Chapter 2

Exercise 2A

1. One solution is $z = 1 = \operatorname{cis} 0$ and the other five solutions divide the complex plane into six equal-sized regions:

$$z \in \left\{ \operatorname{cis} 0, \operatorname{cis} \frac{\pi}{3}, \operatorname{cis} \frac{2\pi}{3}, \operatorname{cis} \pi, \operatorname{cis} -\frac{\pi}{3}, \operatorname{cis} \frac{2\pi}{3} \right\}$$

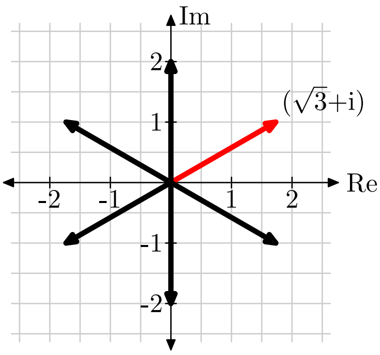
2. One solution is $z = 1 = \operatorname{cis} 0$ and the other five solutions divide the complex plane into eight equal-sized regions:

$$z \in \{\operatorname{cis} 0, \operatorname{cis} \pm 45°, \operatorname{cis} \pm 90°, \operatorname{cis} \pm 135°, \operatorname{cis} 180°\}$$

3. One solution is $z = 1 = \operatorname{cis} 0$ and the other five solutions divide the complex plane into seven equal-sized regions:

$$z \in \left\{ \operatorname{cis} 0, \operatorname{cis} \pm \frac{2\pi}{7}, \operatorname{cis} \pm \frac{4\pi}{7}, \pm \operatorname{cis} \frac{6\pi}{7} \right\}$$

4. Start with the one known root and mark in the other five, each rotated $\frac{\pi}{3}$ from the previous.

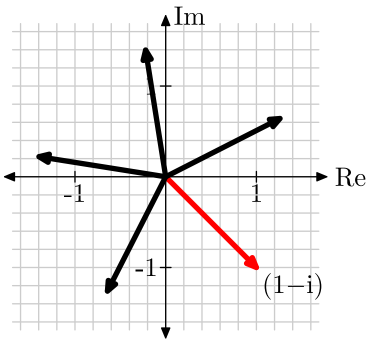


$$\sqrt{3} + \mathrm{i} = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} \operatorname{cis} \left(\tan^{-1} \frac{1}{\sqrt{3}} \right)$$
$$= 2 \operatorname{cis} \frac{\pi}{6}$$

$\therefore z^6 = -64$ has roots

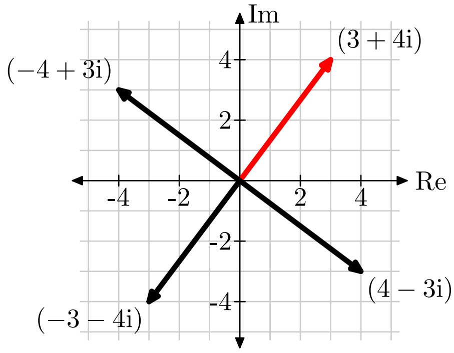
$$z \in \left\{ 2 \operatorname{cis} \pm \frac{\pi}{6}, 2 \operatorname{cis} \pm \frac{\pi}{2}, 2 \operatorname{cis} \pm \frac{5\pi}{6} \right\}$$

5. Start with the one known root and mark in the other four, each rotated $\frac{360}{5}^\circ = 72°$ from the previous.



$$1 - \mathrm{i} = \sqrt{2} \operatorname{cis} -45°$$
$$\therefore \quad z \in \left\{ \sqrt{2} \operatorname{cis} -45°, \sqrt{2} \operatorname{cis} -117°, \sqrt{2} \operatorname{cis} 27°, \sqrt{2} \operatorname{cis} 99°, \sqrt{2} \operatorname{cis} 171° \right\}$$

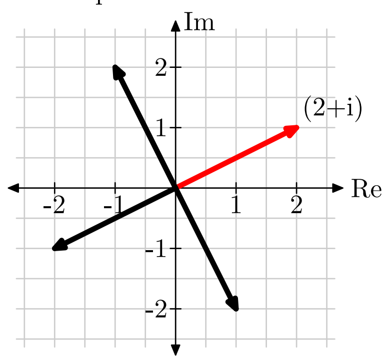
6. Start with the one known root and mark in the other four, each rotated 90° from the previous.



