Solutions to

Unit 3C Specialist Mathematics

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Preface

The answers in the Sadler text book sometimes are not enough. For those times when your really need to see a fully worked solution, look here.

It is essential that you use this sparingly!

You should not look here until you have given your best effort to a problem. Understand the problem here, then go away and do it on your own.

Errors

If you encounter any discrepancies between the work here and the solutions given in the Sadler text book, it is very likely that the error is mine. I have yet to find any errors in Sadler's solutions. Mine, however, have not been proofread as thoroughly and it is likely that there are errors in this work. Caveat discipulus!

Contents

Chapter 1

Exercise 1A

1. (a) $$\begin{aligned}AB &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}\\ &= \sqrt{4^2 + 3^2 - 2\times4\times3\cos(70^\circ - 160^\circ)}\\ &= \sqrt{25 - 24\cos(-90^\circ)}\\ &= \sqrt{25 - 24\times0}\\ &= \sqrt{25}\\ &= 5\end{aligned}$$
If we had recognised before we began that A and B form a right angle we could have simplified this by using Pythagoras rather than the cosine rule.

(b) $$\begin{aligned}AB &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}\\ &= \sqrt{4^2 + 4^2 - 2\times4\times4\cos(310^\circ - 10^\circ)}\\ &= \sqrt{32 - 32\cos(300^\circ)}\\ &= \sqrt{32 - 32\times\frac{1}{2}}\\ &= \sqrt{16}\\ &= 4\end{aligned}$$

(c) $$\begin{aligned}AB &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}\\ &= \sqrt{2^2 + 7^2 - 2\times2\times7\cos(225^\circ - 75^\circ)}\\ &= \sqrt{53 - 28\cos(150^\circ)}\\ &= \sqrt{53 - 28\times\frac{-\sqrt{3}}{2}}\\ &= \sqrt{53 + 14\sqrt{3}}\end{aligned}$$

(d) $$\begin{aligned}AB &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}\\ &= \sqrt{3^2 + 2^2 - 2\times3\times2\cos(165^\circ - -150^\circ)}\\ &= \sqrt{13 - 12\cos(315^\circ)}\\ &= \sqrt{13 - 12\frac{\sqrt{2}}{2}}\\ &= \sqrt{13 - 6\sqrt{2}}\end{aligned}$$

2. (a) If we recognise that $\theta_1 - \theta_2$ is a right angle there is no need to use the cosine rule as Pythagoras will do.
$$\begin{aligned}PQ &= \sqrt{r_1^2 + r_2^2}\\ &= \sqrt{5^2 + 12^2}\\ &= 13\end{aligned}$$

(b) Here $\theta_2 - \theta_1 = \frac{3\pi}{4} - -\frac{3\pi}{4} = \frac{3\pi}{2}$ which also represents a right angle, so we can use Pythagoras again.
$$\begin{aligned}PQ &= \sqrt{r_1^2 + r_2^2}\\ &= \sqrt{3^2 + 2^2}\\ &= \sqrt{13}\end{aligned}$$

(c) $$\begin{aligned}PQ &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}\\ &= \sqrt{4^2 + 3^2 - 2\times4\times3\cos(\frac{\pi}{12} - -\frac{7\pi}{12})}\\ &= \sqrt{25 - 24\cos(\frac{8\pi}{12})}\\ &= \sqrt{25 - 24\cos(\frac{2\pi}{3})}\\ &= \sqrt{25 - 24\times-\frac{1}{2}}\\ &= \sqrt{25 + 12}\\ &= \sqrt{37}\end{aligned}$$

(d) $$\begin{aligned}PQ &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}\\ &= \sqrt{3^2 + 4^2 - 2\times3\times4\cos(\frac{11\pi}{10} - \frac{\pi}{10})}\\ &= \sqrt{25 - 24\cos(\frac{10\pi}{10})}\\ &= \sqrt{25 - 24\cos(\pi)}\\ &= \sqrt{25 - 24\times-1}\\ &= \sqrt{25 + 24}\\ &= \sqrt{49}\\ &= 7\end{aligned}$$
If we had recognised before we began that P and Q formed a straight angle we could simply have done
$$\begin{aligned}PQ &= r_1 + r_2\\ &= 3 + 4\\ &= 7\end{aligned}$$

3. $$\begin{aligned}AB &= \sqrt{r_A^2 + r_B^2 - 2r_Ar_B\cos(\theta_B - \theta_A)}\\ &= \sqrt{2^2 + 3^2 - 2\times2\times3\cos(\frac{2\pi}{3} - \frac{\pi}{3})}\\ &= \sqrt{13 - 12\cos(\frac{\pi}{3})}\\ &= \sqrt{13 - 12\times\frac{1}{2}}\\ &= \sqrt{7}\end{aligned}$$
$$\begin{aligned}AC &= \sqrt{r_A^2 + r_C^2 - 2r_Ar_C\cos(\theta_A - \theta_C)}\\ &= \sqrt{2^2 + (3\sqrt{2})^2 - 2\times2\times3\sqrt{2}\cos(\frac{\pi}{3} - \frac{\pi}{12})}\\ &= \sqrt{22 - 12\sqrt{2}\cos(\frac{\pi}{4})}\\ &= \sqrt{22 - 12\sqrt{2}\times\frac{1}{\sqrt{2}}}\\ &= \sqrt{10}\end{aligned}$$
AC exceeds AB by $\sqrt{10} - \sqrt{7}$.

4. (a) Cartesian coordinates of A:

$$(4\cos 10°, 4\sin 10°)$$

Cartesian coordinates of B:

$$(3\cos 130°, 3\sin 130°)$$

$$\begin{aligned}
\text{AB} &= \sqrt{\begin{array}{l}(4\cos 10° - 3\cos 130°)^2\\ +(4\sin 10° - 3\sin 130°)^2\end{array}}\\
&= \sqrt{\begin{array}{l}16\cos^2 10° - 24\cos 10°\cos 130° + 9\cos^2 130°\\ +16\sin^2 10° - 24\sin 10°\sin 130° + 9\sin^2 130°\end{array}}\\
&= \sqrt{16 - 24(\cos 10°\cos 130° + \sin 10°\sin 130°) + 9}\\
&= \sqrt{25 - 24\cos(10° - 130°)}\\
&= \sqrt{25 - 24\cos(-120°)}\\
&= \sqrt{25 - 24 \times -\frac{1}{2}}\\
&= \sqrt{25 + 12}\\
&= \sqrt{37}
\end{aligned}$$

(b)

$$\begin{aligned}
\text{AB} &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}\\
&= \sqrt{4^2 + 3^2 - 2 \times 4 \times 3\cos(130° - 10°)}\\
&= \sqrt{25 - 24\cos(120°)}\\
&= \sqrt{25 - 24 \times -\frac{1}{2}}\\
&= \sqrt{25 + 12}\\
&= \sqrt{37}
\end{aligned}$$

5. (a) Cartesian coordinates of P:

$$(3\sqrt{2}\cos\frac{4\pi}{5}, 3\sqrt{2}\sin\frac{4\pi}{5})$$

Cartesian coordinates of Q:

$$\begin{aligned}
&(4\cos -\frac{19\pi}{20}, 4\sin -\frac{19\pi}{20})\\
=&(4\cos\frac{19\pi}{20}, -4\sin\frac{19\pi}{20})
\end{aligned}$$

$$\begin{aligned}
\text{PQ} &= \sqrt{\begin{array}{l}(3\sqrt{2}\cos\frac{4\pi}{5} - 4\cos\frac{19\pi}{20})^2\\ +(3\sqrt{2}\sin\frac{4\pi}{5} - -4\sin\frac{19\pi}{20})^2\end{array}}\\
&= \sqrt{\begin{array}{l}(3\sqrt{2}\cos\frac{4\pi}{5} - 4\cos\frac{19\pi}{20})^2\\ +(3\sqrt{2}\sin\frac{4\pi}{5} + 4\sin\frac{19\pi}{20})^2\end{array}}\\
&= \sqrt{\begin{array}{l}18\cos^2\frac{4\pi}{5} - 24\sqrt{2}\cos\frac{4\pi}{5}\cos\frac{19\pi}{20} + 16\cos^2\frac{19\pi}{20}\\ +18\sin^2\frac{4\pi}{5} + 24\sqrt{2}\sin\frac{4\pi}{5}\sin\frac{19\pi}{20} + 16\sin^2\frac{19\pi}{20}\end{array}}\\
&= \sqrt{18 - 24\sqrt{2}(\cos\frac{4\pi}{5}\cos\frac{19\pi}{20} - \sin\frac{4\pi}{5}\sin\frac{19\pi}{20}) + 16}\\
&= \sqrt{34 - 24\sqrt{2}\cos(\frac{4\pi}{5} + \frac{19\pi}{20})}\\
&= \sqrt{34 - 24\sqrt{2}\cos(\frac{16\pi}{20} + \frac{19\pi}{20})}\\
&= \sqrt{34 - 24\sqrt{2}\cos(\frac{35\pi}{20})}\\
&= \sqrt{34 - 24\sqrt{2}\cos(\frac{7\pi}{4})}\\
&= \sqrt{34 - 24\sqrt{2} \times \frac{1}{\sqrt{2}}}\\
&= \sqrt{34 - 24}\\
&= \sqrt{10}
\end{aligned}$$

(b)

$$\begin{aligned}
\text{PQ} &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}\\
&= \sqrt{(3\sqrt{2})^2 + 4^2 - 2 \times 3\sqrt{2} \times 4\cos(\frac{4\pi}{5} - -\frac{19\pi}{20})}\\
&= \sqrt{18 + 16 - 24\sqrt{2}\cos(\frac{16\pi}{20} + \frac{19\pi}{20})}\\
&= \sqrt{34 - 24\sqrt{2}\cos(\frac{35\pi}{20})}\\
&= \sqrt{34 - 24\sqrt{2}\cos(\frac{7\pi}{4})}\\
&= \sqrt{34 - 24\sqrt{2} \times \frac{1}{\sqrt{2}}}\\
&= \sqrt{34 - 24}\\
&= \sqrt{10}
\end{aligned}$$

Exercise 1B

Questions 1 to 11 need no working. Refer to the solutions in Sadler.

12. (a) For the given point $(0, \frac{3\pi}{2})$,

$$\begin{aligned}
r &= \frac{3\pi}{2}\\
\theta &= \frac{\pi}{2}\\
\therefore \quad r &= 3\theta\\
k &= 3
\end{aligned}$$

For point A

$$\begin{aligned}
\theta &= \pi\\
r &= 3\theta\\
&= 3\pi\\
x &= -3\pi\\
y &= 0
\end{aligned}$$

Point A has Cartesian coordinates $(-3\pi, 0)$.

(b) For the given point $(-5\pi, 0)$,

$$\begin{aligned} r &= 5\pi \\ \theta &= \pi \\ \therefore \quad r &= 5\theta \\ k &= 5 \end{aligned}$$

For point B

$$\begin{aligned} \theta &= \frac{\pi}{2} \\ r &= 5\theta \\ &= \frac{5\pi}{2} \\ x &= 0 \\ y &= \frac{5\pi}{2} \end{aligned}$$

Point B has Cartesian coordinates $(0, \frac{5\pi}{2})$.

(c) For the given point $(0, 3)$,

$$\begin{aligned} r &= 3 \\ \theta &= \frac{\pi}{2} \\ \therefore \quad r &= \frac{3}{\pi/2}\theta \\ &= \frac{6}{\pi}\theta \\ k &= \frac{6}{\pi} \end{aligned}$$

For point C

$$\begin{aligned} \theta &= \pi \\ r &= \frac{6}{\pi}\theta \\ &= 6 \\ x &= -6 \\ y &= 0 \end{aligned}$$

Point C has Cartesian coordinates $(-6, 0)$.

(d) For the given point $(-10, 0)$,

$$\begin{aligned} r &= 10 \\ \theta &= \pi \\ \therefore \quad r &= \frac{10}{\pi}\theta \\ k &= \frac{10}{\pi} \end{aligned}$$

For point D

$$\begin{aligned} \theta &= \frac{\pi}{2} \\ r &= \frac{10}{\pi}\theta \\ &= 5 \\ x &= 0 \\ y &= 5 \end{aligned}$$

Point D has Cartesian coordinates $(0, 5)$.

Note regarding the extension work: these polar graphs are collectively called petal plots (for obvious reasons).

Miscellaneous Exercise 1

1–3 No working required.

4. (a) $2\mathbf{a} = 6\mathbf{i} - 2\mathbf{j}$

(b)
$$\begin{aligned} \frac{|\mathbf{a}|}{|\mathbf{b}|}\mathbf{b} &= \frac{\sqrt{10}}{\sqrt{20}}(2\mathbf{i} + 4\mathbf{j}) \\ &= \frac{2}{\sqrt{2}}(\mathbf{i} + 2\mathbf{j}) \\ &= \sqrt{2}(\mathbf{i} + 2\mathbf{j}) \end{aligned}$$

(c)
$$\begin{aligned} |d\mathbf{i} - 9\mathbf{j}| &= |3\mathbf{a}| \\ d^2 + 9^2 &= 9^2 + 3^2 \\ d &= \pm 3 \end{aligned}$$

(d) $\mathbf{a} \cdot \mathbf{b} = 3 \times 2 + -1 \times 4 = 2$

(e)
$$\begin{aligned} \cos\theta &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \\ &= \frac{2}{\sqrt{10}\sqrt{20}} \\ &= \frac{\sqrt{2}}{10} \\ \theta &= \cos^{-1}\frac{\sqrt{2}}{10} \\ &\approx 82^\circ \end{aligned}$$

5.
$$\begin{aligned}\mathbf{p}\cdot\mathbf{q} &= 0\\ a(\mathbf{i}+\mathbf{j})\cdot(2\mathbf{i}-b\mathbf{j}) &= 0\\ 2a-ab &= 0\\ b &= 2\end{aligned}$$
(reject $a = 0$ since $\mathbf{p}$ is non zero.)

$$\begin{aligned}a\sqrt{1^2+1^2} &= \sqrt{2^2+(-b)^2}\\ 2a^2 &= 4+b^2\\ &= 4+2^2\\ a^2 &= 4\\ a &= \pm 2\end{aligned}$$

$$\begin{aligned}(2\mathbf{i}-b\mathbf{j})-3(c\mathbf{i}+d\mathbf{j}) &= 23\mathbf{i}-5\mathbf{j}\\ (2-3c)\mathbf{i}+(-2-3d)\mathbf{j} &= 23\mathbf{i}-5\mathbf{j}\\ -3c\mathbf{i}-3d\mathbf{j} &= 21\mathbf{i}-3\mathbf{j}\\ c &= -7\\ d &= 1\end{aligned}$$

$$\begin{aligned}e\mathbf{i}+f\mathbf{j} &= k(2\mathbf{i}-2\mathbf{j})\\ f &= -e\end{aligned}$$

$$\begin{aligned}\sqrt{e^2+f^2} &= \sqrt{c^2+d^2}\\ 2e^2 &= c^2+d^2\\ &= (-7)^2+1^2\\ &= 50\\ e^2 &= 5\\ e &= 5\\ f &= -5\end{aligned}$$
(rejecting $e = -5, f = 5$ because $\mathbf{s}$ must be in the same direction as $\mathbf{q}$ which has a positive $\mathbf{i}$ component.)

