 56 &= 325e^{8k} \\ k &= \frac{\ln\frac{56}{325}}{8} \\ &= -0.220 \end{aligned}$$

Population declined by about 22% per annum.

22. $$\begin{aligned} P_8 &= P_0e^{kt} \\ 1250 &= 200e^{8k} \\ k &= \frac{\ln\frac{1250}{200}}{8} \\ &= 0.229 P_{12} \quad = 200e^{12k} \\ &= 3125 \end{aligned}$$

23. $$\begin{aligned} e^{5k} &= 2 \\ k &= \frac{\ln 2}{5} \\ &= 0.139 \end{aligned}$$

The claim amounts to a 13.9%p.a. interest rate, compounding continuously.

24. $$\begin{aligned} \frac{P}{P_0} &= e^{-0.022t} \\ 0.6 &= e^{-0.022t} \\ t &= \frac{\ln(0.6)}{-0.022} \\ &= 23.22 \end{aligned}$$

A top-up dose will be required after 23 minutes.

25. $$\begin{aligned} \frac{C}{C_0} &= e^{kt} \\ 0.5 &= e^{5700k} \\ k &= \frac{\ln 0.5}{5700} \\ &= -0.0001216 \\ 0.6 &= e^{kt} \\ t &= \frac{\ln 0.6}{k} \\ &\approx 4\,200 \text{ years} \end{aligned}$$

26. $$\begin{aligned} \frac{M}{M_0} &= e^{kt} \\ 0.5 &= e^{30k} \\ k &= \frac{\ln(0.5)}{30} \\ &= -0.0231 \\ \frac{1}{15} &= e^{kt} \\ t &= \frac{\ln\frac{1}{15}}{k} \\ &= 117.2 \end{aligned}$$

The area should be considered unsafe for 118 years. (It becomes 'safe' a couple of months into the 118th year. In this situation it makes sense to round answers up rather than to the nearest year.)

27. (a)

$$2 = e^{\frac{p}{100}t}$$

$$t = \frac{\ln 2}{\frac{p}{100}} = \frac{100 \ln 2}{p}$$

$$100 \ln 2 \approx 69.3$$

$$\therefore \quad t \approx \frac{69.3}{p}$$

(b) Because 72 is a multiple of 2, 3, 4, 6, 8, 9, 12 and 18. This makes it easy to divide by common interest rates and this ease of calculation is important in a rule of thumb.

28.

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - 28)$$

$$\int \frac{dT}{T-28} = -k \int dt$$

$$\ln(T - 28) = -kt + c$$

when $t = 0$

$$c = \ln(T_0 - 28)$$

$$e^c = T_0 - 28$$

$$\therefore \quad T - 28 = e^{-kt+c} = e^c e^{-kt} = (T_0 - 28)e^{-kt}$$

$$T = (T_0 - 28)e^{-kt} + 28$$

Let $t = 0$ represent the time the object was first placed. Let $t = x$ be the time of the first measurement of 135°C. The time of the second measurement of 91°C is then $t = x + 10$.

$$\begin{aligned}
135 - 28 &= (240 - 28)e^{-kx} \\
107 &= 212e^{-kx} \\
91 - 28 &= (240 - 28)e^{-k(x+10)} \\
63 &= 212e^{-kx-10k} \\
63 &= 212e^{-kx}e^{-10k} \\
63 &= 107e^{-10k} \\
e^{-10k} &= \frac{63}{107} \\
k &= \frac{\ln \frac{63}{107}}{-10} \\
&= 0.0530 \\
e^{-kx} &= \frac{107}{212} \\
x &= \frac{\ln \frac{107}{212}}{-k} \\
&= 12.91 \text{ minutes}
\end{aligned}$$

The item was in the 28°C environment for about 13 minutes before the 135°C temperature was recorded.