7. (a) $(2+\mathrm{i})^2 = 4 + 4\mathrm{i} + \mathrm{i} \quad 2$
$= 3 + 4\mathrm{i}$

(b) $(2+\mathrm{i})^4 = \left((2+\mathrm{i})^2\right)^2$
$= (3+4\mathrm{i})^2$
$= 9 + 24\mathrm{i} + 16\mathrm{i}^2$
$= -7 + 24\mathrm{i}$

(c) We know that $z = 2 + \mathrm{i}$ is a solution from the preceding work. Start with this solution and find the other three, each rotated 90° from the previous.



(d) Solutions are

$$z \in \{(2+\mathrm{i}), (-1+2\mathrm{i}), (1-2\mathrm{i}), (-2-\mathrm{i})\}$$

8. $k = (2 \operatorname{cis} 20°)^5$
$= (2 \operatorname{cis} 20°)(2 \operatorname{cis} 20°)(2 \operatorname{cis} 20°)^3$
$= (4 \operatorname{cis} 40°)(2 \operatorname{cis} 20°)^2(2 \operatorname{cis} 20°)$
$= (4 \operatorname{cis} 40°)(4 \operatorname{cis} 40°)(2 \operatorname{cis} 20°)$
$= (16 \operatorname{cis} 80°)(2 \operatorname{cis} 20°)$
$= 32 \operatorname{cis}(100)°$

(This is unnecessarily long-winded, but I haven't wanted to anticipate the next section. Students should be able to see how to do this in a single step.)

Now the other four solutions are simply 72° rotations of the first. (The first is also included for completeness.)

$$z \in \{2\operatorname{cis} 20°, 2\operatorname{cis} 92°, 2\operatorname{cis} 164°, 2\operatorname{cis}(-52°), 2\operatorname{cis}(-124)°\}$$

9. The four solutions will be separated by 90°. From previous work it should be clear how this relates to the complex numbers in $a + b\mathrm{i}$ form so that no working is required. (See, for example, question 7.)

Exercise 2B

It is useful for much of the work in this section to be familiar with Pascal's triangle and its role in binomial expansions:

0:							1						
1:						1		1					
2:					1		2		1				
3:				1		3		3		1			
4:			1		4		6		4		1		
5:		1		5		10		10		5		1	
6:	1		6		15		20		15		6		1

etc.

1. To prove:

$$(\cos\theta + \mathrm{i}\sin\theta)^{-1} = \cos -\theta + \mathrm{i}\sin -\theta$$

Proof:

$$\begin{aligned}
\text{L.H.S.} &= (\cos\theta + \mathrm{i}\sin\theta)^{-1}\\
&= \frac{1}{\cos\theta + \mathrm{i}\sin\theta}\\
&= \frac{\cos\theta - \mathrm{i}\sin\theta}{(\cos\theta + \mathrm{i}\sin\theta)(\cos\theta - \mathrm{i}\sin\theta)}\\
&= \frac{\cos\theta - \mathrm{i}\sin\theta}{\cos^2\theta - \mathrm{i}^2\sin^2\theta}\\
&= \frac{\cos\theta - \mathrm{i}\sin\theta}{\cos^2\theta + \sin^2\theta}\\
&= \cos\theta - \mathrm{i}\sin\theta\\
&= \cos -\theta + \mathrm{i}\sin -\theta\\
&= \text{R.H.S.}
\end{aligned}$$

□

2. $$\begin{aligned} z^4 &= \cos\frac{4\pi}{6} + \mathrm{i}\sin\frac{4\pi}{6}\\ &= \cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\end{aligned}$$

3. $$\begin{aligned} z^5 &= 2^5\operatorname{cis}\frac{5\pi}{6}\\ &= 32\operatorname{cis}\frac{5\pi}{6}\end{aligned}$$

4. $$\begin{aligned} z &= 3^5\operatorname{cis}\frac{5\pi}{3}\\ &= 243\operatorname{cis}\left(-\frac{\pi}{3}\right)\\ &= 243\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)\end{aligned}$$

(matching the style of our answer to that of the question.)