6. (a) $\text{OP} = \sqrt{7^2+(-24)^2} = 25\,\text{m}$

(b) $3|\,(-5\mathbf{i}+12\mathbf{j})\,| = 3\sqrt{(-5)^2+12^2} = 39\,\text{m}$

(c) $7\mathbf{i}-24\mathbf{j}+3\,(-5\mathbf{i}+12\mathbf{j}) = (-8\mathbf{i}+12\mathbf{j})$ m

(d)
$$\begin{aligned}|\,(-8\mathbf{i}+12\mathbf{j})\,| &= \sqrt{(-8)^2+12^2}\\ &= 4\sqrt{2^2+3^2}\\ &= 4\sqrt{13}\,\text{m}\end{aligned}$$

7.
$$\begin{aligned}\overrightarrow{\text{OP}} &= \overrightarrow{\text{OA}}+\frac{2}{5}\overrightarrow{\text{AB}}\\ &= \overrightarrow{\text{OA}}+\frac{2}{5}(\overrightarrow{\text{OB}}-\overrightarrow{\text{OA}})\\ &= \frac{3}{5}\overrightarrow{\text{OA}}+\frac{2}{5}\overrightarrow{\text{OB}}\\ &= \frac{3}{5}\,(-7\mathbf{i}+7\mathbf{j})+\frac{2}{5}\,(8\mathbf{i}+2\mathbf{j})\\ &= \frac{-21+16}{5}\mathbf{i}+\frac{21+4}{5}\mathbf{j}\\ &= -\mathbf{i}+5\mathbf{j}\end{aligned}$$

8.
$$\begin{aligned}2z-\bar{z} &= -5+6\text{i}\\ 2(a+b\text{i})-(a-b\text{i}) &= -5+6\text{i}\\ a+3b\text{i} &= -5+6\text{i}\\ a &= -5\\ b &= 2\end{aligned}$$

9.
$$\begin{aligned}\sin 75^\circ &= \sin(45^\circ+30^\circ)\\ &= \sin 45^\circ\cos 30^\circ+\cos 45^\circ\sin 30^\circ\\ &= \frac{\sqrt{2}}{2}\,\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\,\frac{1}{2}\\ &= \frac{\sqrt{2}\left(\sqrt{3}+1\right)}{4}\end{aligned}$$

10.
$$\begin{aligned}(a+b\text{i})(-1+2\text{i}) &= 1+2\text{i}\\ -a+2a\text{i}-b\text{i}-2b &= 1+2\text{i}\\ (-a-2b)+(2a-b)\text{i} &= 1+2\text{i}\end{aligned}$$
This gives the following pair of equations by equating real and imaginary components:

$$\begin{aligned}-a-2b &= 1\\ 2a-b &= 2\end{aligned}$$

Solving simultaneously gives

$$\begin{aligned}a &= \frac{3}{5}\\ b &= -\frac{4}{5}\\ \therefore\qquad z &= \frac{3}{5}-\frac{4}{5}\text{i}\end{aligned}$$

11.
$$\begin{aligned}(a+b\text{i})^2 &= -5+12\text{i}\\ a^2-b^2+2ab\text{i} &= -5+12\text{i}\\ a^2-b^2 &= -5\\ 2ab &= 12\\ b &= \frac{6}{a}\\ a^2-\left(\frac{6}{a}\right)^2 &= -5\\ a^2-\frac{36}{a^2} &= -5\\ a^4+5a^2-36 &= 0\\ (a^2-4)(a^2+9) &= 0\\ a &= \pm 2\\ b &= \pm 3\\ z &= \pm(2+3\text{i})\end{aligned}$$
(Note that we disregard $(a^2+9) = 0$ as leading to any solutions because a must be real.)

12.
$$\begin{aligned}(a+bi)^2 &= 5+12i\\ a^2-b^2+2abi &= 5+12i\\ a^2-b^2 &= 5\\ 2ab &= 12\\ b &= \frac{6}{a}\\ a^2-\left(\frac{6}{a}\right)^2 &= 5\\ a^2-\frac{36}{a^2} &= 5\\ a^4-5a^2-36 &= 0\\ (a^2+4)(a^2-9) &= 0\\ a &= \pm 3\\ b &= \pm 2\\ \therefore \quad \sqrt{5+12i} &= \pm(3+2i)\end{aligned}$$

(Note that the square root of a real number is defined uniquely to mean the positive square root. No such unique definition exists for the square root of complex numbers.)

13.
$$\begin{aligned}2\sin\left(x+\frac{\pi}{6}\right) &= \sqrt{2}\\ \sin\left(x+\frac{\pi}{6}\right) &= \frac{\sqrt{2}}{2}\\ x+\frac{\pi}{6} &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}\\ x &= \frac{\pi}{12}, \frac{7\pi}{12}, \frac{25\pi}{12}, \frac{31\pi}{12}\end{aligned}$$

14.
$$y' = 2x-3$$

when $x=-1$

$$\begin{aligned}y' &= 2(-1)-3\\ &= -5\end{aligned}$$

The equation of the tangent line is

$$\begin{aligned}y-y_0 &= m(x-x_0)\\ y-9 &= -5(x--1)\\ y &= -5x-5+9\\ y &= -5x+4\end{aligned}$$

15.
$$\begin{aligned}y' &= 2(3x^2-x+1)+(6x-1)(2x-3)\\ &= 6x^2-2x+2+12x^2-18x-2x+3\\ &= 18x^2-22x+5 \text{when } x=2\\ y' &= 18(2)^2-22(2)+5\\ &= 33\end{aligned}$$

The equation of the tangent line is

$$\begin{aligned}y-y_0 &= m(x-x_0)\\ y-11 &= 33(x-2)\\ y &= 33x-66+11\\ y &= 33x-55\end{aligned}$$

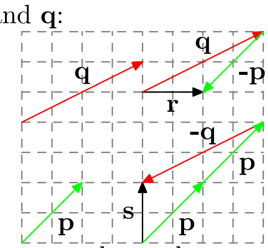
16.
$$\begin{aligned}4\theta &= 3\pi\\ \theta &= \frac{3\pi}{4}\end{aligned}$$

Polar coordinates: $\left(3\pi, \angle\frac{3\pi}{4}\right)$. (Assuming that we restrict r and θ to positive values as usual.)

The Cartesian coordinates of this point are

$$\left(3\pi\cos\frac{3\pi}{4}, 3\pi\sin\frac{3\pi}{4}\right) = \left(-\frac{3\sqrt{2}\pi}{2}, \frac{3\sqrt{2}\pi}{2}\right)$$

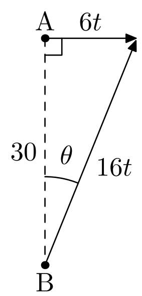
17. It may be simplest to first express **r** and **s** in terms of **p** and **q**:



then use these to work out the rest of the vectors:

$$\begin{aligned}\mathbf{t} &= \mathbf{r}+2\mathbf{s}\\ &= -\mathbf{p}+\mathbf{q}+2(2\mathbf{p}-\mathbf{q})\\ &= 3\mathbf{p}-\mathbf{q}\\ \mathbf{u} &= -\mathbf{t}\\ &= -3\mathbf{p}+\mathbf{q}\\ \mathbf{v} &= 2.5\mathbf{r}+2\mathbf{s}\\ &= 2.5(-\mathbf{p}+\mathbf{q})+2(2\mathbf{p}-\mathbf{q})\\ &= 1.5\mathbf{p}+.5\mathbf{q}\\ \mathbf{w} &= 3.5\mathbf{r}+\mathbf{s}\\ &= 3.5(-\mathbf{p}+\mathbf{q})+2\mathbf{p}-\mathbf{q}\\ &= -1.5\mathbf{p}+2.5\mathbf{q}\\ \mathbf{x} &= 2\mathbf{r}+0.5\mathbf{s}\\ &= 2(-\mathbf{p}+\mathbf{q})+0.5(2\mathbf{p}-\mathbf{q})\\ &= -\mathbf{p}+1.5\mathbf{q}\\ \mathbf{y} &= \mathbf{r}-\mathbf{s}\\ &= -\mathbf{p}+\mathbf{q}-(2\mathbf{p}-\mathbf{q})\\ &= -3\mathbf{p}+2\mathbf{q}\end{aligned}$$

18.



$$\begin{aligned}\sin\theta &= \frac{6t}{16t}\\ \theta &= \sin^{-1}\frac{3}{8}\\ &= 22^\circ\end{aligned}$$

Chapter 2

Exercise 2A

1. (a) $|z| = \sqrt{4^2 + (-3)^2} = 5$

   (b) $|z| = \sqrt{12^2 + 5^2} = 13$

   (c) $|z| = \sqrt{3^2 + 2^2} = \sqrt{13}$

   (d) $|z| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$

   (e) $|z| = \sqrt{1^2 + 5^2} = \sqrt{26}$

   (f) $|z| = \sqrt{0^2 + 5^2} = 5$

2. Process:

   - Determine which quadrant z lies in on the Argand diagram by examining the sign of the real and imaginary parts
   - Use inverse tangent to determine the angle made with the real axis on an argand diagram.
   - Combine these to obtain the principal argument.

   (a) 1^{st} Quadrant: $\tan\theta = \frac{2}{2}$, $\therefore \arg z = \frac{\pi}{4}$

   (b) 4^{th} Quadrant: $\tan\theta = \frac{-2}{2}$, $\therefore \arg z = -\frac{\pi}{4}$

   (c) 2^{nd} Quadrant: $\tan\theta = \frac{2}{-2}$, $\therefore \arg z = \frac{3\pi}{4}$

   (d) 3^{rd} Quadrant: $\tan\theta = \frac{-2}{-2}$, $\therefore \arg z = -\frac{3\pi}{4}$

   (e) 2^{nd} Quadrant: $\tan\theta = \frac{2\sqrt{3}}{-2}$, $\therefore \arg z = \frac{2\pi}{3}$

   (f) 4^{th} Quadrant: $\tan\theta = \frac{-3\sqrt{3}}{3}$, $\therefore \arg z = -\frac{\pi}{3}$

3. There seems little point in showing working for these problems.
   z_1, z_2 and z_3: subtract 2π to obtain the principal argument.
   z_4: add 2π to obtain the principal argument.
   z_5 to z_{12}: read r from the magnitude shown on the diagram, and determine θ as the directed angle measured anticlockwise from the positive real axis.

4. There seems little point in showing working for these problems.
   Determine r by Pythagoras as for question 1.
   Determine θ as for question 2.

5. There seems little point in showing working for these problems. You should be able to do them in a single step.
   Determine the real and imaginary components by evaluating the trigonometric expressions exactly and then multiplying by r.

Exercise 2B

1. Read r from the magnitude of z and θ from the directed angle (converted to radians) measured anticlockwise from the positive real axis.

2–9 No working needed. You should be able to do these questions in a single step, converting to radians and adding or subtracting a multiple of 2π where necessary.

10. $$7\operatorname{cis}\frac{\pi}{2} = 7\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right) = 7(0 + \mathrm{i}) = 7\mathrm{i}$$

11. $5\operatorname{cis}\left(-\frac{\pi}{2}\right) = 5\left(\cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right)\right)$
$= 5(0 - \mathrm{i})$
$= -5\mathrm{i}$

12. $\operatorname{cis}\pi = \cos\pi + \mathrm{i}\sin\pi$
$= -1 + 0\mathrm{i}$
$= -1$

13. $3\operatorname{cis}2\pi = 3(\cos 2\pi + \mathrm{i}\sin 2\pi)$
$= 3(1 + 0\mathrm{i})$
$= 3$

(With a little practice you may find you can do questions like these by simply sketching or visualising an Argand diagram.)

14. $10\operatorname{cis}\frac{\pi}{4} = 10\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)$
$= 5\sqrt{2} + 5\sqrt{2}\mathrm{i}$

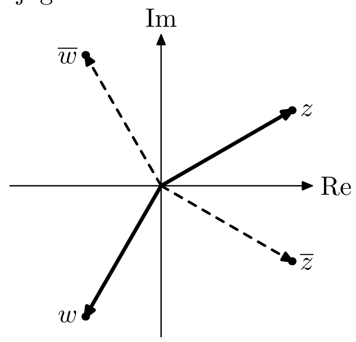
15. $4\operatorname{cis}\frac{2\pi}{3} = 4\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right)$
$= -2 + 2\sqrt{3}\mathrm{i}$

16. $4\operatorname{cis}\left(-\frac{2\pi}{3}\right) = 4\left(\cos\left(-\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{3}\right)\right)$
$= -2 - 2\sqrt{3}\mathrm{i}$

17. $12\operatorname{cis}\left(-\frac{4\pi}{3}\right) = 12\operatorname{cis}\frac{2\pi}{3}$
$= 12\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right)$
$= -6 + 6\sqrt{3}\mathrm{i}$

18–21 Do these questions in the same way as question 4 of exercise 2A (i.e. using Pythagoras to find r and inverse tangent to determine θ, adding or subtracting π for z in the second or third quadrant respectively).