Exercise 6C

1. With the product rule:

$$\begin{aligned}
y &= x^3(2x+1)^5 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= 3x^2(2x+1)^5 + x^3(5)(2x+1)^4(2) \\
&= 3x^2(2x+1)^5 + 10x^3(2x+1)^4 \\
&= x^2(2x+1)^4(3(2x+1) + 10x) \\
&= x^2(2x+1)^4(6x + 3 + 10x) \\
&= x^2(2x+1)^4(16x+3)
\end{aligned}$$

Using logarithmic differentiation:

$$\begin{aligned}
y &= x^3(2x+1)^5 \\
\ln y &= \ln(x^3(2x+1)^5) \\
&= \ln(x^3) + \ln((2x+1)^5) \\
&= 3\ln(x) + 5\ln(2x+1) \\
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3}{x} + \frac{5 \times 2}{2x+1} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= y\left(\frac{3}{x} + \frac{10}{2x+1}\right) \\
&= y\left(\frac{3(2x+1) + 10x}{x(2x+1)}\right) \\
&= (x^3(2x+1)^5)\left(\frac{16x+3}{x(2x+1)}\right) \\
&= x^2(2x+1)^4(16x+3)
\end{aligned}$$

2. With the chain rule:

$$\begin{aligned} y &= (3x^2-2)^5 \\ &= u^5 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \\ &= 5u^4(6x) \\ &= 30xu^4 \\ &= 30x(3x^2-2)^4 \end{aligned}$$

Using logarithmic differentiation:

$$\begin{aligned} y &= (3x^2-2)^5 \\ \ln y &= \ln((3x^2-2)^5) \\ &= 5\ln(3x^2-2) \\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{5}{3x^2-2}6x \\ &= \frac{30x}{3x^2-2} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{30xy}{3x^2-2} \\ &= \frac{30x(3x^2-2)^5}{3x^2-2} \\ &= 30x(3x^2-2)^4 \end{aligned}$$

3. With the quotient rule:

$$\begin{aligned} y &= \frac{x^3}{x^2+1} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3x^2(x^2+1)-x^3(2x)}{(x^2+1)^2} \\ &= \frac{3x^4+3x^2-2x^4}{(x^2+1)^2} \\ &= \frac{x^4+3x^2}{(x^2+1)^2} \end{aligned}$$

Using logarithmic differentiation:

$$\begin{aligned} y &= \frac{x^3}{x^2+1} \\ \ln y &= \ln\frac{x^3}{x^2+1} \\ &= \ln(x^3)-\ln(x^2+1) \\ &= 3\ln(x)-\ln(x^2+1) \\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3}{x}-\frac{2x}{x^2+1} \\ &= \frac{3(x^2+1)-2x^2}{x(x^2+1)} \\ &= \frac{3x^2+3-2x^2}{x(x^2+1)} \\ &= \frac{x^2+3}{x(x^2+1)} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= y\left(\frac{x^2+3}{x(x^2+1)}\right) \\ &= \frac{x^3}{x^2+1}\left(\frac{x^2+3}{x(x^2+1)}\right) \\ &= \frac{x^2(x^2+3)}{(x^2+1)^2} \\ &= \frac{x^4+3x^2}{(x^2+1)^2} \end{aligned}$$

4. (a)

$$\begin{aligned} y &= x^x \\ \ln y &= x\ln x \\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \ln(x)+\frac{x}{x} \\ &= \ln(x)+1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= y(\ln(x)+1) \\ &= x^x(\ln(x)+1) \end{aligned}$$

(b)

$$\begin{aligned} y &= x^{2x} \\ \ln y &= 2x\ln x \\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= 2\ln(x)+\frac{2x}{x} \\ &= 2\ln(x)+2 \\ &= 2(\ln(x)+1) \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= 2y(\ln(x)+1) \\ &= 2x^{2x}(\ln(x)+1) \end{aligned}$$

(c)

$$\begin{aligned} y &= x^{\cos x} \\ \ln y &= \cos(x)\ln x \\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\sin(x)\ln(x)+\frac{\cos(x)}{x} \\ &= \frac{\cos(x)-x\sin(x)\ln(x)}{x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y(\cos(x)-x\sin(x)\ln(x))}{x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x^{cos(x)}(\cos(x)-x\sin(x)\ln(x))}{x} \\ &= x^{cos(x)-1}(\cos(x)-x\sin(x)\ln(x)) \end{aligned}$$