5. $$\begin{aligned} \cos 2\theta + \mathrm{i}\sin 2\theta &= (\cos\theta + \mathrm{i}\sin\theta)^2\\ &= \cos^2\theta + 2\mathrm{i}\sin\theta\cos\theta + \mathrm{i}^2\sin^2\theta\\ &= \cos^2\theta - \sin^2\theta + 2\mathrm{i}\sin\theta\cos\theta\end{aligned}$$

Now equating real and imaginary parts we obtain

$$\begin{aligned} \cos 2\theta &= \sin^2\theta - \cos^2\theta\\ \text{and}\ \ \sin 2\theta &= 2\sin\theta\cos\theta\end{aligned}$$

6. $$\begin{aligned} \cos 3\theta + \mathrm{i}\sin 3\theta &= (\cos\theta + \mathrm{i}\sin\theta)^3\\ &= \cos^3\theta + 3\mathrm{i}\sin\theta\cos^2\theta\\ &\quad + 3\mathrm{i}^2\sin^2\theta\cos\theta + \mathrm{i}^3\sin^3\theta\\ &= \cos^3\theta + 3\mathrm{i}\sin\theta\cos^2\theta\\ &\quad - 3\sin^2\theta\cos\theta - \mathrm{i}\sin^3\theta\\ &= \cos^3\theta - 3\sin^2\theta\cos\theta\\ &\quad + \mathrm{i}(3\sin\theta\cos^2\theta - \sin^3\theta)\end{aligned}$$

Now equating real and imaginary parts we obtain

$$\begin{aligned} \sin 3\theta &= 3\sin\theta\cos^2\theta - \sin^3\theta\\ \text{and}\ \ \cos 3\theta &= \cos^3\theta - 3\sin^2\theta\\ &= \cos^3\theta - 3(1 - \cos^2\theta)\\ &= \cos^3\theta + 3\cos^2\theta - 3\end{aligned}$$

7. $$\begin{aligned}\cos 5\theta + \mathrm{i}\sin 5\theta &= (\cos\theta + \mathrm{i}\sin\theta)^5\\ &= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta\\ &\quad - 10\mathrm{i}\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + \mathrm{i}\sin^5\theta\\ &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta\\ &\quad + \mathrm{i}(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)\end{aligned}$$
$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$
$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

8. $$\begin{aligned}(1+\mathrm{i})^6 &= \left(\sqrt{2}\,\mathrm{cis}\,\frac{\pi}{4}\right)^6\\ &= \left(\sqrt{2}\right)^6 \mathrm{cis}\,\frac{6\pi}{4}\\ &= 8\,\mathrm{cis}\,\frac{3\pi}{2}\\ &= 8\,\mathrm{cis}\,\frac{-\pi}{2}\end{aligned}$$

9. $$\begin{aligned}(\sqrt{3}+\mathrm{i})^5 &= \left(\sqrt{\left(\sqrt{3}\right)^2 + 1^2}\,\mathrm{cis}\left(\tan^{-1}\frac{1}{\sqrt{3}}\right)\right)^5\\ &= \left(2\,\mathrm{cis}\,\frac{\pi}{6}\right)^5\\ &= 2^5\,\mathrm{cis}\,\frac{5\pi}{6}\\ &= 32\,\mathrm{cis}\,\frac{5\pi}{6}\end{aligned}$$

10. $$\begin{aligned}(-3+3\sqrt{3}\mathrm{i})^4 &= \left(3(-1+\sqrt{3})\right)^4\\ &= 3^4(-1+\sqrt{3})^4\\ &= 81\left(2\,\mathrm{cis}\left(\tan^{-1}\frac{\sqrt{3}}{-1}\right)\right)^4\\ &= 81\left(2\,\mathrm{cis}\,\frac{2\pi}{3}\right)^4\\ &= 81\times 2^4\,\mathrm{cis}\,\frac{8\pi}{3}\\ &= 1296\,\mathrm{cis}\,\frac{2\pi}{3}\end{aligned}$$

11. $$\begin{aligned}\left(4-4\sqrt{3}\right)^{\frac{1}{3}} &= \left(\sqrt{16+48}\,\mathrm{cis}\left(\tan^{-1}\frac{-4\sqrt{3}}{4}\right)\right)^{\frac{1}{3}}\\ &= \left(8\,\mathrm{cis}\left(-\frac{\pi}{3}\right)\right)^{\frac{1}{3}}\\ &= 2\,\mathrm{cis}\left(-\frac{\pi}{9}\right)\end{aligned}$$
The other two cube roots are rotated by $\frac{2\pi}{3}$, i.e. $2\,\mathrm{cis}\left(-\frac{7\pi}{9}\right)$ and $2\,\mathrm{cis}\,\frac{5\pi}{9}$.