22. (a) The conjugate is a reflection in the real axis:



(b) $\overline{z} = r_1\operatorname{cis}(-\alpha)$
$\overline{w} = r_2\operatorname{cis}(-\beta)$

23. No working needed.

24. No working needed.

25. First subtract 360° to get the principal argument, then multiply this by −1 to obtain the conjugate.

26. First add 360° to get the principal argument, then multiply this by −1 to obtain the conjugate.

27. No working needed.

28. No working needed.

29. No working needed.

30. First subtract 4π to get the principal argument, then multiply this by −1 to obtain the conjugate.

Exercise 2C

1. $zw = (2 + 3\mathrm{i})(5 - 2\mathrm{i})$
$= 10 - 4\mathrm{i} + 15\mathrm{i} + 6$
$= 16 + 11\mathrm{i}$

2. $zw = (3 + 2\mathrm{i})(-1 + 2\mathrm{i})$
$= -3 + 6\mathrm{i} - 2\mathrm{i} - 4$
$= -7 + 4\mathrm{i}$

3. $zw = 3 \times 5\operatorname{cis}(60° + 20°)$
$= 15\operatorname{cis}80°$

4. $zw = 3 \times 3\operatorname{cis}(120° + 150°)$
$= 9\operatorname{cis}270°$
$= 9\operatorname{cis}(-90°)$

5. $zw = 3 \times 3\operatorname{cis}(30° - 80°)$
$= 9\operatorname{cis}(-50°)$

6. $zw = 5 \times 2\operatorname{cis}\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$
$= 10\operatorname{cis}\frac{7\pi}{12}$

7. $zw = 4 \times 2\operatorname{cis}\left(\frac{\pi}{4} - \frac{4\pi}{4}\right)$
$= 8\operatorname{cis}\left(-\frac{\pi}{2}\right)$

8. $zw = 2 \times 1\left(\cos(50° + 60°) + \mathrm{i}\sin(50° + 60°)\right)$
$= 2\left(\cos 110° + \mathrm{i}\sin 110°\right)$

9. $zw = 2 \times 3\left(\cos(170° + 150°) + \mathrm{i}\sin(170° + 150°)\right)$
$= 6\left(\cos 320° + \mathrm{i}\sin 320°\right)$
$= 6\left(\cos(-40°) + \mathrm{i}\sin(-40°)`\right)$

10. $\frac{z}{w} = \frac{6-3\mathrm{i}}{3-4\mathrm{i}}$
$= \frac{(6-3\mathrm{i})(3+4\mathrm{i})}{(3-4\mathrm{i})(3+4\mathrm{i})}$
$= \frac{18+24\mathrm{i}-9\mathrm{i}+12}{9+16}$
$= \frac{30+15\mathrm{i}}{25}$
$= 1.2 + 0.6\mathrm{i}$

11. $\frac{z}{w} = \frac{-6+3\mathrm{i}}{-3+4\mathrm{i}}$
$= \frac{-1(6-3\mathrm{i})}{-1(3-4\mathrm{i})}$
$= 1.2 + 0.6\mathrm{i}$

12. $\frac{z}{w} = \frac{8}{2}\operatorname{cis}(60° - 40°)$
$= 4\operatorname{cis} 20°$

13. $\frac{z}{w} = \frac{5}{1}\operatorname{cis}(120° - 150°)$
$= 5\operatorname{cis}(-30°)$

14. $\frac{z}{w} = \frac{3}{3}\operatorname{cis}(-150° - 80°)$
$= \operatorname{cis}(-230°)$
$= \operatorname{cis}(-230° + 360°)$
$= \operatorname{cis} 130°$

15. $\frac{z}{w} = \frac{2}{2}\operatorname{cis}\left(\frac{3\pi}{5} - \frac{2\pi}{5}\right)$
$= \operatorname{cis}\frac{\pi}{5}$

16. $\frac{z}{w} = \frac{4}{2}\operatorname{cis}\left(\frac{\pi}{4} - -\frac{3\pi}{4}\right)$
$= 2\operatorname{cis}\pi$

17. $\frac{z}{w} = \frac{5}{2}\left(\cos\left(\frac{3\pi}{4} - \frac{\pi}{2}\right) + \mathrm{i}\sin\left(\frac{3\pi}{4} - \frac{\pi}{2}\right)\right)$
$= 2.5\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)$

18. $\frac{z}{w} = \frac{2}{5}\left(\cos(50° - 50°) + \mathrm{i}\sin(50° - 50°)\right)$
$= 0.4\left(\cos 0 + \mathrm{i}\sin 0\right)$

19. From z, zw has been rotated 40° and scaled by 2, so
$$w = 2\operatorname{cis} 40°$$

20. From z, zw has been rotated $180-30-50 = 100°$ and scaled by 3, so
$$w = 3\operatorname{cis} 100°$$

21. From z, zw has been rotated $-90°$ and scaled by 2, so
$$w = 2\operatorname{cis}(-90°)$$

22. From z, zw has been rotated $180 - 110 + 50 = 120°$ and scaled by 2, so
$$w = 2\operatorname{cis} 120°$$

23. From z, zw has been rotated $180 - 110 + 90 = 160°$ and scaled by 1, so
$$w = \operatorname{cis} 160°$$

24. From z, zw has been rotated $-140°$ and scaled by 2, so
$$w = 2\operatorname{cis}(-140°)$$

25. From z, $\frac{z}{w}$ has been rotated $-120°$ and scaled by 1, so
$$w = \operatorname{cis} 120°$$

26. From z, $\frac{z}{w}$ has been rotated $-80°$ and scaled by $\frac{1}{2}$, so
$$w = 2\operatorname{cis} 80°$$

27. From z, $\frac{z}{w}$ has been rotated 100° and scaled by $\frac{1}{2}$, so
$$w = 2\operatorname{cis}(-100°)$$

28. (a) $2z = 2(6\operatorname{cis} 40°)$
$= 12\operatorname{cis} 40°$

(b) $3w = 3(2\operatorname{cis} 30°)$
$= 6\operatorname{cis} 30°$

(c) $zw = 6 \times 2\operatorname{cis}(40° + 30°)$
$= 12\operatorname{cis} 70°$

(d) $wz = zw$
$= 12\operatorname{cis} 70°$

(e) $iz = 6\operatorname{cis}(40° + 90°)$
$= 6\operatorname{cis} 130°$

(f) $iw = 2\operatorname{cis}(30° + 90°)$
$= 2\operatorname{cis} 120°$

(g) $\frac{w}{z} = \frac{2}{6}\operatorname{cis}(30° - 40°)$
$= \frac{1}{3}\operatorname{cis}(-10°)$

(h) $\frac{1}{z} = \frac{1}{6}\operatorname{cis}(0 - 40°)$
$= \frac{1}{6}\operatorname{cis}(-40°)$

29. (a) $zw = 8 \times 4\operatorname{cis}(\frac{2\pi}{3} + \frac{3\pi}{4})$
$= 32\operatorname{cis}\frac{17\pi}{12}$
$= 32\operatorname{cis}\left(-\frac{7\pi}{12}\right)$

(b) $wz = zw$

$= 32 \operatorname{cis}\left(-\frac{7\pi}{12}\right)$

(c) $\frac{w}{z} = \frac{4}{8}\operatorname{cis}(\frac{3\pi}{4} - \frac{2\pi}{3})$

$= 0.5 \operatorname{cis}\frac{\pi}{12}$

(d) $\frac{z}{w} = \frac{8}{4}\operatorname{cis}(\frac{2\pi}{3} - \frac{3\pi}{4})$

$= 2 \operatorname{cis}\left(-\frac{\pi}{12}\right)$

(e) $\overline{z} = 8 \operatorname{cis}\left(-\frac{2\pi}{3}\right)$

(f) $\overline{w} = 4 \operatorname{cis}\left(-\frac{3\pi}{4}\right)$

(g) $\frac{1}{z} = \frac{1}{8}\operatorname{cis}\left(0 - \frac{2\pi}{3}\right)$

$= 0.125 \operatorname{cis}\left(-\frac{2\pi}{3}\right)$

(h) $\frac{\mathrm{i}}{w} = \frac{1}{4}\operatorname{cis}\left(\frac{\pi}{2} - \frac{3\pi}{4}\right)$

$= 0.25 \operatorname{cis}\left(-\frac{\pi}{4}\right)$

Exercise 2D

1-5 No working required.

6. First rewrite as $z : |z - (3 - 3\mathrm{i})| = 3$

7. Think of this as points equidistant between $(-8 + 0\mathrm{i})$ and $(0 + 4\mathrm{i})$. Plot those points on the Argand diagram, and the locus is the perpendicular bisector of the line segment between them.

8. Think of this as points equidistant between $(-2 - 3\mathrm{i})$ and $(4 - \mathrm{i})$. Plot those points on the Argand diagram, and the locus is the perpendicular bisector of the line segment between them.

9. Since $x = \operatorname{Re}(z)$, $z : \operatorname{Re}(z) = 5$ is the vertical line $x = 5$.

10. Since $y = \operatorname{Im}(z)$, $z : \operatorname{Im}(z) = -4$ is the horizontal line $y = -4$.

11.
$$\begin{aligned} \theta &= \arg z \\ \tan\theta &= \frac{y}{x} \\ \therefore \quad \tan\frac{\pi}{3} &= \frac{y}{x} \\ y &= \sqrt{3}x \ : \ x \geq 0 \end{aligned}$$
We need to specify $x \geq 0$ to exclude the 4^{th} quadrant.

12.
$$\begin{aligned} \theta &= \arg z \\ \tan\theta &= \frac{y}{x} \\ \therefore \quad \tan\left(-\frac{\pi}{3}\right) &= \frac{y}{x} \\ y &= -\sqrt{3}x \ : \ x \geq 0 \end{aligned}$$
We need to specify $x \geq 0$ to exclude the 2^{nd} quadrant.

13. Simply substitute x for $\operatorname{Re}(z)$ and y for $\operatorname{Im}(z)$ to give $x + y = 6$.

14.
$$\begin{aligned} |z| &= 6 \\ \sqrt{x^2 + y^2} &= 6 \\ x^2 + y^2 &= 36 \end{aligned}$$

15. This is a circular region centred at $(0, 4)$ having radius 3. You should be able to produce a Cartesian equation from this description, or you can do it algebraically:
$$\begin{aligned} |z - 4\mathrm{i}| &\leq 3 \\ |x + (y - 4)\mathrm{i}| &\leq 3 \\ \sqrt{x^2 + (y - 4)^2} &\leq 3 \\ x^2 + (y - 4)^2 &\leq 9 \end{aligned}$$
You have to be careful with inequalities like this because multiplying both sides of the equation by a negative changes the direction of the inequality. It can be particularly tricky when you square both sides of an inequation as we have in the last step here: you have to make sure you know with certainty that both sides are positive. For example, $a < 2$ is not the same as $a^2 < 4$. We are safe here, however, since the square root on the left hand side is always positive and so is 3 on the right hand side.

16. This is a circle radius 4 centred at $(2, 3)$.
$$\begin{aligned} |z - (2 + 3\mathrm{i})| &= 4 \\ |(x - 2) + (y - 3)\mathrm{i}| &= 4 \\ (x - 2)^2 + (y - 3)^2 &= 16 \end{aligned}$$

17. This is a circle radius 4 centred at $(2, -3)$.

$$\begin{aligned}|z - 2 + 3\mathrm{i}| &= 4\\ |(x-2) + (y+3)\mathrm{i}| &= 4\\ (x-2)^2 + (y+3)^2 &= 16\end{aligned}$$

18. This is a line defined by the locus of points equidistant from $(2, 0)$ and $(6, 0)$, i.e. the vertical line $x = 4$. Showing this algebraically:

$$\begin{aligned}|z-2| &= |z-6|\\ |(x-2)+y\mathrm{i}| &= |(x-6)+y\mathrm{i}|\\ (x-2)^2+y^2 &= (x-6)^2+y^2\\ x^2-4x+4+y^2 &= x^2-12x+36+y^2\\ -4x+4 &= -12x+36\\ 8x &= 32\\ x &= 4\end{aligned}$$

19. This is a line defined by the locus of points equidistant from $(0, 6)$ and $(2, 0)$:

$$\begin{aligned}|z-6\mathrm{i}| &= |z-2|\\ |x+(y-6)\mathrm{i}| &= |(x-2)+y\mathrm{i}|\\ x^2+(y-6)^2 &= (x-2)^2+y^2\\ x^2+y^2-12y+36 &= x^2-4x+4+y^2\\ -12y+36 &= -4x+4\\ 4x-12y &= -32\\ x-3y &= -8\end{aligned}$$

20. This is a line defined by the locus of points equidistant from $(2, 1)$ and $(4, -5)$:

$$\begin{aligned}|z-(2+\mathrm{i})| &= |z-(4-5\mathrm{i})|\\ |(x-2)+(y-1)\mathrm{i}| &= |(x-4)+(y+5)\mathrm{i}|\\ (x-2)^2+(y-1)^2 &= (x-4)^2+(y+5)^2\\ x^2-4x+4+y^2-2y+1 &= x^2-8x+16+y^2+10y+25\\ 4x-12y &= 36\\ x-3y &= 9\end{aligned}$$

21. This is a doughnut-shaped region centred at the origin, including all points on or inside the 5-unit circle and on or outside the 3-unit circle. For the Cartesian equation, substitute $\sqrt{x^2+y^2}$ for $|z|$.

22. This is the region in the first quadrant bounded below by the line $z = \frac{\pi}{6}$ and above by the line $z = \frac{\pi}{3}$.

$$\begin{aligned}\tfrac{\pi}{6} \le &\quad \theta &\le \tfrac{\pi}{3}\\ \tan\tfrac{\pi}{6} \le &\quad \tan\theta &\le \tan\tfrac{\pi}{3}\\ \tfrac{1}{\sqrt{3}} \le &\quad \tfrac{y}{x} &\le \sqrt{3}\\ \tfrac{x}{\sqrt{3}} \le &\quad y &\le \sqrt{3}x \quad : \quad x \ge 0\end{aligned}$$

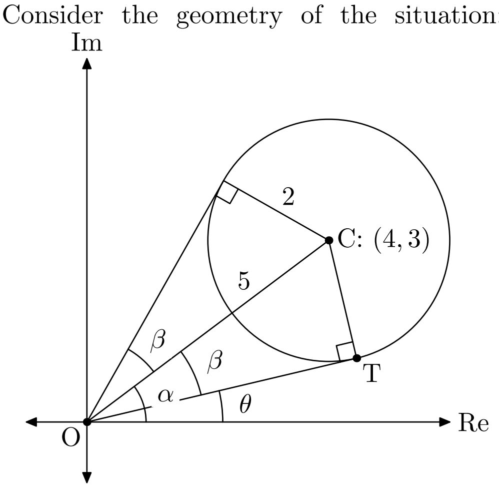
Note that the second line above (where we take the tangent) is only valid because the tangent function is "strictly increasing" in the first quadrant, so $a > b$ implies $\tan a > \tan b$. This is only true for functions that have positive gradient in the given domain.