(d)

$$\begin{aligned} y &= x^{\sin x} \\ \ln y &= \sin(x)\ln x \\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \cos(x)\ln(x)+\frac{\sin(x)}{x} \\ &= \frac{x\cos(x)\ln(x)+\sin(x)}{x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y(x\cos(x)\ln(x)+\sin(x))}{x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x^{sin(x)}(x\cos(x)\ln(x)+\sin(x))}{x} \\ &= x^{sin(x)-1}(x\cos(x)ln(x)+\sin(x)) \end{aligned}$$

(e)
$$\begin{aligned}
y &= \sqrt{\frac{3x+1}{3x-1}}\\
\ln y &= \frac{1}{2}\left(\ln(3x+1) - \ln(3x-1)\right)\\
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2}\left(\frac{3}{3x+1} - \frac{3}{3x-1}\right)\\
&= \frac{3}{2}\left(\frac{1}{3x+1} - \frac{1}{3x-1}\right)\\
&= \frac{3}{2}\left(\frac{(3x-1)-(3x+1)}{(3x+1)(3x-1)}\right)\\
&= \frac{3}{2}\left(\frac{-2}{(3x+1)(3x-1)}\right)\\
&= \frac{-3}{(3x+1)(3x-1)}\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= y\left(\frac{-3}{(3x+1)(3x-1)}\right)\\
&= \sqrt{\frac{3x+1}{3x-1}}\left(\frac{-3}{(3x+1)(3x-1)}\right)\\
&= \frac{-3\sqrt{3x+1}}{(3x+1)(3x-1)\sqrt{3x-1}}\\
&= \frac{-3\sqrt{(3x+1)(3x-1)}}{(3x+1)(3x-1)^2}\\
&= -3(3x+1)^{-0.5}(3x-1)^{-1.5}
\end{aligned}$$

(f)
$$\begin{aligned}
y &= \sqrt{\frac{1+x}{2-x}}\\
\ln y &= \frac{1}{2}\left(\ln(1+x) - \ln(2-x)\right)\\
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2}\left(\frac{1}{1+x} - \frac{-1}{2-x}\right)\\
&= \frac{1}{2}\left(\frac{(2-x)+(1+x)}{(1+x)(2-x)}\right)\\
&= \frac{3}{2(1+x)(2-x)}\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= y\left(\frac{3}{2(1+x)(2-x)}\right)\\
&= \sqrt{\frac{1+x}{2-x}}\left(\frac{3}{2(1+x)(2-x)}\right)\\
&= 1.5(1+x)^{-0.5}(2-x)^{-1.5}
\end{aligned}$$

Miscellaneous Exercise 6

1. (a) No working needed.

(b) No working needed.

(c)
$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2(x+3)-(2x-1)}{(x+3)^2}\\
&= \frac{2x+6-2x+1}{(x+3)^2}\\
&= \frac{7}{(x+3)^2}
\end{aligned}$$

(d) No working needed.

(e) No working needed.

(f)
$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= 2\cos(x)(-\sin x)\\
&= -2\cos x\sin x\\
&= -\sin 2x
\end{aligned}$$

(g) No working needed.

(h) No working needed.

(i) No working needed.

(j)
$$\begin{aligned}
y &= x(\sin^2 x + \cos^2 x)\\
&= x\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= 1
\end{aligned}$$

(k) No working needed.

(l)
$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= e^{\sin x} + xe^{\sin x}(\cos x)\\
&= e^{\sin x}(1 + x\cos x)
\end{aligned}$$

(m)
$$\begin{aligned}
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= 6x\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= 6xy
\end{aligned}$$

(n)
$$\begin{aligned}
4y + 4x\frac{\mathrm{d}y}{\mathrm{d}x} + 5y^4\frac{\mathrm{d}y}{\mathrm{d}x} - 15 &= 8\cos 2x\\
\frac{\mathrm{d}y}{\mathrm{d}x}(4x+5y^4) &= 8\cos(2x) - 4y + 15\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{8\cos(2x) - 4y + 15}{4x+5y^4}
\end{aligned}$$

(o)
$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= 4t^3\\
\frac{\mathrm{d}x}{\mathrm{d}t} &= 2t-3\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}\\
&= \frac{4t^3}{2t-3}
\end{aligned}$$