12. $$\begin{aligned}z^4 &= 16\mathrm{i}\\ &= 16\,\mathrm{cis}\,\frac{\pi}{2}\\ z &= \left(16\,\mathrm{cis}\,\frac{\pi}{2}\right)^{\frac{1}{4}}\\ &= 2\,\mathrm{cis}\,\frac{\pi}{8}\end{aligned}$$
The other three roots are rotated by $\frac{\pi}{2}$, i.e. $2\,\mathrm{cis}\left(-\frac{3\pi}{8}\right)$, $2\,\mathrm{cis}\left(-\frac{7\pi}{8}\right)$ and $2\,\mathrm{cis}\,\frac{5\pi}{8}$.

13. $$\begin{aligned}\sqrt{\left(8\sqrt{2}\right)^2 + \left(8\sqrt{2}\right)^2} &= 16\\ z^4 &= 16\,\mathrm{cis}\,\frac{3\pi}{4}\\ z &= 2\,\mathrm{cis}\,\frac{3\pi}{16}\end{aligned}$$
The other three roots are rotated by $\frac{\pi}{2}$, i.e. $2\,\mathrm{cis}\left(-\frac{5\pi}{16}\right)$, $2\,\mathrm{cis}\left(-\frac{13\pi}{16}\right)$ and $2\,\mathrm{cis}\,\frac{11\pi}{16}$.

14. $$\begin{aligned}z^4 + 4 &= 0\\ z^4 &= -4\\ &= 4\,\mathrm{cis}\,\pi\\ z &= \sqrt{2}\,\mathrm{cis}\,\frac{\pi}{4}\end{aligned}$$
The other three roots are rotated by $\frac{\pi}{2}$, i.e. $\sqrt{2}\,\mathrm{cis}\left(-\frac{\pi}{4}\right)$, $\sqrt{2}\,\mathrm{cis}\left(-\frac{3\pi}{4}\right)$ and $\sqrt{2}\,\mathrm{cis}\,\frac{3\pi}{4}$.

15. $$\begin{aligned}|z_1| &= \frac{1}{2}\sqrt{\left(\sqrt{2}\right)^2 + \left(\sqrt{6}\right)^2}\\ &= \frac{1}{2}\sqrt{8}\\ &= \sqrt{2}\\ \arg(z_1) &= \tan^{-1}\frac{\sqrt{6}}{\sqrt{2}}\\ &= \tan^{-1}\sqrt{3}\\ &= \frac{\pi}{3}\\ \therefore\quad z_1 &= \sqrt{2}\,\mathrm{cis}\,\frac{\pi}{3}\\ |z_2| &= \frac{1}{2}\sqrt{\left(\sqrt{6}\right)^2 + \left(\sqrt{2}\right)^2}\\ &= \sqrt{2}\\ \arg(z_2) &= \tan^{-1}\frac{\sqrt{2}}{\sqrt{6}}\\ &= \tan^{-1}\frac{1}{\sqrt{3}}\\ &= \frac{\pi}{6}\\ \therefore\quad z_2 &= \sqrt{2}\,\mathrm{cis}\,\frac{\pi}{6}\\ z_1^6 &= \left(\sqrt{2}\right)^6\mathrm{cis}\,\frac{6\pi}{3}\\ &= 8\,\mathrm{cis}\,0\\ z_2^3 &= \left(\sqrt{2}\right)^3\mathrm{cis}\,\frac{3\pi}{6}\\ &= 2\sqrt{2}\,\mathrm{cis}\,\frac{\pi}{2}\\ z_3^4 &= 2^4\,\mathrm{cis}\,\frac{4\pi}{8}\\ &= 16\,\mathrm{cis}\,\frac{\pi}{2}\\ \frac{z_1^6 z_2^3}{z_3^4} &= \frac{(8\,\mathrm{cis}\,0)\left(2\sqrt{2}\,\mathrm{cis}\,\frac{\pi}{2}\right)}{16\,\mathrm{cis}\,\frac{\pi}{2}}\\ &= \frac{16\sqrt{2}\,\mathrm{cis}\,\frac{\pi}{2}}{16\,\mathrm{cis}\,\frac{\pi}{2}}\\ &= \sqrt{2}\end{aligned}$$

Exercise 2C

1–6 No working required.