The step taken in the last step (multiplication by x) is only valid because we restrict $x \ge 0$. (If x was negative, this step would change the direction of the inequalities.

An alternative, possibly simpler, approach to this question would be to first observe that the specified region lies above (or on) the line $y = \frac{x}{\sqrt{3}}$ and below (or on) the line $y = \sqrt{3}x$, (again using $\tan\theta = \frac{y}{x}$ to obtain these equations). This is essentially doing the same thing as the above, but it's perhaps a more intuitive way of looking at it.

23. We have a circle centred at $(-3, 3)$ having a radius of 2.
    (a) The minimum value of $\mathrm{Im}(z)$ is $3 - 2 = 1$.
    (b) The maximum value of $\mathrm{Re}(z)$ is $-3+2 = -1$ and the minimum is $-3-2 = -5$ so the maximum value of $|\mathrm{Re}(z)|$ is $|-5| = 5$.
    (c) The distance between the origin and the centre of the circle is $\sqrt{(-3)^2+3^2} = 3\sqrt{2}$. The point on the circle nearest the origin is 2 units nearer, so the minimum $|z|$ is $3\sqrt{2}-2$.
    (d) The point on the circle furthest from the origin is 2 units further away than the centre, so the maximum $|z|$ is $3\sqrt{2}+2$.
    (e) The locus of $\overline{z}$ is the reflection in the real axis of the locus of z. The point on this image furthest from the origin is the same distance from the origin as the corresponding point on the original: $3\sqrt{2}+2$.

24. We have a circle centred at $(4, 3)$ having a radius of 2.
    (a) The minimum value of $\mathrm{Im}(z)$ is $3 - 2 = 1$.
    (b) The maximum value of $\mathrm{Re}(z)$ is $4 + 2 = 6$.
    (c) The distance between the origin and the centre of the circle is $\sqrt{4^2+3^2} = 5$. The point on the circle furthest from the origin is 2 units further, so the maximum $|z|$ is $5 + 2 = 7$.
    (d) The point on the circle nearest the origin is 2 units nearer than the centre, so the minimum $|z|$ is $5 - 2 = 3$.
    (e) Consider the geometry of the situation:



The minimum value of arg z corresponds to the angle for the lower tangent, T.

$$\begin{aligned}\tan\alpha &= \frac{3}{4}\\ \alpha &= 0.644\\ \sin\beta &= \frac{2}{5}\\ \beta &= 0.412\\ \theta &= \alpha - \beta\\ &= 0.23\end{aligned}$$

(f) The maximum value of arg z corresponds to the angle for the upper tangent.

$$\begin{aligned}\theta &= \alpha + \beta\\ &= 1.06\end{aligned}$$

Miscellaneous Exercise 2

1. (a) $$\begin{aligned}z + w &= 3 + 2 + (-4 + 3)\mathrm{i}\\ &= 5 - \mathrm{i}\end{aligned}$$

(b) $$\begin{aligned}z - w &= 3 - 2 + (-4 - 3)\mathrm{i}\\ &= 1 - 7\mathrm{i}\end{aligned}$$

(c) $$\begin{aligned}zw &= (3 - 4\mathrm{i})(2 + 3\mathrm{i})\\ &= 6 + 9\mathrm{i} - 8\mathrm{i} + 12\\ &= 18 + \mathrm{i}\end{aligned}$$

(d) $$\begin{aligned}z^2 &= (3 - 4\mathrm{i})^2\\ &= 9 - 24\mathrm{i} - 16\\ &= -7 - 24\mathrm{i}\end{aligned}$$

(e) $$\begin{aligned}\frac{z}{w} &= \frac{3 - 4\mathrm{i}}{2 + 3\mathrm{i}}\\ &= \frac{(3 - 4\mathrm{i})(2 - 3\mathrm{i})}{(2 + 3\mathrm{i})(2 - 3\mathrm{i})}\\ &= \frac{6 - 9\mathrm{i} - 8\mathrm{i} - 12}{4 + 9}\\ &= \frac{-6 - 17\mathrm{i}}{13}\\ &= -\frac{6}{13} - \frac{17}{13}\mathrm{i}\end{aligned}$$

(f) $$\begin{aligned}\frac{w}{z} &= \frac{2 + 3\mathrm{i}}{3 - 4\mathrm{i}}\\ &= \frac{(2 + 3\mathrm{i})(3 + 4\mathrm{i})}{(3 - 4\mathrm{i})(3 + 4\mathrm{i})}\\ &= \frac{6 + 8\mathrm{i} + 9\mathrm{i} - 12}{9 + 16}\\ &= \frac{-6 + 17\mathrm{i}}{25}\\ &= -\frac{6}{25} + \frac{17}{25}\mathrm{i}\end{aligned}$$

2. (a) $\overrightarrow{\mathrm{AB}} = \mathbf{c}$

(b) $\overrightarrow{\mathrm{AD}} = \frac{1}{4}\overrightarrow{\mathrm{AB}} = \frac{1}{4}\mathbf{c}$

(c) $\overrightarrow{\mathrm{DB}} = \frac{3}{4}\overrightarrow{\mathrm{AB}} = \frac{3}{4}\mathbf{c}$

(d) $$\begin{aligned}\overrightarrow{\mathrm{DE}} &= \overrightarrow{\mathrm{DB}} + \overrightarrow{\mathrm{BE}}\\ &= \frac{3}{4}\mathbf{c} + \frac{1}{2}\mathbf{c}\\ &= \frac{5}{4}\mathbf{c}\end{aligned}$$

(e) $$\begin{aligned}\overrightarrow{\mathrm{OB}} &= \overrightarrow{\mathrm{OA}} + \overrightarrow{\mathrm{AB}}\\ &= \mathbf{a} + \mathbf{c}\end{aligned}$$

(f) $$\begin{aligned}\overrightarrow{\mathrm{OD}} &= \overrightarrow{\mathrm{OA}} + \overrightarrow{\mathrm{AD}}\\ &= \mathbf{a} + \frac{1}{4}\mathbf{c}\end{aligned}$$

(g) $$\begin{aligned}\overrightarrow{\mathrm{CE}} &= \overrightarrow{\mathrm{CB}} + \overrightarrow{\mathrm{BE}}\\ &= \mathbf{a} + \frac{1}{2}\mathbf{c}\end{aligned}$$

(h) $$\begin{aligned}\overrightarrow{\mathrm{OE}} &= \overrightarrow{\mathrm{OA}} + \overrightarrow{\mathrm{AE}}\\ &= \mathbf{a} + \frac{3}{2}\mathbf{c}\end{aligned}$$

alternatively

$$\begin{aligned}\overrightarrow{\mathrm{OE}} &= \overrightarrow{\mathrm{OC}} + \overrightarrow{\mathrm{CE}}\\ &= \mathbf{c} + \mathbf{a} + \frac{1}{2}\mathbf{c}\\ &= \mathbf{a} + \frac{3}{2}\mathbf{c}\end{aligned}$$

3. (a) $$\begin{aligned}r &= \sqrt{(-3)^2 + (-3\sqrt{3})^2}\\ &= \sqrt{9 + 27}\\ &= 6\\ \tan\theta &= \frac{-3\sqrt{3}}{-3}\\ \theta &= -\frac{2\pi}{3} \qquad \text{(third quadrant)}\\ -3 - 3\sqrt{3}\mathrm{i} &= 6\,\mathrm{cis}\left(-\frac{2\pi}{3}\right)\end{aligned}$$

(b) $8\operatorname{cis}\left(-\frac{5\pi}{6}\right)$

$$= 8\left(\cos\left(-\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right)$$
$$= 8\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\mathrm{i}\right)$$
$$= -4\sqrt{3} - 4\mathrm{i}$$

4. (a) $(2\cos\frac{\pi}{2}, 2\sin\frac{\pi}{2}) = (0, 2)$

   (b) $(5\cos\pi, 5\sin\pi) = (-5, 0)$

   (c) $(4\cos\frac{-3\pi}{4}, 4\sin\frac{-3\pi}{4}) = (-2\sqrt{2}, -2\sqrt{2})$

5.
$$\begin{aligned}
z &= \sqrt{1^2 + 1^2}\operatorname{cis}\tan^{-1}\frac{1}{1}\\
&= \sqrt{2}\operatorname{cis}\frac{\pi}{4}\\
w &= \sqrt{(-1)^2 + 1^2}\operatorname{cis}\tan^{-1}\frac{1}{-1}\\
&= \sqrt{2}\operatorname{cis}\frac{3\pi}{4}\\
zw &= \sqrt{2}\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4} + \frac{3\pi}{4}\right)\\
&= 2\operatorname{cis}\pi\\
\frac{z}{w} &= \frac{\sqrt{2}}{\sqrt{2}}\operatorname{cis}\left(\frac{\pi}{4} - \frac{3\pi}{4}\right)\\
&= \operatorname{cis}\left(-\frac{\pi}{2}\right)
\end{aligned}$$

6. Show that the variable part of each expression is a scalar multiple of that of the other, i.e. $(2\mathbf{i} - 7\mathbf{j}) = -1(-2\mathbf{i} + 7\mathbf{j})$.

7. Show that the variable parts are perpendicular vectors, using the scalar product, i.e. $(6\mathbf{i} - 4\mathbf{j}) \cdot (2\mathbf{i} + 3\mathbf{j}) = 12 - 12 = 0$.

8. (a) The only scalar multiples of non-parallel, non-zero vectors that equates them is zero, so $p = q = 0$.

   (b) $p - 3 = 0$ so $p = 3$, and $q = 0$.

   (c) $p + 2 = 0$ so $p = -2$, and $q - 1 = 0$ so $q = 1$.

   (d)
$$\begin{aligned}
p\mathbf{a} + 2\mathbf{b} &= 3\mathbf{a} - q\mathbf{b}\\
p\mathbf{a} - 3\mathbf{a} &= -q\mathbf{b} - 2\mathbf{b}\\
(p-3)\mathbf{a} &= (-q-2)\mathbf{b}\\
p &= 3\\
q &= -2
\end{aligned}$$

   (e)
$$\begin{aligned}
p\mathbf{a} + q\mathbf{a} + p\mathbf{b} - 2q\mathbf{b} &= 3\mathbf{a} + 6\mathbf{b}\\
(p+q-3)\mathbf{a} &= (6-p+2q)\mathbf{b}\\
p + q &= 3\\
p - 2q &= 6\\
p &= 4\\
q &= -1
\end{aligned}$$

   (f)
$$\begin{aligned}
p\mathbf{a} + 2\mathbf{a} - 2p\mathbf{b} &= \mathbf{b} + 5q\mathbf{b} - q\mathbf{a}\\
(p+2+q)\mathbf{a} &= (1+5q+2p)\mathbf{b}\\
p + q &= -2\\
2p + 5q &= -1\\
-2p - 2q &= 4\\
3q &= 3\\
q &= 1\\
p &= -3
\end{aligned}$$

9.
$$\begin{aligned}
3\mathbf{i} + \mathbf{j} + \lambda(7\mathbf{i} - 5\mathbf{j}) &= 5\mathbf{i} - 6\mathbf{j} + \mu(4\mathbf{i} - \mathbf{j})\\
(3 + 7\lambda - 5 - 4\mu)\mathbf{i} &= (-6 - \mu - 1 + 5\lambda)\mathbf{j}\\
(7\lambda - 4\mu - 2)\mathbf{i} &= (5\lambda - \mu - 7)\mathbf{j}\\
7\lambda - 4\mu &= 2\\
5\lambda - \mu &= 7\\
-20\lambda + 4\mu &= -28\\
-13\lambda &= -26\\
\lambda &= 2\\
\mathbf{r} &= 3\mathbf{i} + \mathbf{j} + 2(7\mathbf{i} - 5\mathbf{j})\\
&= 17\mathbf{i} - 9\mathbf{j}
\end{aligned}$$

10. (a) $\frac{\mathrm{d}}{\mathrm{d}x}x^3 = 3x^2$

    (b) $\frac{\mathrm{d}}{\mathrm{d}x}6x^5 = 30x^4$

    (c) $\frac{\mathrm{d}}{\mathrm{d}x}(x^2 - 7x + 3) = 2x - 7$

    (d) $\frac{\mathrm{d}}{\mathrm{d}x}((x+4)(x+2)) = \frac{\mathrm{d}}{\mathrm{d}x}(x^2+6x+8) = 2x+6$
    (You could use the product rule here if you preferred.)

    (e) $\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^3+x}{x} = \frac{\mathrm{d}}{\mathrm{d}x}(x^2 + 1) = 2x$
    (You could use the quotient rule here if you preferred, but this approach is probably simpler.)