(p) $\dfrac{dy}{dt} = 15\cos 5t$

$\dfrac{dx}{dt} = 2\cos t$

$\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx}$

$= \dfrac{15\cos 5t}{2\cos t}$

2. $e^{kt} = \dfrac{N}{N_0}$

$e^{5k} = 2$

$5k = \ln 2$

$k = 0.2\ln 2$

≈ 0.139

3. For any matrices X and Y, for XY to be a possible product, we need columns(X)=rows(Y). Thus

(a) AB is possible $(1 = 1)$
(b) AC is not possible $(1 \neq 2)$
(c) BC is possible $(2 = 2)$
(d) CB is not possible $(2 \neq 1)$
(e) BD is possible $(2 = 2)$
(f) CD is possible $(2 = 2)$
(g) AD is not possible $(1 \neq 2)$
(h) DA is possible $(3 = 3)$

4. (a) $\mathrm{AB} = \begin{bmatrix} 5+0 & -5+9 \\ -2+0 & 2-3 \end{bmatrix}$

$= \begin{bmatrix} 5 & 4 \\ -2 & -1 \end{bmatrix}$

(b) $\det \mathrm{A} = (5)(-1) - (3)(-2)$

$= 1$

(c) $\mathrm{A}^{-1} = \dfrac{1}{\det \mathrm{A}} \begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix}$

$= \begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix}$

(d) $\mathrm{B}^{-1} = \dfrac{1}{(1)(3) - (-1)(0)} \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$

(e) $\mathrm{C} = \mathrm{A}^{-1}\mathrm{B}$

$= \begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$

$= \begin{bmatrix} -1+0 & 1-9 \\ 2+0 & -2+15 \end{bmatrix}$

$= \begin{bmatrix} -1 & -8 \\ 2 & 13 \end{bmatrix}$

(f) $\mathrm{D} = \mathrm{BA}^{-1}$

$= \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix}$

$= \begin{bmatrix} -1-2 & -3-5 \\ 0+6 & 0+15 \end{bmatrix}$

$= \begin{bmatrix} -3 & -8 \\ 6 & 15 \end{bmatrix}$

5. (a) No working required.

(b) $\mathrm{T} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 0 \end{bmatrix}$

$\mathrm{T} = \begin{bmatrix} 5 & 4 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 5 & 4 \\ 3 & 0 \end{bmatrix} \dfrac{1}{-3} \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}$

$= \dfrac{1}{3} \begin{bmatrix} 5 & 4 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$

$= \dfrac{1}{3} \begin{bmatrix} 5+4 & 5-8 \\ 3+0 & 3+0 \end{bmatrix}$

$= \dfrac{1}{3} \begin{bmatrix} 9 & -3 \\ 3 & 3 \end{bmatrix}$

$= \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$

6. $\mathrm{PQ} = \mathrm{R}$

$\mathrm{P} = \mathrm{RQ}^{-1}$

$= \begin{bmatrix} 6 & 1 & 4 \\ 7 & 5 & 3 \\ -3 & 3 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & 3 \\ -2 & -1 & 1 \end{bmatrix}$

7. (a) No working required.

(b) No working required.

8. (a) $\begin{bmatrix} 4 \\ 1 \end{bmatrix} \begin{bmatrix} -3, 5 \end{bmatrix} = \begin{bmatrix} -12 & 20 \\ -3 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} -3, 5 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -12+5 \end{bmatrix} = \begin{bmatrix} -7 \end{bmatrix}$

9.
- AB is not possible (A has 1 column; B has 2 rows)
- AC is possible (columns(A)=rows(C)=1) and has size rows(A)×columns(C)= (3×4)
- BA is possible (columns(B)=rows(A)=3 and has size rows(B)×columns(A)= (2×1)
- BC is not possible (B has 3 columns; C has 1 row)
- CA is not possible (C has 4 columns; A has 3 rows)
- CB is not possible (C has 4 columns; B has 2 rows)

Thus A can pre-multiply C and B can pre-multiply A so BAC is a possible product and has dimensions rows(B)×columns(C)= (2×4).