7. $$3e^{\frac{4\pi i}{3}} = 3\operatorname{cis}\frac{4\pi}{3} = 3\operatorname{cis}\left(-\frac{2\pi}{3}\right)$$

8. $$2e^{2+\frac{\pi i}{3}} = 2\left(e^2\right)\left(e^{\frac{\pi i}{3}}\right) = 2e^2\operatorname{cis}\frac{\pi}{3}$$

9. $$\begin{aligned}9\sqrt{2}e^{\frac{-\pi i}{4}} &= 9\sqrt{2}\operatorname{cis}\frac{-\pi}{4}\\ &= 9\sqrt{2}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)\\ &= 9\sqrt{2}\left(\cos\frac{\pi}{4} i\sin\frac{\pi}{4}\right)\\ &= 9\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)\\ &= 9 - 9i\end{aligned}$$

10. $$\begin{aligned}2e^{\frac{-5\pi i}{6}} &= 2\operatorname{cis}\frac{-5\pi}{6}\\ &= 2\left(\cos\frac{-5\pi}{6} + i\sin\frac{-5\pi}{6}\right)\\ &= 2\left(-\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)\\ &= 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)\\ &= -\sqrt{3} - i\end{aligned}$$

11. $$\begin{aligned}10e^{\frac{2\pi i}{3}} &= 10\operatorname{cis}\frac{2\pi}{3}\\ &= 10\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\\ &= 10\left(-\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\\ &= 10\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\\ &= (-5 + 5\sqrt{3}i)\end{aligned}$$

12. $$\begin{aligned}10e^{\frac{3\pi i}{4}} &= 10\operatorname{cis}\frac{3\pi}{4}\\ &= 10\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\\ &= 10\left(-\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\\ &= 10\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\\ &= -5\sqrt{2} + 5\sqrt{2}i\end{aligned}$$

13. $$\begin{aligned}|6\sqrt{3} + 6i| &= \sqrt{6^2 \times 3 + 6^2}\\ &= 12\\ \arg(6\sqrt{3} + 6i) &= \tan^{-1}\frac{6}{6\sqrt{3}}\\ &= \frac{\pi}{6}\\ 6\sqrt{3} + 6i &= 12e^{\frac{i\pi}{6}}\end{aligned}$$

14. $$\begin{aligned}|-1 - \sqrt{3}i| &= \sqrt{1+3}\\ &= 2\\ \arg(-1 - \sqrt{3}i) &= \tan^{-1} -\sqrt{3}-1\\ &= -\frac{2\pi}{3} \qquad \text{(3rd quadrant)}\\ -1 - \sqrt{3}i &= 2e^{-\frac{2i\pi}{3}}\end{aligned}$$

15. $$3i = 3\operatorname{cis}\frac{\pi}{2} = 3e^{\frac{i\pi}{2}}$$

16. $$-2 = 2(-1) = 2e^{i\pi}$$

17. $$\begin{aligned}r &= \sqrt{5^2 + 12^2}\\ &= 13\\ \theta &= \tan^{-1}\frac{12}{5}\\ &= 1.18 \quad \text{(2d.p.)}\\ 5 + 12i &= 13e^{1.18i}\end{aligned}$$

18. $$\begin{aligned}r &= \sqrt{2^2 + 7^2}\\ &= \sqrt{53}\\ \theta &= \tan^{-1}\frac{-7}{2}\\ &= -1.29 \quad \text{(2d.p.)}\\ 2 - 7i &= \sqrt{53}e^{-1.29i}\end{aligned}$$

19. To find a complex conjugate, keep the modulus unchanged, and take the opposite of the argument. Thus if $z = re^{i\theta}$ then $\bar{z} = re^{-i\theta}$.

20. (a) $$\bar{z} = 4e^{\frac{-\pi i}{3}} = 4\cos\frac{\pi}{3} - 4i\sin\frac{\pi}{3} = 2 - 2\sqrt{3}i$$

(b) $$\begin{aligned}z^2 &= \left(4e^{\frac{\pi i}{3}}\right)^2\\ &= 16e^{\frac{2\pi i}{3}}\\ &= -16\cos\frac{\pi}{3} + 16i\sin\frac{\pi}{3}\\ &= -8 + 8\sqrt{3}i\end{aligned}$$