    (f) $\frac{\mathrm{d}}{\mathrm{d}x}(2x+3)^3 = 3(2x+3)(2) = 12x + 18$

11.
$$\begin{aligned}
4\pi &= 6\theta\\
\theta &= \frac{2\pi}{3}
\end{aligned}$$
polar coordinates : $\left(4\pi, \frac{2\pi}{3}\right)$
$$\begin{aligned}
x &= 4\pi\cos\frac{2\pi}{3}\\
&= -2\pi\\
y &= 4\pi\sin\frac{2\pi}{3}\\
&= 2\sqrt{3}\pi
\end{aligned}$$
Cartesian coordinates : $(-2\pi, 2\sqrt{3}\pi)$

12. (a)
$$\begin{aligned} p(2\mathbf{i}+4\mathbf{j})+q(5\mathbf{i}-3\mathbf{j}) &= -9\mathbf{i}+21\mathbf{j} \\ 2p+5q &= -9 \\ 4p-3q &= 21 \\ -4p-10q &= 18 \\ -13q &= 39 \\ q &= -3 \\ 2p+5(-3) &= -9 \\ 2p &= 6 \\ p &= 3 \\ \therefore \quad -9\mathbf{i}+21\mathbf{j} &= 3\mathbf{a}-3\mathbf{b} \end{aligned}$$

(b)
$$\begin{aligned} p(2\mathbf{i}+4\mathbf{j})+q(5\mathbf{i}-3\mathbf{j}) &= 4\mathbf{i}-18\mathbf{j} \\ 2p+5q &= 4 \\ 4p-3q &= -18 \\ -4p-10q &= -8 \\ -13q &= -26 \\ q &= 2 \\ 2p+5(2) &= 4 \\ 2p &= -6 \\ p &= -3 \\ \therefore \quad 4\mathbf{i}-18\mathbf{j} &= -3\mathbf{a}+2\mathbf{b} \end{aligned}$$

(c)
$$\begin{aligned} p(2\mathbf{i}+4\mathbf{j})+q(5\mathbf{i}-3\mathbf{j}) &= -7\mathbf{i}+12\mathbf{j} \\ 2p+5q &= -7 \\ 4p-3q &= 12 \\ -4p-10q &= 14 \\ -13q &= 26 \\ q &= -2 \\ 2p+5(-2) &= -7 \\ 2p &= 3 \\ p &= 1.5 \\ \therefore \quad -7\mathbf{i}+12\mathbf{j} &= 1.5\mathbf{a}-2\mathbf{b} \end{aligned}$$

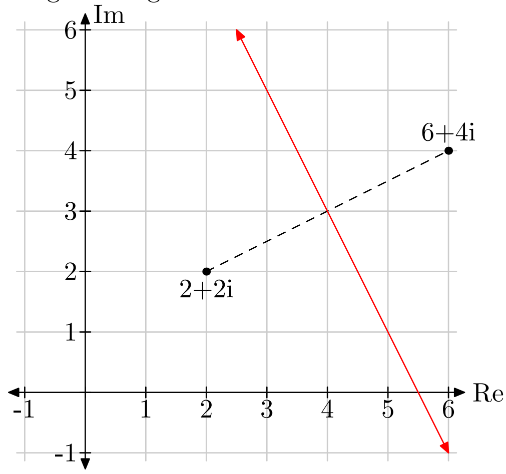
(d)
$$\begin{aligned} p(2\mathbf{i}+4\mathbf{j})+q(5\mathbf{i}-3\mathbf{j}) &= -34\mathbf{i}+23\mathbf{j} \\ 2p+5q &= -34 \\ 4p-3q &= 23 \\ -4p-10q &= 68 \\ -13q &= 91 \\ q &= -7 \\ 2p+5(-7) &= -34 \\ 2p &= 1 \\ p &= 0.5 \\ \therefore \quad -34\mathbf{i}+23\mathbf{j} &= 0.5\mathbf{a}-7\mathbf{b} \end{aligned}$$

13. $\sin^2 \mathrm{A} \cos \mathrm{A} = (1-\cos^2 \mathrm{A}) \cos \mathrm{A} = \cos \mathrm{A} - \cos^3 \mathrm{A}$

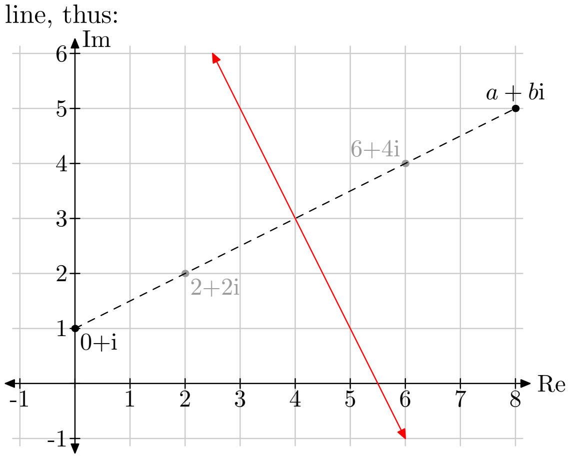
14.
$$\begin{aligned} \sin 2\theta \cos\theta &= (2\sin\theta\cos\theta)\cos\theta \\ &= 2\sin\theta\cos^2\theta \\ &= 2\sin\theta(1-\sin^2\theta) \\ &= 2\sin\theta - 2\sin^3\theta \end{aligned}$$

15.
$$\begin{aligned} 2\cos^2 x &= 2-\sin x \\ 2(1-\sin^2 x) &= 2-\sin x \\ 2-2\sin^2 x &= 2-\sin x \\ -2\sin^2 x &= -\sin x \end{aligned}$$
$$2\sin x = 1 \quad \text{or} \quad \sin x = 0$$
$$\sin x = 0.5 \qquad x \in \{0, \pi, 2\pi\}$$
$$x \in \left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$$
Solutions are $0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, 2\pi$.

16. From the first set of points given we can obtain the Argand diagram:

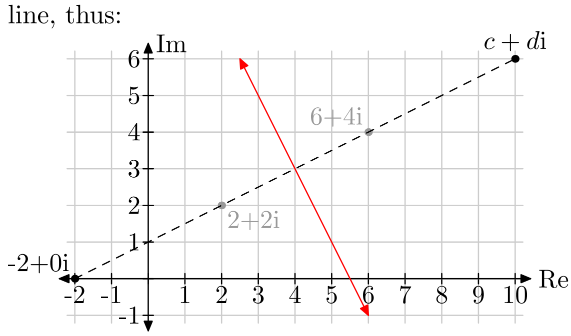


From the second set we see that the line is the set of points equidistant from $(0+\mathrm{i})$ and $(a+b\mathrm{i})$, hence $(a+b\mathrm{i})$ is the reflection of $(0+\mathrm{i})$ in the line, thus:



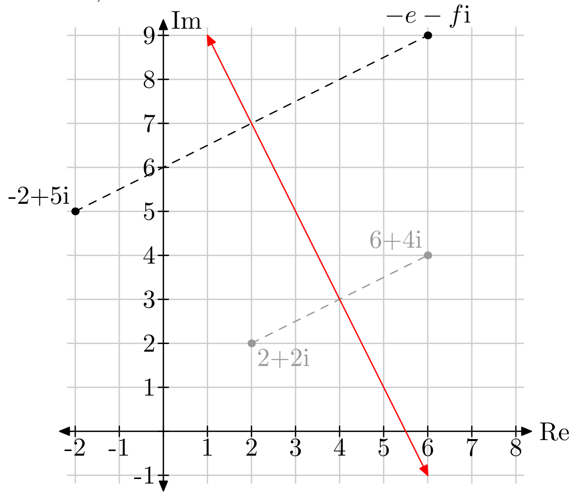
giving us $a = 8$ and $b = 5$.

From the third set we see that the line is the set of points equidistant from $(-2+0\mathrm{i})$ and $(c+d\mathrm{i})$, hence $(c+d\mathrm{i})$ is the reflection of $(-2+0\mathrm{i})$ in the line, thus:



giving us $c = 10$ and $d = 6$.

From the fourth set we see that the line is the set of points equidistant from $(-2+5\mathrm{i})$ and $(-e-f\mathrm{i})$, hence $(-e-f\mathrm{i})$ is the reflection of $(-2+5\mathrm{i})$ in the line, thus:



giving us $e = -6$ and $f = -9$.

17. $\begin{pmatrix} 14 \\ 7 \end{pmatrix} + t\begin{pmatrix} -8 \\ 6 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix} + t\begin{pmatrix} 4 \\ 10 \end{pmatrix}$

i components:

$$\begin{aligned} 14 - 8t &= -4 + 4t \\ 12t &= 18 \\ t &= 1.5 \end{aligned}$$

j components:

$$7 + 6(1.5) = 1 + 10(1.5)$$

Since the value of $t = 1.5$ satisfies both **i** and **j** components, a collision occurs at 12:30pm + 1.5 hours, i.e. at 2pm. The location of the collision is given by

$$\begin{aligned} \mathbf{r} &= \begin{pmatrix} 14 \\ 7 \end{pmatrix} + 1.5\begin{pmatrix} -8 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 14 \\ 7 \end{pmatrix} + \begin{pmatrix} -12 \\ 9 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 16 \end{pmatrix} \text{km} \end{aligned}$$

18. (a) $$\begin{aligned} {}_{\mathrm{A}}\mathbf{v}_{\mathrm{B}} &= \mathbf{v}_{\mathrm{A}} - \mathbf{v}_{\mathrm{B}} \\ &= 13\mathbf{i} + 2\mathbf{j} - (5\mathbf{i} + 8\mathbf{j}) \\ &= 8\mathbf{i} - 6\mathbf{j} \end{aligned}$$

$$\begin{aligned} \overrightarrow{\mathrm{BA}} &= \mathbf{r}_{\mathrm{A}} - \mathbf{r}_{\mathrm{B}} \\ &= (12\mathbf{i} + 15\mathbf{j}) - (19\mathbf{i} + 16\mathbf{j}) \\ &= -7\mathbf{i} - \mathbf{j} \end{aligned}$$

$$\begin{aligned} \overrightarrow{\mathrm{BP}} &= \overrightarrow{\mathrm{BA}} + t\,{}_{\mathrm{A}}\mathbf{v}_{\mathrm{B}} \\ &= -7\mathbf{i} - \mathbf{j} + t(8\mathbf{i} - 6\mathbf{j}) \\ &= (-7 + 8t)\mathbf{i} + (-1 - 6t)\mathbf{j} \end{aligned}$$

$$\begin{aligned} \overrightarrow{\mathrm{BP}} \cdot {}_{\mathrm{A}}\mathbf{v}_{\mathrm{B}} &= 0 \\ ((-7 + 8t)\mathbf{i} + (-1 - 6t)\mathbf{j}) \cdot (8\mathbf{i} - 6\mathbf{j}) &= 0 \\ 8(-7 + 8t) - 6(-1 - 6t) &= 0 \\ -56 + 64t + 6 + 36t &= 0 \\ 100t - 50 &= 0 \\ t &= 0.5 \end{aligned}$$

$$\begin{aligned} \left|\overrightarrow{\mathrm{BP}}\right| &= |(-7 + 8 \times 0.5)\mathbf{i} + (-1 - 6 \times 0.5)\mathbf{j}| \\ &= |-3\mathbf{i} - 4\mathbf{j}| \\ &= 5\text{km} \end{aligned}$$

The minimum distance is 0.5 km and occurs at 3:30pm.

(b) The position of the ships at time t is:

$$\begin{aligned} \mathbf{r}_{\mathrm{A}}(t) &= 12\mathbf{i} + 15\mathbf{j} + t(13\mathbf{i} + 2\mathbf{j}) \\ &= (12 + 13t)\mathbf{i} + (15 + 2t)\mathbf{j} \\ \mathbf{r}_{\mathrm{B}}(t) &= 19\mathbf{i} + 16\mathbf{j} + t(5\mathbf{i} + 8\mathbf{j}) \\ &= (19 + 5t)\mathbf{i} + (16 + 8t)\mathbf{j} \end{aligned}$$

The displacement $\overrightarrow{\mathrm{AB}}$ at time t is given by

$$\begin{aligned} \overrightarrow{\mathrm{AB}}(t) &= \mathbf{r}_{\mathrm{B}}(t) - \mathbf{r}_{\mathrm{A}}(t) \\ &= (19 + 5t)\mathbf{i} + (16 + 8t)\mathbf{j} \\ &\quad - ((12 + 13t)\mathbf{i} + (15 + 2t)\mathbf{j}) \\ &= (7 - 8t)\mathbf{i} + (1 + 6t)\mathbf{j} \\ d &= |\overrightarrow{\mathrm{AB}}| \\ d^2 &= (7 - 8t)^2 + (1 + 6t)^2 \\ \frac{\mathrm{d}}{\mathrm{d}t}d^2 &= 2(7 - 8t)(-8) + 2(1 + 6t)(6) \\ &= -112 + 128t + 12 + 72t \\ &= 200t - 100 \end{aligned}$$

At the turning point,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}d^2 &= 0 \\ 200t - 100 &= 0 \\ t &= 0.5 \end{aligned}$$

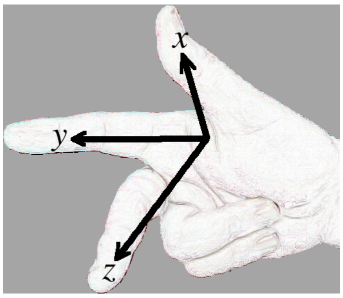
$$\begin{aligned} d &= \sqrt{(7 - 8 \times 0.5)^2 + (1 + 6 \times 0.5)^2} \\ &= \sqrt{3^2 + 4^2} \\ &= 5\text{km} \end{aligned}$$

The minimum distance is 0.5 km and occurs at 3:30pm.

Chapter 3

Exercise 3A

1. For students with limited experience with hardware, the right hand screw rule explained in Sadler is not a very helpful mnemonic. An alternative method of remembering the relationships between axes in three dimensions is the "Right Hand Rule". In this rule, the x, y and z axes correspond to thumb, index finger and middle finger respectively as illustrated below.