$\mathrm{BAC} = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 10 \\ 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 10 & 0 & 10 & 10 \\ 8 & 0 & 8 & 8 \end{bmatrix}$

10.
$$AB = BA$$
$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 3x & 3y \\ 0 & z \end{bmatrix} = \begin{bmatrix} 3x & y \\ 0 & z \end{bmatrix}$$

This gives us no restriction on x or z (since $3x = 3x$ is true for all x, and $z = z$ for all z), but y must be zero (since $3y = y$ is only true for $y = 0$).

11.
$$\begin{aligned}
M^{-1} &= \frac{1}{2}\begin{bmatrix} a & 1 \\ -2 & 0 \end{bmatrix} \\
M^{-1}M^{-1} &= \frac{1}{2}\begin{bmatrix} a & 1 \\ -2 & 0 \end{bmatrix}\frac{1}{2}\begin{bmatrix} a & 1 \\ -2 & 0 \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} a^2-2 & a \\ -2a & -2 \end{bmatrix} \\
\therefore \quad \begin{bmatrix} b & 1 \\ c & d \end{bmatrix} &= \frac{1}{4}\begin{bmatrix} a^2-2 & a \\ -2a & -2 \end{bmatrix} \\
\begin{bmatrix} 4b & 4 \\ 4c & 4d \end{bmatrix} &= \begin{bmatrix} a^2-2 & a \\ -2a & -2 \end{bmatrix} \\
\therefore \quad a &= 4 \\
\therefore \quad \begin{bmatrix} 4b & 4 \\ 4c & 4d \end{bmatrix} &= \begin{bmatrix} 14 & 4 \\ -8 & -2 \end{bmatrix} \\
b &= \frac{14}{4} = \frac{7}{2} \\
c &= -2 \\
d &= -\frac{2}{4} = -\frac{1}{2}
\end{aligned}$$

12. Without loss of generality, consider just one point:

$$\begin{aligned}
P' &= \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} P \\
P'' &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} P' \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} P \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 5 \end{bmatrix} P
\end{aligned}$$

Thus the single matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 5 \end{bmatrix}$

13.
$$\begin{aligned}
\begin{bmatrix} 2x & x \\ 4 & y \end{bmatrix}^2 &= \begin{bmatrix} 24 & p \\ 0 & q \end{bmatrix} \\
\begin{bmatrix} 2x & x \\ 4 & y \end{bmatrix}\begin{bmatrix} 2x & x \\ 4 & y \end{bmatrix} &= \begin{bmatrix} 24 & p \\ 0 & q \end{bmatrix} \\
\begin{bmatrix} 4x^2+4x & 2x^2+xy \\ 8x+4y & 4x+y^2 \end{bmatrix} &= \begin{bmatrix} 24 & p \\ 0 & q \end{bmatrix} \\
4x^2+4x &= 24 \\
4x^2+4x-24 &= 0 \\
x^2+x-6 &= 0 \\
(x+3)(x-2) &= 0 \\
x &= -3 \\
\text{or} \qquad x &= 2
\end{aligned}$$

for $x = -3$:
$$\begin{aligned}
\begin{bmatrix} 4(-3)^2+4(-3) & 2(-3)^2+(-3)y \\ 8(-3)+4y & 4(-3)+y^2 \end{bmatrix} &= \begin{bmatrix} 24 & p \\ 0 & q \end{bmatrix} \\
\begin{bmatrix} 24 & 18-3y \\ 4y-24 & y^2-12 \end{bmatrix} &= \begin{bmatrix} 24 & p \\ 0 & q \end{bmatrix} \\
4y-24 &= 0 \\
y-6 &= 0 \\
y &= 6 \\
\begin{bmatrix} 24 & 18-3(6) \\ 4(6)-24 & (6)^2-12 \end{bmatrix} &= \begin{bmatrix} 24 & p \\ 0 & q \end{bmatrix} \\
\begin{bmatrix} 24 & 0 \\ 0 & 24 \end{bmatrix} &= \begin{bmatrix} 24 & p \\ 0 & q \end{bmatrix} \\
p &= 0 \\
q &= 24
\end{aligned}$$