(c) $$\frac{1}{z} = \left(4e^{\frac{\pi i}{3}}\right)^{-1} = \frac{1}{4}e^{\frac{-\pi i}{3}} = \frac{1}{4}\cos\frac{\pi}{3} - \frac{1}{4}i\sin\frac{\pi}{3} = \frac{1}{8} - \frac{\sqrt{3}}{8}i$$

21. (a) $$\text{LHS} = \frac{1}{\operatorname{cis} n\theta} = (\operatorname{cis} n\theta)^{-1} = \operatorname{cis}(-n\theta) = \text{RHS}$$

□

(b) $$\text{LHS} = \frac{1}{\operatorname{cis} n\theta} = \frac{1}{e^{n\theta i}} = e^{-n\theta i} = \operatorname{cis}(-n\theta) = \text{RHS}$$

□

(c) $$\begin{aligned}\text{LHS} &= \frac{1}{\operatorname{cis} n\theta}\\ &= \frac{1}{\cos n\theta + i\sin n\theta}\\ &= \frac{\cos n\theta - i\sin n\theta}{(\cos n\theta + i\sin n\theta)(\cos n\theta - i\sin n\theta)}\\ &= \frac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta - i^2\sin^2 n\theta}\\ &= \frac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta + \sin^2 n\theta}\\ &= \cos n\theta - i\sin n\theta\\ &= \cos(-n\theta) + i\sin(-n\theta)\\ &= \operatorname{cis}(-n\theta)\\ &= \text{RHS}\end{aligned}$$

□

22. (a) First cosine:

$$\begin{aligned}\text{RHS} &= \frac{e^{i\theta} + e^{-i\theta}}{2}\\ &= \frac{\cos\theta + i\sin\theta + \cos -\theta + i\sin -\theta}{2}\\ &= \frac{\cos\theta + i\sin\theta + \cos\theta - i\sin\theta}{2}\\ &= \frac{2\cos\theta}{2}\\ &= \cos\theta\\ &= \text{LHS}\end{aligned}$$

□

then sine:

$$\begin{aligned}\text{RHS} &= \frac{e^{i\theta} - e^{-i\theta}}{2i}\\ &= \frac{\cos\theta + i\sin\theta - \cos -\theta - i\sin -\theta}{2i}\\ &= \frac{\cos\theta + i\sin\theta - \cos\theta + i\sin\theta}{2i}\\ &= \frac{2i\sin\theta}{2i}\\ &= \sin\theta\\ &= \text{LHS}\end{aligned}$$

□

23. (a) No working required: simple application of the chain rule.

(b) No working required: simple application of the chain rule.

(c) $$\frac{d}{d\theta}\left(e^{i\theta}e^2\right) = \frac{d}{d\theta}\left(e^{i\theta+2}\right) = ie^{2+i\theta}$$

Alternatively, bearing in mind that e^2 is a constant, we can do

$$\frac{d}{d\theta}\left(e^{i\theta}e^2\right) = ie^2e^{i\theta}$$

in a single step.

24. (a) $$\int e^{2ix}\,dx = \frac{e^{2ix}}{2i} + c = \frac{ie^{2ix}}{-2} + c = -\frac{1}{2}ie^{2ix} + c$$

(b) $$\int e^{3ix}\,dx = \frac{e^{3ix}}{3i} + c = \frac{ie^{3ix}}{-3} + c = -\frac{1}{3}ie^{3ix} + c$$

(c) $$\int e^{3+ix}\,dx = \frac{e^{3+ix}}{i} + c = \frac{ie^{3+ix}}{-1} + c = -ie^{3+ix} + c$$