2. (a) $\mathbf{a}+\mathbf{b} = (2+3)\mathbf{i} + (6+8)\mathbf{j} + (3-1)\mathbf{k} = 5\mathbf{i} + 14\mathbf{j} + 2\mathbf{k}$

   (b) $\mathbf{a}-\mathbf{b} = (2-3)\mathbf{i} + (6-8)\mathbf{j} + (3--1)\mathbf{k} = -\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$

   (c) $2\mathbf{a}+\mathbf{b} = (2\times 2+3)\mathbf{i} + (2\times 6+8)\mathbf{j} + (2\times 3-1)\mathbf{k} = 7\mathbf{i} + 20\mathbf{j} + 5\mathbf{k}$

   (d) $2(\mathbf{a}+\mathbf{b}) = 2(5\mathbf{i} + 14\mathbf{j} + 2\mathbf{k}) = 10\mathbf{i} + 28\mathbf{j} + 4\mathbf{k}$

   (e) $\mathbf{a}\cdot\mathbf{b} = 2\times 3 + 6\times 8 + 3\times(-1) = 51$

   (f) $\mathbf{b}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{b} = 51$

   (g) $|\mathbf{a}| = \sqrt{2^2+6^2+3^2} = \sqrt{49} = 7$

   (h) $|\mathbf{a}+\mathbf{b}| = \sqrt{5^2+14^2+2^2} = \sqrt{225} = 15$

3. (a) $\mathbf{c}+\mathbf{d} = \begin{pmatrix} -1+2 \\ 4+0 \\ 3+4 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$

   (b) $\mathbf{c}-\mathbf{d} = \begin{pmatrix} -1-2 \\ 4-0 \\ 3-4 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix}$

   (c) $2\mathbf{c}+\mathbf{d} = \begin{pmatrix} 2\times -1+2 \\ 2\times 4+0 \\ 2\times 3+4 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 10 \end{pmatrix}$

   (d) $2(\mathbf{c}+\mathbf{d}) = 2\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 14 \end{pmatrix}$

   (e) $\mathbf{c}\cdot\mathbf{d} = -1\times 2 + 4\times 0 + 3\times 4 = 10$

   (f) $\mathbf{d}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{d} = 10$

   (g) $|\mathbf{c}| = \sqrt{(-1)^2+4^2+3^2} = \sqrt{26}$

   (h) $|\mathbf{c}+\mathbf{d}| = \sqrt{1^2+4^2+7^2} = \sqrt{66}$

4. (a) $\mathbf{e}-\mathbf{f} = \langle 1--1, 4-2, -3-0\rangle = \langle 2, 2, -3\rangle$

   (b) $\mathbf{e}-2\mathbf{f} = \langle 1-2\times -1, 4-2\times 2, -3-2\times 0\rangle = \langle 3, 0, -3\rangle$

   (c) $2\mathbf{e}+\mathbf{f} = \langle 2\times 1+-1, 2\times 4+2, 2\times -3+0\rangle = \langle 1, 10, -6\rangle$

   (d) $\mathbf{e}+\mathbf{f} = \langle 1+-1, 4+2, -3+0\rangle = \langle 0, 6, -3\rangle$

   (e) $\mathbf{e}\cdot\mathbf{f} = 1\times -1 + 4\times 2 + -3\times 0 = 7$

   (f) $(2\mathbf{e})\cdot(3\mathbf{f}) = 2\times 3\times \mathbf{e}\cdot\mathbf{f} = 6\times 7 = 42$

   (g) $(\mathbf{e}-\mathbf{f})\cdot(\mathbf{e}-\mathbf{f}) = 2^2+2^2+(-3)^2 = 17$

   (h) $|\mathbf{e}-\mathbf{f}| = \sqrt{2^2+2^2+(-3)^2} = \sqrt{17}$

5. (a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (3\mathbf{i}+2\mathbf{j}+\mathbf{k}) - (2\mathbf{i}+3\mathbf{j}+-4\mathbf{k}) = \mathbf{i}-\mathbf{j}+5\mathbf{k}$

   (b) $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (-5\mathbf{i}-\mathbf{j}+\mathbf{k}) - (3\mathbf{i}+2\mathbf{j}+\mathbf{k}) = -8\mathbf{i}-3\mathbf{j}$

(c) $\overrightarrow{\mathrm{CA}} = \overrightarrow{\mathrm{OA}} - \overrightarrow{\mathrm{OC}}$
$= (2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) - (-5\mathbf{i} - \mathbf{j} + \mathbf{k})$
$= 7\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$

(d) $\overrightarrow{\mathrm{AC}} = -\overrightarrow{\mathrm{AC}}$
$= -(7\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$
$= -7\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$

6. (a) $\mathbf{p} + \mathbf{q} = \begin{pmatrix} 3+4 \\ 2-1 \\ 1+3 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 4 \end{pmatrix}$

(b) $\mathbf{q} + \mathbf{r} = \begin{pmatrix} 4+2 \\ -1+0 \\ 3+1 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 4 \end{pmatrix}$

(c) $(\mathbf{p} + \mathbf{q}) \cdot (\mathbf{q} + \mathbf{r}) = \begin{pmatrix} 7 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -1 \\ 4 \end{pmatrix}$
$= 7 \times 6 + 1 \times -1 + 4 \times 4$
$= 57$

7. (a) $|\mathbf{u}| = \sqrt{3^2 + (-2)^2 + 6^2}$
$= 7$

(b) $|\mathbf{v}| = \sqrt{2^2 + 14^2 + 5^2}$
$= 15$

(c) $\mathbf{u} \cdot \mathbf{v} = 3 \times 2 + -2 \times 14 + 6 \times 5$
$= 8$

(d) $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$
$8 = 7 \times 15 \cos\theta$
$\theta = \cos^{-1} \dfrac{8}{105}$
$= 85.6°$

8. $\cos\angle\mathrm{AOB} = \dfrac{\overrightarrow{\mathrm{OA}} \cdot \overrightarrow{\mathrm{OB}}}{|\overrightarrow{\mathrm{OA}}||\overrightarrow{\mathrm{OB}}|}$
$= \dfrac{1 \times 2 + 1 \times -1 + -1 \times 2}{\sqrt{1^2 + 1^2 + (-1)^2}\sqrt{2^2 + (-1)^2 + 2^2}}$
$= \dfrac{-1}{3\sqrt{3}}$
$\angle\mathrm{AOB} = \cos^{-1} \dfrac{-1}{3\sqrt{3}}$
$= 101°$

9. $\cos\theta = \dfrac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{p}||\mathbf{q}|}$
$= \dfrac{-1 \times -1 + 2 \times 1 + 1 \times -2}{\sqrt{(-1)^2 + 2^2 + 1^2}\sqrt{(-1)^2 + 1^2 + (-2)^2}}$
$= \dfrac{1}{6}$
$\theta = \cos^{-1} \dfrac{1}{6}$
$= 80°$

10. $\cos\theta = \dfrac{\mathbf{s} \cdot \mathbf{t}}{|\mathbf{s}||\mathbf{t}|}$
$= \dfrac{2 \times 3 + 1 \times 0 + -1 \times 3}{\sqrt{2^2 + 1^2 + (-1)^2}\sqrt{3^2 + 0^2 + 3^2}}$
$= \dfrac{3}{\sqrt{6} \times 3\sqrt{2}}$
$= \dfrac{1}{\sqrt{6}\sqrt{2}}$
$\theta = \cos^{-1} \dfrac{1}{\sqrt{6}\sqrt{2}}$
$= 73°$

11. (a) Let **u** be a scalar multiple of **r** having unit magnitude.
$|\mathbf{r}| = \sqrt{2^2 + (-3)^2 + 6^2}$
$= 7$
$\therefore \quad \mathbf{u} = \dfrac{1}{7}\mathbf{r}$
$= \dfrac{1}{7}(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$

(b) Let **v** be a scalar multiple of **r** having the same magnitude as **s**. If it's a scalar multiple of **r** then it's also a scalar multiple of **u**, so
$|\mathbf{s}| = \sqrt{3^2 + 4^2}$
$= 5$
$\therefore \quad \mathbf{v} = 5\mathbf{u}$
$= \dfrac{5}{7}(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$

(c) Scale **s** by the ratio of the magnitudes of **r** and **s**.
$\dfrac{|\mathbf{r}|}{|\mathbf{s}|}\mathbf{s} = \dfrac{7}{5}(3\mathbf{i} + 4\mathbf{k})$

(d) $\cos\theta = \dfrac{\mathbf{r} \cdot \mathbf{s}}{|\mathbf{r}||\mathbf{s}|}$
$= \dfrac{2 \times 3 - 3 \times 0 + 6 \times 4}{7 \times 5}$
$= \dfrac{30}{35}$
$= \dfrac{6}{7}$
$\theta = \cos^{-1} \dfrac{6}{7}$
$= 31°$

12. (a) Vectors are scalar multiples of each other (the second is double the first) so they are parallel.

(b) Not scalar multiples, so not parallel.
$(3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 3 - 2 - 3$
$= -2$
Dot product is not zero, so not perpendicular.

(c) Not scalar multiples, so not parallel.
$\langle 1, 3, -2 \rangle \cdot \langle -2, 3, 1 \rangle = -2 + 9 - 2$
$= 5$
Dot product is not zero, so not perpendicular.

(d) Not scalar multiples, so not parallel.

$$\begin{aligned}\langle 1,2,3\rangle \cdot \langle 3,3,-3\rangle &= 3+6-9\\ &= 0\end{aligned}$$

Dot product is zero, so vectors are perpendicular.

(e) Not scalar multiples, so not parallel.

$$\begin{aligned}\begin{pmatrix}3\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}5\\-7\\1\end{pmatrix} &= 15-14-1\\ &= 0\end{aligned}$$

Dot product is zero, so vectors are perpendicular.

(f) Not scalar multiples, so not parallel.

$$\begin{aligned}\begin{pmatrix}-2\\6\\8\end{pmatrix}\cdot\begin{pmatrix}1\\-3\\4\end{pmatrix} &= -2-18+32\\ &= 12\end{aligned}$$

Dot product is not zero, so vectors are not perpendicular.

(g) Not scalar multiples, so not parallel.

$$\begin{aligned}\begin{pmatrix}3\\1\\5\end{pmatrix}\cdot\begin{pmatrix}6\\2\\-4\end{pmatrix} &= 18+2-20\\ &= 0\end{aligned}$$

Dot product is zero, so vectors are perpendicular.

13.
$$\begin{aligned}\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3 &= (5+3-2)\mathbf{i}+(10+5+3)\mathbf{j}\\ &\quad+(5+5-1)\mathbf{k}\\ &= (6\mathbf{i}+18\mathbf{j}+9\mathbf{k})\,\text{N}\\ |\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3| &= \sqrt{6^2+18^2+9^2}\\ &= 21\text{N}\end{aligned}$$

14.
$$\begin{aligned}\overrightarrow{\text{BA}} &= \overrightarrow{\text{OA}}-\overrightarrow{\text{OB}}\\ -\mathbf{i}+3\mathbf{j}+4\mathbf{k} &= (2\mathbf{i}+3\mathbf{j}-4\mathbf{k})-\overrightarrow{\text{OB}}\\ \overrightarrow{\text{OB}} &= (2\mathbf{i}+3\mathbf{j}-4\mathbf{k})-(-\mathbf{i}+3\mathbf{j}+4\mathbf{k})\\ &= 3\mathbf{i}-8\mathbf{k}\end{aligned}$$

The position vector of B is $3\mathbf{i}-8\mathbf{k}$.

15. $(\mathbf{a}+\mathbf{b})+(\mathbf{a}-\mathbf{b})=2\mathbf{a}$

$$\begin{aligned}\begin{pmatrix}7\\1\\2\end{pmatrix}+\begin{pmatrix}3\\3\\-4\end{pmatrix} &= \begin{pmatrix}10\\4\\-2\end{pmatrix}\\ \therefore\quad \mathbf{a} &= \frac{1}{2}\begin{pmatrix}10\\4\\-2\end{pmatrix}\\ &= \begin{pmatrix}5\\2\\-1\end{pmatrix}\end{aligned}$$

$(\mathbf{a}+\mathbf{b})-(\mathbf{a}-\mathbf{b})=2\mathbf{b}$

$$\begin{aligned}\begin{pmatrix}7\\1\\2\end{pmatrix}-\begin{pmatrix}3\\3\\-4\end{pmatrix} &= \begin{pmatrix}4\\-2\\6\end{pmatrix}\\ \therefore\quad \mathbf{b} &= \frac{1}{2}\begin{pmatrix}4\\-2\\6\end{pmatrix}\\ &= \begin{pmatrix}2\\-1\\3\end{pmatrix}\end{aligned}$$

16. $\mathbf{b}$ is parallel to $\mathbf{a}$ so they are scalar multiples, i.e. $\mathbf{b}=k\mathbf{a}$. By examining the $\mathbf{i}$ and $\mathbf{j}$ components we can see that $k=2$ so $p=2\times 1=2$.

$\mathbf{c}$ is perpendicular to $\mathbf{a}$ so they have zero dot product, i.e.

$$\begin{aligned}2\times 7+3q+1\times -2 &= 0\\ 12+3q &= 0\\ q &= -4\end{aligned}$$

$\mathbf{d}$ is perpendicular to $\mathbf{b}$, but $\mathbf{b}$ is parallel to $\mathbf{a}$ so $\mathbf{d}$ is perpendicular to $\mathbf{a}$.

$$\begin{aligned}\mathbf{d}\cdot\mathbf{a} &= 0\\ 3\times 2-4\times 3+r\times 1 &= 0\\ -6+r &= 0\\ r &= 6\end{aligned}$$

Note that even though both are perpendicular to $\mathbf{a}$, $\mathbf{c}$ is not parallel to $\mathbf{d}$ (as they would be in two dimensions).

17. (a) $(-4\mathbf{i}-4\mathbf{j}+11\mathbf{k})+(2\mathbf{i}+4\mathbf{j}-2\mathbf{k})=(-2\mathbf{i}+9\mathbf{k})\,\text{m}$

(b) $(-4\mathbf{i}-4\mathbf{j}+11\mathbf{k})+2\,(2\mathbf{i}+4\mathbf{j}-2\mathbf{k})=(4\mathbf{j}+7\mathbf{k})\,\text{m}$

(c)
$$\begin{aligned}(-4\mathbf{i}-4\mathbf{j}+11\mathbf{k})+3\,(2\mathbf{i}+4\mathbf{j}-2\mathbf{k}) &= (2\mathbf{i}+8\mathbf{j}+5\mathbf{k})\,\text{m}\\ |\,(2\mathbf{i}+8\mathbf{j}+5\mathbf{k})\,| &= \sqrt{2^2+8^2+5^2}\\ &= 9.6\text{m}\end{aligned}$$

(d)
$$\begin{aligned}|\,(-4\mathbf{i}-4\mathbf{j}+11\mathbf{k})+t\,(2\mathbf{i}+4\mathbf{j}-2\mathbf{k})\,| &= 15\\ |(-4+2t)\mathbf{i}+(-4+4t)\mathbf{j}+(11-2t)\mathbf{k}| &= 15\\ \sqrt{(-4+2t)^2+(-4+4t)^2+(11-2t)^2} &= 15\\ (-4+2t)^2+(-4+4t)^2+(11-2t)^2 &= 15^2\\ 24t^2-92t+153 &= 225\end{aligned}$$

$t=4.5\text{s}$ (ignoring the negative root)

18. A, B and C are collinear if $\overrightarrow{AB} = k\overrightarrow{AC}$.

$$\begin{aligned}\overrightarrow{AB} &= (3\mathbf{i}+\mathbf{j}-4\mathbf{k})-(7\mathbf{i}+5\mathbf{j})\\ &= -4\mathbf{i}-4\mathbf{j}-4\mathbf{k}\\ \overrightarrow{AC} &= (2\mathbf{i}-5\mathbf{k})-(7\mathbf{i}+5\mathbf{j})\\ &= -5\mathbf{i}-5\mathbf{j}-5\mathbf{k}\\ \overrightarrow{AB} &= \frac{4}{5}\overrightarrow{AC}\end{aligned}$$