for $x = 2$:
$$\begin{aligned}
\begin{bmatrix} 4(2)^2+4(2) & 2(2)^2+(2)y \\ 8(2)+4y & 4(2)+y^2 \end{bmatrix} &= \begin{bmatrix} 24 & p \\ 0 & q \end{bmatrix} \\
\begin{bmatrix} 24 & 8+2y \\ 16+4y & 8+y^2 \end{bmatrix} &= \begin{bmatrix} 24 & p \\ 0 & q \end{bmatrix} \\
16+4y &= 0 \\
4+y &= 0 \\
y &= -4 \\
\begin{bmatrix} 24 & 8+2(-4) \\ 16+4(-4) & 8+(-4)^2 \end{bmatrix} &= \begin{bmatrix} 24 & p \\ 0 & q \end{bmatrix} \\
\begin{bmatrix} 24 & 0 \\ 0 & 24 \end{bmatrix} &= \begin{bmatrix} 24 & p \\ 0 & q \end{bmatrix} \\
p &= 0 \\
q &= 24
\end{aligned}$$

Thus $p = 0$ and $q = 24$ and $(x, y) \in \{(-3, 6), (2, -4)\}$

14. (a)
$$\begin{aligned}
A^2 &= BCB^{-1}BCB^{-1} \\
&= BCCB^{-1} \\
&= BC^2B^{-1}
\end{aligned}$$

(b)
$$\begin{aligned}
A^3 &= A^2A \\
&= BC^2B^{-1}BCB^{-1} \\
&= BC^2CB^{-1} \\
&= BC^3B^{-1}
\end{aligned}$$

(c) $A^n = BC^nB^{-1}$

(You should be able to see how you could use mathematical induction to prove this quite simply.)

15. $$L = \begin{bmatrix} 0 & 1.7 & 2.8 & 0.2 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \end{bmatrix}$$

$$P_0 = \begin{bmatrix} 1020 \\ 1560 \\ 1100 \\ 540 \end{bmatrix}$$

(a) $$P_6 = L^3 P_0 = \begin{bmatrix} 4750 \\ 1370 \\ 1402 \\ 122 \end{bmatrix}$$

There will be about 122 4th generation females in 6 years.

(b) $$P_{10} = L^5 P_0 = \begin{bmatrix} 5672 \\ 2511 \\ 1140 \\ 411 \end{bmatrix}$$

There will be about 2511 2nd generation females in 10 years.

16. To prove: $2^{n-1} + 3^{2n+1}$ is a multiple of 7 for all integer n, $n \geq 1$.

For $n = 1$,

$$\begin{aligned} 2^{n-1} + 3^{2n+1} &= 2^{1-1} + 3^{2(1)+1} \\ &= 2^0 + 3^3 \\ &= 28 \\ &= 7 \times 4 \end{aligned}$$

Assume the proposition is true for $n = k$, i.e.

$$2^{k-1} + 3^{2k+1} = 7a$$

for some integer a.

Then for $n = k + 1$ we need to demonstrate that

$$2^{k+1-1} + 3^{2(k+1)+1} = 2^k + 3^{2k+3}$$

is a multiple of 7.

$$\begin{aligned} 2^k + 3^{2k+3} &= 2(2^{k-1}) + 9(3^{2k+1}) \\ &= 2(2^{k-1}) + (2+7)(3^{2k+1}) \\ &= 2(2^{k-1}) + 2(3^{2k+1}) + 7(3^{2k+1}) \\ &= 2(2^{k-1} + 3^{2k+1}) + 7(3^{2k+1}) \\ &= 2(7a) + 7(3^{2k+1}) \\ &= 7(2a + 3^{2k+1}) \end{aligned}$$

which is a multiple of 7 as required.