25. $$\begin{aligned}
&\int \mathrm{e}^x \cos x\,\mathrm{d}x + \mathrm{i}\int \mathrm{e}^x \sin x\,\mathrm{d}x\\
&= \int \mathrm{e}^x(\cos x + \mathrm{i}\sin x)\,\mathrm{d}x\\
&= \int \mathrm{e}^x\mathrm{e}^{\mathrm{i}x}\,\mathrm{d}x\\
&= \int \mathrm{e}^{(1+\mathrm{i})x}\,\mathrm{d}x\\
&= \frac{\mathrm{e}^{(1+\mathrm{i})x}}{1+\mathrm{i}} + c\\
&= \frac{(1-\mathrm{i})\mathrm{e}^{(1+\mathrm{i})x}}{(1-\mathrm{i})(1+\mathrm{i})} + c\\
&= \frac{(1-\mathrm{i})\mathrm{e}^{x+\mathrm{i}x}}{1-\mathrm{i}^2} + c\\
&= \frac{(1-\mathrm{i})\mathrm{e}^{x}\mathrm{e}^{\mathrm{i}x}}{2} + c\\
&= \frac{\mathrm{e}^x}{2}(1-\mathrm{i})\operatorname{cis} x + c\\
&= \frac{\mathrm{e}^x}{2}(1-\mathrm{i})(\cos x + \mathrm{i}\sin x) + c\\
&= \frac{\mathrm{e}^x}{2}\left((\cos x + \mathrm{i}\sin x) - \mathrm{i}(\cos x + \mathrm{i}\sin x)\right) + c\\
&= \frac{\mathrm{e}^x}{2}\big(\cos x + \mathrm{i}\sin x - \mathrm{i}\cos x - \mathrm{i}^2\sin x)\big) + c\\
&= \frac{\mathrm{e}^x}{2}(\cos x + \mathrm{i}\sin x - \mathrm{i}\cos x + \sin x)) + c\\
&= \frac{\mathrm{e}^x(\sin x + \cos x)}{2} + \mathrm{i}\frac{\mathrm{e}^x(\sin x - \cos x)}{2} + c
\end{aligned}$$

Equating real and imaginary parts (and bearing in mind that the constant of integration can also have real and imaginary parts) gives

$$\int \mathrm{e}^x \cos x\,\mathrm{d}x = \frac{\mathrm{e}^x(\sin x + \cos x)}{2} + c$$

and $$\int \mathrm{e}^x \sin x\,\mathrm{d}x = \frac{\mathrm{e}^x(\sin x - \cos x)}{2} + c$$

Miscellaneous Exercise 2

1. $$\begin{aligned}
\cos\theta + \mathrm{i}\sin\theta &= \mathrm{e}^{\mathrm{i}\theta}\\
\therefore\quad \cos n\theta + \mathrm{i}\sin n\theta &= \mathrm{e}^{\mathrm{i}n\theta}\\
&= \left(e^{\mathrm{i}}\theta\right)^n\\
&= (\cos\theta + \mathrm{i}\sin\theta)^n
\end{aligned}$$

2. $$\begin{aligned}
\cos\theta + \mathrm{i}\sin\theta &= \mathrm{e}^{\mathrm{i}\theta}\\
\therefore\quad \cos(-n\theta) + \mathrm{i}\sin(-n\theta) &= \mathrm{e}^{-\mathrm{i}n\theta}\\
&= \left(e^{\mathrm{i}}\theta\right)^{-n}\\
&= (\cos\theta + \mathrm{i}\sin\theta)^{-n}
\end{aligned}$$

3. (a) $$\begin{aligned} z + \bar{z} &= a + b\mathrm{i} + a - b\mathrm{i}\\ &= 2a \end{aligned}$$

(b) $$\begin{aligned} z + \bar{z} &= a + b\mathrm{i} - (a - b\mathrm{i})\\ &= 2b\mathrm{i} \end{aligned}$$

(c) $$\begin{aligned} z\bar{z} &= (a + b\mathrm{i})(a - b\mathrm{i})\\ &= a^2 - b^2\mathrm{i}^2\\ &= a^2 + b^2 \end{aligned}$$

(d) $$\begin{aligned} \frac{z}{\bar{z}} &= \frac{z^2}{z\bar{z}}\\ &= \frac{(a+b\mathrm{i})^2}{a^2+b^2}\\ &= \frac{a^2 + 2ab\mathrm{i} + b^2\mathrm{i}^2}{a^2+b^2}\\ &= \frac{a^2 - b^2 + 2ab\mathrm{i}\mathrm{i}^2}{a^2+b^2} \end{aligned}$$

4. $$\begin{aligned} \frac{z}{\bar{z}} &= \frac{r\mathrm{e}^{\mathrm{i}\theta}}{r\mathrm{e}^{-\mathrm{i}\theta}}\\ &= \frac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{-\mathrm{i}\theta}}\\ &= \mathrm{e}^{\mathrm{i}\theta+\mathrm{i}\theta}\\ &= \mathrm{e}^{2\mathrm{i}\theta} \end{aligned}$$

5. Most of these require no working. About half of them need the chain rule, but in such a straightforward way that you should be able to differentiate them in a single line.