$\therefore$ A, B and C are collinear. □

19. Let P be the point that divides AB in the ratio 2:3.

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA} + \frac{2}{5}\overrightarrow{AB}\\ &= \begin{pmatrix}3\\4\\4\end{pmatrix} + \frac{2}{5}\left(\begin{pmatrix}-2\\9\\-1\end{pmatrix} - \begin{pmatrix}3\\4\\4\end{pmatrix}\right)\\ &= \begin{pmatrix}3\\4\\4\end{pmatrix} + \frac{2}{5}\begin{pmatrix}-5\\5\\-5\end{pmatrix}\\ &= \begin{pmatrix}3\\4\\4\end{pmatrix} + \begin{pmatrix}-2\\2\\-2\end{pmatrix}\\ &= \begin{pmatrix}1\\6\\2\end{pmatrix}\end{aligned}$$

20.
$$\begin{aligned}\overrightarrow{AB} &= (4\mathbf{i}-\mathbf{j}+\mathbf{k})-(3\mathbf{i}+2\mathbf{j}-\mathbf{k})\\ &= \mathbf{i}-3\mathbf{j}+2\mathbf{k}\\ \overrightarrow{OP} &= \overrightarrow{OB}+\overrightarrow{BP}\\ &= \overrightarrow{OB}+\overrightarrow{AB}\\ &= (4\mathbf{i}-\mathbf{j}+\mathbf{k})+(\mathbf{i}-3\mathbf{j}+2\mathbf{k})\\ &= 5\mathbf{i}-4\mathbf{j}+3\mathbf{k}\end{aligned}$$

21.
$$\begin{aligned}\overrightarrow{AB} &= (9\mathbf{i}+6\mathbf{j}-9\mathbf{k})-(5\mathbf{i}-2\mathbf{j}+3\mathbf{k})\\ &= 4\mathbf{i}+8\mathbf{j}-12\mathbf{k}\\ \overrightarrow{OP} &= \overrightarrow{OA}+\frac{3}{4}\overrightarrow{AB}\\ &= (5\mathbf{i}-2\mathbf{j}+3\mathbf{k})+\frac{3}{4}(4\mathbf{i}+8\mathbf{j}-12\mathbf{k})\\ &= (5\mathbf{i}-2\mathbf{j}+3\mathbf{k})+(3\mathbf{i}+6\mathbf{j}-9\mathbf{k})\\ &= 8\mathbf{i}+4\mathbf{j}-6\mathbf{k}\end{aligned}$$

22.
$$\begin{aligned}\overrightarrow{AB} &= (3\mathbf{k})-(2\mathbf{i}+3\mathbf{j}+2\mathbf{k})\\ &= -2\mathbf{i}-3\mathbf{j}+\mathbf{k}\\ \overrightarrow{AC} &= (4\mathbf{i}-3\mathbf{j}+2\mathbf{k})-(2\mathbf{i}+3\mathbf{j}+2\mathbf{k})\\ &= 2\mathbf{i}-6\mathbf{j}\\ \overrightarrow{BC} &= (4\mathbf{i}-3\mathbf{j}+2\mathbf{k})-(3\mathbf{k})\\ &= 4\mathbf{i}-3\mathbf{j}-\mathbf{k}\\ \overrightarrow{AB}\cdot\overrightarrow{AC} &= -4+18+0\\ &= 14\\ \overrightarrow{AB}\cdot\overrightarrow{BC} &= -8+9-1\\ &= 0\end{aligned}$$

$\therefore$ AB $\perp$ BC and $\triangle$ABC is right angled at B. □

23. Let α be the angle $\mathbf{a}$ makes with the $x-$axis.

$$\begin{aligned}\cos\alpha &= \frac{\mathbf{a}\cdot\mathbf{i}}{|\mathbf{a}||\mathbf{i}|}\\ &= \frac{(2\mathbf{i}+3\mathbf{j}-\mathbf{k})\cdot(\mathbf{i})}{\sqrt{2^2+3^2+(-1)^2}\times 1}\\ &= \frac{2}{\sqrt{14}}\\ \alpha &= \cos^{-1}\frac{2}{\sqrt{14}}\\ &= 57.7^\circ\end{aligned}$$

Let β be the angle $\mathbf{a}$ makes with the $y-$axis.

$$\begin{aligned}\cos\beta &= \frac{\mathbf{a}\cdot\mathbf{j}}{|\mathbf{a}||\mathbf{j}|}\\ &= \frac{(2\mathbf{i}+3\mathbf{j}-\mathbf{k})\cdot(\mathbf{j})}{\sqrt{14}}\\ &= \frac{3}{\sqrt{14}}\\ \beta &= \cos^{-1}\frac{3}{\sqrt{14}}\\ &= 36.7^\circ\end{aligned}$$

Let γ be the angle $\mathbf{a}$ makes with the $z-$axis.

$$\begin{aligned}\cos\gamma &= \frac{\mathbf{a}\cdot\mathbf{k}}{|\mathbf{a}||\mathbf{k}|}\\ &= \frac{(2\mathbf{i}+3\mathbf{j}-\mathbf{k})\cdot(\mathbf{k})}{\sqrt{14}}\\ &= \frac{-1}{\sqrt{14}}\\ \gamma &= \cos^{-1}\frac{-1}{\sqrt{14}}\\ &= 105.5^\circ\end{aligned}$$

We want the acute angle, so we need the supplementary angle:
$180 - 105.5 = 74.5^\circ$

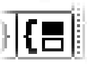
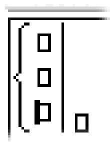
24. **Vector d:**

$$\mathbf{d} = \lambda\mathbf{a}+\mu\mathbf{b}+\eta\mathbf{c}$$

$$\begin{aligned}7\mathbf{i}-5\mathbf{j}+10\mathbf{k} = \ &\lambda(\mathbf{i}-2\mathbf{j}+3\mathbf{k})\\ &+\mu(2\mathbf{i}+\mathbf{j}-\mathbf{k})\\ &+\eta(4\mathbf{i}-\mathbf{j}+3\mathbf{k})\end{aligned}$$

Equating $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ components gives three equations in three unknowns:

$$\begin{aligned}\lambda \quad +2\mu \quad +4\eta &= 7\\ -2\lambda \quad +\mu \quad -\eta &= -5\\ 3\lambda \quad -\mu \quad +3\eta &= 10\end{aligned}$$

Solving this using the Classpad. On the 2D tab, tap on . Tap the same icon a second time to expand to three lines . Fill in the three equations. Use x, y, z in place of the Greek letters:

```
⎧ x+2y+4z=7
⎨ -2x+y-z=-5
⎩ 3x-y+3z=10 | x,y,z
        {x=1,y=-1,z=2}
```

$\mathbf{d} = \mathbf{a} - \mathbf{b} + 2\mathbf{c}$

Vector e:

$$\begin{aligned} \mathbf{e} &= \lambda\mathbf{a} + \mu\mathbf{b} + \eta\mathbf{c} \\ \mathbf{i} - 5\mathbf{j} + 8\mathbf{k} &= \lambda(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \\ &\quad + \mu(2\mathbf{i} + \mathbf{j} - \mathbf{k}) \\ &\quad + \eta(4\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \end{aligned}$$

Equating $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ components gives three equations in three unknowns:

$$\begin{aligned} \lambda \quad +2\mu \quad +4\eta &= 1 \\ -2\lambda \quad +\mu \quad -\eta &= -5 \\ 3\lambda \quad -\mu \quad +3\eta &= 8 \end{aligned}$$

Solving this using the Classpad gives $\lambda = 1, \mu = -2, \eta = 1$.

$\mathbf{e} = \mathbf{a} - 2\mathbf{b} + \mathbf{c}$

Vector f:

$$\begin{aligned} \mathbf{f} &= \lambda\mathbf{a} + \mu\mathbf{b} + \eta\mathbf{c} \\ 2\mathbf{j} - 2\mathbf{k} &= \lambda(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \\ &\quad + \mu(2\mathbf{i} + \mathbf{j} - \mathbf{k}) \\ &\quad + \eta(4\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \end{aligned}$$

Equating $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ components gives three equations in three unknowns:

$$\begin{aligned} \lambda \quad +2\mu \quad +4\eta &= 0 \\ -2\lambda \quad +\mu \quad -\eta &= 2 \\ 3\lambda \quad -\mu \quad +3\eta &= -2 \end{aligned}$$

Solving this using the Classpad gives $\lambda = -2, \mu = -1, \eta = 1$.

$\mathbf{f} = -2\mathbf{a} - \mathbf{b} + \mathbf{c}$

25. (a)
$$\begin{aligned} \overrightarrow{DC} &= (10\mathbf{i})\,\text{cm} \\ \overrightarrow{DB} &= (10\mathbf{i} + 4\mathbf{k})\,\text{cm} \\ \overrightarrow{DI} &= (3\mathbf{j} + \mathbf{k})\,\text{cm} \end{aligned}$$

(b)
$$\begin{aligned} \cos\angle IDB &= \frac{\overrightarrow{DI} \cdot \overrightarrow{DB}}{|\overrightarrow{DI}||\overrightarrow{DB}|} \\ &= \frac{(3\mathbf{j} + \mathbf{k}) \cdot (10\mathbf{i} + 4\mathbf{k})}{|(3\mathbf{j} + \mathbf{k})||(10\mathbf{i} + 4\mathbf{k})|} \\ &= \frac{0 + 0 + 4}{\sqrt{0^2 + 3^2 + 1^2}\sqrt{10^2 + 0^2 + 4^2}} \\ &= \frac{4}{\sqrt{10}\sqrt{116}} \\ \angle IDB &= 83^\circ \end{aligned}$$

26. (a)
$$\begin{aligned} \overrightarrow{AO} &= (\mathbf{0}) - (4\mathbf{i} + 2\mathbf{j}) \\ &= (-4\mathbf{i} - 2\mathbf{j}) \\ |\overrightarrow{AO}| &= \sqrt{(-4)^2 + (-2)^2 + 0^2} \\ &= \sqrt{20} \\ \overrightarrow{AE} &= (8\mathbf{k}) - (4\mathbf{i} + 2\mathbf{j}) \\ &= (-4\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}) \\ |\overrightarrow{AE}| &= \sqrt{(-4)^2 + (-2)^2 + 8^2} \\ &= \sqrt{84} \\ \cos\angle OAE &= \frac{\overrightarrow{AO} \cdot \overrightarrow{AE}}{|\overrightarrow{AO}||\overrightarrow{AE}|} \\ &= \frac{16 + 4 + 0}{\sqrt{20}\sqrt{84}} \\ &= \frac{20}{\sqrt{20}\sqrt{84}} \\ \angle OAE &= 60.8^\circ \end{aligned}$$

(b)
$$\begin{aligned} \overrightarrow{DB} &= (-4\mathbf{i} + 2\mathbf{j}) - (4\mathbf{i} - 2\mathbf{j}) \\ &= (-8\mathbf{i} + 4\mathbf{j}) \\ |\overrightarrow{DB}| &= \sqrt{(-8)^2 + 4^2 + 0^2} \\ &= \sqrt{80} \\ \cos\theta &= \frac{\overrightarrow{AE} \cdot \overrightarrow{DB}}{|\overrightarrow{AE}||\overrightarrow{DB}|} \\ &= \frac{32 - 8 + 0}{\sqrt{84}\sqrt{80}} \\ &= \frac{24}{\sqrt{84}\sqrt{80}} \\ \theta &= 73.0^\circ \end{aligned}$$

27. (a)
$$\begin{aligned} \overrightarrow{AB} &= \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix} \\ \overrightarrow{BC} &= \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} \\ \overrightarrow{AC} &= \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix} \end{aligned}$$

(b)
$$\begin{aligned} |\overrightarrow{AB}| &= \sqrt{6^2 + 2^2 + 1^2} \\ &= \sqrt{41} \\ |\overrightarrow{BC}| &= \sqrt{(-1)^2 + (-2)^2 + 3^2} \\ &= \sqrt{14} \\ |\overrightarrow{AC}| &= \sqrt{5^2 + 0^2 + 4^2} \\ &= \sqrt{41} \end{aligned}$$

AB = AC so △ABC is isosceles. □

(c)
$$\begin{aligned} \overrightarrow{AC} \cdot \overrightarrow{AC} &= 25 + 0 + 16 \\ &= 41 \end{aligned}$$

Alternatively,
$$\begin{aligned} \overrightarrow{AC} \cdot \overrightarrow{AC} &= |\overrightarrow{AC}|^2 \\ &= \left(\sqrt{41}\right)^2 \\ &= 41 \end{aligned}$$

(d)
$$\begin{aligned}
\cos\angle A &= \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|}\\
&= \frac{30+0+4}{\sqrt{41}\sqrt{41}}\\
&= \frac{34}{41}\\
\angle A &= 34.0^\circ\\
\angle A+\angle B+\angle C &= 180^\circ\\
\angle C &= \angle B \text{ (isosceles)}\\
\therefore\quad \angle A+2\angle B &= 180^\circ\\
2\angle B &= 180^\circ - 34.0^\circ\\
&= 146.0^\circ\\
\therefore\quad \angle B = \angle C &= 73^\circ
\end{aligned}$$

28. Let $\mathbf{v}_A$ be the velocity of the first bird, and $\mathbf{v}_B$ that of the second. The apparent velocity of the second from the point of view of the first is

$$\begin{aligned}
{}_B\mathbf{v}_A &= \mathbf{v}_B - \mathbf{v}_A\\
-\mathbf{i}+3\mathbf{j}-\mathbf{k} &= \mathbf{v}_B - (4\mathbf{i}+\mathbf{j}+\mathbf{k})\\
\mathbf{v}_B &= (-\mathbf{i}+3\mathbf{j}-\mathbf{k}) + (4\mathbf{i}+\mathbf{j}+\mathbf{k})\\
&= (3\mathbf{i}+4\mathbf{j})\text{ m/s}\\
|\mathbf{v}_B| &= \sqrt{3^2+4^2+0^2}\\
&= 5 \text{ m/s}
\end{aligned}$$

29. If the aircraft are following the same path, their velocity vectors must be scalar multiples. B is behind A but is closing the gap at 35 m/s, so it must be flying 35 m/s faster than A.

$$\begin{aligned}
|\mathbf{v}_A| &= \sqrt{60^2+(-120^2)+40^2}\\
&= 140 \text{ m/s}\\
|\mathbf{v}_B| &= 140+35\\
&= 175 \text{ m/s}\\
\mathbf{v}_B &= \frac{175}{140}\mathbf{v}_A\\
&= \frac{5}{4}\mathbf{v}_A\\
&= \frac{5}{4}(60\mathbf{i}-120\mathbf{j}+40\mathbf{k})\\
&= (75\mathbf{i}-150\mathbf{j}+50\mathbf{k})\text{ m/s}
\end{aligned}$$

Exercise 3B

1. (a) $\mathbf{r} = 3\mathbf{i}+2\mathbf{j}-\mathbf{k}+\lambda(2\mathbf{i}-\mathbf{j}+2\mathbf{k})$

(b) $\begin{cases} x = 3+2\lambda\\ y = 2-\lambda\\ z = -1+2\lambda\end{cases}$

2. Note that there are many possible correct answers to questions like these. This answer is different to Sadler's. Convince yourself that they are both correct. (What are some other possible correct answers?)