Therefore, by mathematical induction $2^{n-1} + 3^{2n+1}$ is a multiple of 7 for all integer n, $n \geq 1$. □

17. $$\begin{aligned} \frac{dy}{dx} &= 0 \\ 4x - \frac{1}{x} &= 0 \\ 4x^2 - 1 &= 0 \\ x^2 &= \frac{1}{4} \\ x &= \frac{1}{2} \\ y &= 2(\frac{1}{2})^2 - \log_e \frac{1}{2} \\ &= \frac{1}{2} + \log_e 2 \\ \frac{d^2y}{dx^2} &= 4 + \frac{1}{x^2} \end{aligned}$$

at $x = \frac{1}{2}$, $\frac{d^2y}{dx^2} = 8$

$\frac{d^2y}{dx^2} > 0 \implies \frac{dy}{dx}$ is increasing so the stationary point at $(\frac{1}{2}, \frac{1}{2} + \log_e 2)$ is a minimum.

18. (a) $$\begin{aligned} \frac{P}{P_0} &= e^{0.1t} \\ t &= 10 \ln \frac{P}{P_0} \\ &= 10 \ln 2 \\ &\approx 6.93 \text{ years} \\ &\approx 6 \text{ years 11 months.} \end{aligned}$$

(b) $$\begin{aligned} t &= 10 \ln \frac{40\,000}{10\,000} \\ &= 10 \ln 4 \\ &\approx 13.86 \text{ years} \\ &\approx 13 \text{ years 10 months.} \end{aligned}$$

(c) $$\begin{aligned} t &= 10 \ln \frac{80\,000}{10\,000} \\ &= 10 \ln 8 \\ &\approx 20.79 \text{ years} \\ &\approx 20 \text{ years 10 months.} \end{aligned}$$

Note that the answer to (b) is double the answer to (a) since the principal has to double twice. Similarly, the answer to (c) is three times the answer to (a) since the principle has to double three times ($8 = 2^3$).

19. $$\begin{aligned} \frac{dN}{dt} &= -0.18N \\ N_0 &= 12000 \\ N &= N_0 e^{-0.18t} \\ e^{-0.18t} &= \frac{2\,000}{12\,000} \\ -0.18t &= \ln \frac{1}{6} \\ 0.18t &= \ln 6 \\ t &= \frac{\ln 6}{0.18} \end{aligned}$$

≈ 9.95

The critical situation will occur in about 10 years time.

20. (a) No working required.

(b) i. $\begin{bmatrix} 5\,465 & 2\,535 \end{bmatrix} \begin{bmatrix} 0.98 & 0.02 \\ 0.03 & 0.97 \end{bmatrix} = \begin{bmatrix} 5\,402 & 2\,568 \end{bmatrix}$

(populations shown in thousands).

ii. $\mathrm{P} \begin{bmatrix} 0.98 & 0.02 \\ 0.03 & 0.97 \end{bmatrix} = \begin{bmatrix} 5\,465 & 2\,535 \end{bmatrix}$

$$\mathrm{P} = \begin{bmatrix} 5\,465 & 2\,535 \end{bmatrix} \begin{bmatrix} 0.98 & 0.02 \\ 0.03 & 0.97 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 5\,469 & 2\,501 \end{bmatrix}$$

(populations shown in thousands).

(c) $$\begin{bmatrix} 0.98 & 0.02 \\ 0.03 & 0.97 \end{bmatrix}^{10} = \begin{bmatrix} 0.84 & 0.16 \\ 0.24 & 0.76 \end{bmatrix}$$

$$\begin{bmatrix} 0.98 & 0.02 \\ 0.03 & 0.97 \end{bmatrix}^{50} = \begin{bmatrix} 0.63 & 0.37 \\ 0.55 & 0.45 \end{bmatrix}$$

$$\begin{bmatrix} 0.98 & 0.02 \\ 0.03 & 0.97 \end{bmatrix}^{100} = \begin{bmatrix} 0.60 & 0.40 \\ 0.60 & 0.40 \end{bmatrix}$$

After about a hundred years, everything else being equal(!), the population would stabilize with 60% of the total in the city and 40% in the country, i.e.

$$\begin{bmatrix} 5\,465 & 2\,535 \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} = \begin{bmatrix} 4\,782 & 3\,188 \end{bmatrix}$$

(Whether or not this makes sense is a different question. Real population modelling for a city would include calculations like this but would be much more complex as many other factors would need to be taken into consideration. Even then, sensibly forecasting 100 years into the future is well beyond current capabilities.)