(i) $$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x+1}{x-1} = \frac{(x-1)-(x+1)}{(x-1)^2} = -\frac{2}{(x-1)^2}$$

6. (a) $$3y + 3x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 7$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}(3x+2y) = 7 - 3y$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{7-3y}{3x+2y}$$

(b) $$2xy + x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 3x^2 = y + x\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} - x\frac{\mathrm{d}y}{\mathrm{d}x} = y - 2xy - 3x^2$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}(x^2-x) = y - 2xy - 3x^2$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y-2xy-3x^2}{x(x-1)}$$

(c) $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}$$

(d) $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{3t^2}{6t-2}$$

7. Let $P(n)$ be the *proposition* that $5^n + 7\times 13^n$ is a multiple of 8. The initial case, where $n=1$:
$$5^1 + 7\times 13^1 = 5 + 7\times 13 = 96 = 8(12)$$
The statement is true for the initial case: we have established $P(1)$.

Assume $P(k)$, that is, that the statement is true for $n=k$, so.
$$5^k + 7\times 13^k = 8a$$
for some integer a.

Then for $n = k+1$
$$\begin{aligned}5^{k+1}&+7\times 13^{k+1}\\ &= 5\times 5^k + 13\times 7\times 13^k\\ &= 5\times 5^k + (5+8)\times 7\times 13^k\\ &= 5\times 5^k + 5\times 7\times 13^k + 8\times 7\times 13^k\\ &= 5(5^k + 7\times 13^k) + 8\times 7\times 13^k\\ &= 5(8a) + 8\times 7\times 13^k\\ &= 8(5a + 7\times 13^k)\end{aligned}$$
which is a multiple of 8.

Thus $P(k) \implies P(k+1)$ (i.e. if the statement is true for $n=k$ it is also true for $n=k+1$).

Hence since the statement is true for $n=1$ it follows by mathematical induction that it is true for all integer $n\geq 1$. □

8. $u = 2x+3 \qquad x = \frac{u-3}{2}$

$\mathrm{d}u = 2\,\mathrm{d}x \qquad \mathrm{d}x = \frac{\mathrm{d}u}{2}$

$$\begin{aligned}\int \frac{5x}{\sqrt{2x+3}}\,\mathrm{d}x &= \int \frac{5(u-3)}{2\sqrt{u}}\frac{\mathrm{d}u}{2}\\ &= \frac{5}{4}\int \frac{u-3}{\sqrt{u}}\,\mathrm{d}u\\ &= \frac{5}{4}\int\left(\sqrt{u} - \frac{3}{\sqrt{u}}\right)\mathrm{d}u\\ &= \frac{5}{4}\left(\frac{2}{3}u^{\frac{3}{2}} - 3\left(2u^{\frac{1}{2}}\right)\right) + c\\ &= \frac{5}{4}\left(\frac{2}{3}u^{\frac{3}{2}} - 3\left(2u^{\frac{1}{2}}\right)\right) + c\\ &= \frac{5}{6}\left(u^{\frac{3}{2}} - 9\sqrt{u}\right) + c\\ &= \frac{5}{6}\sqrt{u}(u-9) + c\\ &= \frac{5}{6}\sqrt{2x+3}(2x+3-9) + c\\ &= \frac{5}{6}\sqrt{2x+3}(2x-6) + c\\ &= \frac{5}{3}\sqrt{2x+3}(x-3) + c\end{aligned}$$

9.
$$\begin{aligned}3|z-5| &= 2|z+5\mathrm{i}|\\ 3|x+\mathrm{i}y-5| &= 2|x+\mathrm{i}y+5\mathrm{i}|\\ 3|x-5+\mathrm{i}y| &= 2|x+(y+5)\mathrm{i}|\\ 3^2|x-5+\mathrm{i}y|^2 &= 2^2|x+(y+5)\mathrm{i}|^2\\ 9\left((x-5)^2+y^2\right) &= 4\left(x^2+(y+5)^2\right)\\ 9\left(x^2-10x+25+y^2\right) &= 4\left(x^2+y^2+10y+25\right)\\ 9x^2-90x+225+9y^2 &= 4x^2+4y^2+40y+100\\ 5x^2-90x+5y^2-40y &= -125\\ x^2-18x+y^2-8y &= -25\\ (x-9)^2-81+(y-4)^2-16 &= -25\\ (x-9)^2+(y-4)^2 &= 72\end{aligned}$$
□