(a)
$$\begin{aligned}
\overrightarrow{AB} &= (3\mathbf{i}+\mathbf{j}+\mathbf{k})-(4\mathbf{i}+2\mathbf{j}+3\mathbf{k})\\
&= -\mathbf{i}-\mathbf{j}-2\mathbf{k}\\
\mathbf{r} &= 4\mathbf{i}+2\mathbf{j}+3\mathbf{k}+\lambda\overrightarrow{AB}\\
&= 4\mathbf{i}+2\mathbf{j}+3\mathbf{k}+\lambda(-\mathbf{i}-\mathbf{j}-2\mathbf{k})
\end{aligned}$$

(b) $\begin{cases} x = 4-\lambda\\ y = 2-\lambda\\ z = 3-2\lambda\end{cases}$

3.
$$\begin{aligned}
\mathbf{r}\cdot(3\mathbf{i}-\mathbf{j}+5\mathbf{k}) &= (2\mathbf{i}-3\mathbf{j}+2\mathbf{k})\cdot(3\mathbf{i}-\mathbf{j}+5\mathbf{k})\\
\mathbf{r}\cdot(3\mathbf{i}-\mathbf{j}+5\mathbf{k}) &= 6+3+10\\
\mathbf{r}\cdot(3\mathbf{i}-\mathbf{j}+5\mathbf{k}) &= 19
\end{aligned}$$

4.
$$\begin{aligned}
\mathbf{r}\cdot\begin{pmatrix}5\\1\\3\end{pmatrix} &= \begin{pmatrix}2\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}5\\1\\3\end{pmatrix}\\
\mathbf{r}\cdot\begin{pmatrix}5\\1\\3\end{pmatrix} &= 10+1-9\\
\mathbf{r}\cdot\begin{pmatrix}5\\1\\3\end{pmatrix} &= 2
\end{aligned}$$

5. No working required. See Sadler for solution.

6. No working required. See Sadler for solution.

7. Substitute the given point as $\mathbf{r}$:

$$a\mathbf{i}+7\mathbf{j}+5\mathbf{k} = 2\mathbf{i}+b\mathbf{j}-\mathbf{k}+\lambda(-3\mathbf{i}+\mathbf{j}+2\mathbf{k})$$

k components:

$$\begin{aligned}
5 &= -1+2\lambda\\
\lambda &= 3
\end{aligned}$$

i components:

$$\begin{aligned}
a &= 2-3\lambda\\
&= 2-3\times 3\\
&= -7
\end{aligned}$$

j components:

$$\begin{aligned}7 &= b+\lambda\\ 7 &= b+3\\ b &= 4\end{aligned}$$

8. Substitute $\mathbf{r} = \begin{pmatrix} x\\ y\\ z\end{pmatrix}$:

$$\begin{pmatrix} x\\ y\\ z\end{pmatrix}\cdot\begin{pmatrix} 3\\ 2\\ -1\end{pmatrix} = 21$$

$$3x+2y-z=21$$

(You should be able to do this by observation in a single step.)

9. No working required. (Use the inverse of the process used for the previous question.)

10. The line is parallel to $-6\mathbf{i}-4\mathbf{j}+2\mathbf{k}$.
$-6\mathbf{i}-4\mathbf{j}+2\mathbf{k} = -2\,(3\mathbf{i}+2\mathbf{j}-\mathbf{k})$
$\therefore$ the line is parallel to $(3\mathbf{i}+2\mathbf{j}-\mathbf{k})$
$(3\mathbf{i}+2\mathbf{j}-\mathbf{k})$ is perpendicular to the plane
$\therefore$ the line is perpendicular to the plane. □

11. Equate the two expressions for **r** and simplify:

$$\begin{pmatrix} 10\\ 5\\ -2\end{pmatrix}+\lambda\begin{pmatrix} 4\\ 1\\ -2\end{pmatrix} = \begin{pmatrix} 0\\ 8\\ -6\end{pmatrix}+\mu\begin{pmatrix} -1\\ -3\\ 5\end{pmatrix}$$

$$\lambda\begin{pmatrix} 4\\ 1\\ -2\end{pmatrix}-\mu\begin{pmatrix} -1\\ -3\\ 5\end{pmatrix} = \begin{pmatrix} -10\\ 3\\ -4\end{pmatrix}$$

Equate corresponding components to obtain three equations:

$$\begin{aligned}4\lambda+\mu &= -10\\ \lambda+3\mu &= 3\\ -2\lambda-5\mu &= -4\end{aligned}$$

Solve the first two of these simultaneously to obtain $\lambda=-3, \mu=2$.

Substitute these values into the third equation. If it is consistent, then the two lines intersect.

$-2(-3)-5(2) = -4$ is consistent, so the lines intersect.

Determine the point of intersection by substituting either λ or μ into its original equation:

$$\mathbf{r} = \begin{pmatrix} 10\\ 5\\ -2\end{pmatrix}-3\begin{pmatrix} 4\\ 1\\ -2\end{pmatrix} = \begin{pmatrix} -2\\ 2\\ 4\end{pmatrix}$$

12. $$\begin{pmatrix} 1\\ -2\\ 3\end{pmatrix}+\lambda\begin{pmatrix} -1\\ 3\\ 2\end{pmatrix} = \begin{pmatrix} 3\\ 13\\ -15\end{pmatrix}+\mu\begin{pmatrix} -1\\ 0\\ 4\end{pmatrix}$$

$$\lambda\begin{pmatrix} -1\\ 3\\ 2\end{pmatrix}-\mu\begin{pmatrix} -1\\ 0\\ 4\end{pmatrix} = \begin{pmatrix} 2\\ 15\\ -18\end{pmatrix}$$

Equating corresponding components:

$$\begin{aligned}-\lambda+\mu &= 2\\ 3\lambda &= 15\\ 2\lambda-4\mu &= -18\end{aligned}$$

Solve the first two of these simultaneously to obtain $\lambda=5, \mu=7$.

$2(5)-4(7) = -18$ is consistent, so the lines intersect.

The position vector of the point of intersection is:

$$\begin{aligned}\mathbf{r} &= \mathbf{i}-2\mathbf{j}+3\mathbf{k}+5\,(-\mathbf{i}+3\mathbf{j}+2\mathbf{k})\\ &= -4\mathbf{i}+13\mathbf{j}+13\mathbf{k}\end{aligned}$$

13. (a) $$\begin{pmatrix} 13\\ 1\\ 8\end{pmatrix}+\lambda\begin{pmatrix} 2\\ -1\\ 3\end{pmatrix} = \begin{pmatrix} 12\\ 2\\ 6\end{pmatrix}+\mu\begin{pmatrix} 5\\ 3\\ -8\end{pmatrix}$$

$$\lambda\begin{pmatrix} 2\\ -1\\ 3\end{pmatrix}-\mu\begin{pmatrix} 5\\ 3\\ -8\end{pmatrix} = \begin{pmatrix} -1\\ 1\\ -2\end{pmatrix}$$

Equating corresponding components:

$$\begin{aligned}2\lambda-5\mu &= -1\\ -\lambda-3\mu &= 1\\ 3\lambda+8\mu &= -2\end{aligned}$$

Solve the first two of these simultaneously to obtain $\lambda=-\frac{8}{11}, \mu=-\frac{1}{11}$.
$3\left(-\frac{8}{11}\right)+8\left(-\frac{1}{11}\right) = -\frac{32}{11} \neq -2$ is inconsistent, so the lines do not intersect.

(b) $$\begin{pmatrix} 13\\ 1\\ 8\end{pmatrix}+\lambda\begin{pmatrix} 2\\ -1\\ 3\end{pmatrix} = \begin{pmatrix} -5\\ 2\\ -3\end{pmatrix}+\beta\begin{pmatrix} 2\\ 1\\ -1\end{pmatrix}$$

$$\lambda\begin{pmatrix} 2\\ -1\\ 3\end{pmatrix}-\beta\begin{pmatrix} 2\\ 1\\ -1\end{pmatrix} = \begin{pmatrix} -18\\ 1\\ -11\end{pmatrix}$$

Equating corresponding components:

$$\begin{aligned}2\lambda-2\beta &= -18\\ -\lambda-\beta &= 1\\ 3\lambda+\beta &= -11\end{aligned}$$

Solve the first two of these simultaneously to obtain $\lambda=-5, \beta=4$.

$3(-5)+(4) = -11$ is consistent, so the lines intersect.

The position vector of the point of intersection is:

$$\begin{aligned}\mathbf{r} &= 13\mathbf{i}+\mathbf{j}+8\mathbf{k}-5\,(2\mathbf{i}-\mathbf{j}+3\mathbf{k})\\ &= 3\mathbf{i}+6\mathbf{j}-7\mathbf{k}\end{aligned}$$

The angle between the lines is given by

$$\begin{aligned}\cos\theta &= \frac{(2\mathbf{i}-\mathbf{j}+3\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}-\mathbf{k})}{|\,(2\mathbf{i}-\mathbf{j}+3\mathbf{k})\,|\,|\,(2\mathbf{i}+\mathbf{j}-\mathbf{k})\,|}\\ &= \frac{4-1-3}{|\,(2\mathbf{i}-\mathbf{j}+3\mathbf{k})\,|\,|\,(2\mathbf{i}+\mathbf{j}-\mathbf{k})\,|}\\ &= 0\end{aligned}$$

The angle between the lines is 90°.

14. (a) Choose one component (say, **i**) and solve for λ, then confirm that the same value of λ satisfies both of the other components.

$$\begin{aligned} \mathbf{i}\text{ components:} \quad -4 &= 1+5\lambda \\ \lambda &= -1 \\ \mathbf{j}\text{ components:} \quad -5 &= -2+3(-1) \quad ✔ \\ \mathbf{k}\text{ components:} \quad 7 &= 5-2(-1) \quad ✔ \end{aligned}$$

The same value of λ satisfies all three components, so point A is on line L.

(b) $$\begin{aligned} \mathbf{i}\text{ components:} \quad 10 &= 1+5\lambda \\ \lambda &= \frac{9}{5} \\ \mathbf{j}\text{ components:} \quad 3 &\neq -2+3(\frac{9}{5}) \quad ✗ \end{aligned}$$

The same value of λ does not satisfy both **i** and **j** components, so point B is not on line L.

(c) Point A:

$$\begin{aligned} (-4\mathbf{i}-5\mathbf{j}+7\mathbf{k})\cdot(-\mathbf{i}+3\mathbf{j}+2\mathbf{k}) &= 4-15+14 \\ &= 3 \quad ✔ \end{aligned}$$

Point B:

$$\begin{aligned} (10\mathbf{i}+3\mathbf{j}+2\mathbf{k})\cdot(-\mathbf{i}+3\mathbf{j}+2\mathbf{k}) &= -10+9+4 \\ &= 3 \quad ✔ \end{aligned}$$

(d) If line L lies on plane Π then every point on L must satisfy the defining equation for Π.

$$\begin{aligned} \left(\begin{pmatrix}1\\-2\\5\end{pmatrix}+\lambda\begin{pmatrix}5\\3\\-2\end{pmatrix}\right)\cdot\begin{pmatrix}-1\\3\\2\end{pmatrix} &= 3 \\ \begin{pmatrix}1\\-2\\5\end{pmatrix}\cdot\begin{pmatrix}-1\\3\\2\end{pmatrix}+\lambda\begin{pmatrix}5\\3\\-2\end{pmatrix}\cdot\begin{pmatrix}-1\\3\\2\end{pmatrix} &= 3 \\ -1-6+10+\lambda(-5+9-4) &= 3 \\ 3+\lambda(0) &= 3 \end{aligned}$$

which is true for all λ.

15. $$\begin{aligned} \overrightarrow{\mathrm{AB}} &= t_{\mathrm{A}}\mathbf{v}_{\mathrm{B}} \\ \begin{pmatrix}-3\\-8\\2\end{pmatrix}-\begin{pmatrix}-10\\20\\-12\end{pmatrix} &= t\left(\begin{pmatrix}5\\-10\\6\end{pmatrix}-\begin{pmatrix}4\\-6\\4\end{pmatrix}\right) \\ \begin{pmatrix}7\\-28\\14\end{pmatrix} &= t\begin{pmatrix}1\\-4\\2\end{pmatrix} \\ t &= 7\text{ s} \\ \mathbf{r} &= \mathbf{r}_{\mathrm{A}}+t\mathbf{v}_{\mathrm{A}} \\ &= \begin{pmatrix}-10\\20\\-12\end{pmatrix}+7\begin{pmatrix}5\\-10\\6\end{pmatrix} \\ &= \begin{pmatrix}25\\-50\\30\end{pmatrix}\text{ m} \end{aligned}$$

16. $$\begin{aligned} \mathbf{c} &= \begin{pmatrix}2\\2\\1\end{pmatrix}-\begin{pmatrix}1\\2\\0\end{pmatrix} \\ &= \begin{pmatrix}1\\0\\1\end{pmatrix} \end{aligned}$$

Check that $\begin{pmatrix}1\\0\\1\end{pmatrix}$ is not parallel to $\begin{pmatrix}1\\-3\\-5\end{pmatrix}$ ✔

$$\mathbf{r} = \begin{pmatrix}2\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-3\\-5\end{pmatrix}+\mu\begin{pmatrix}1\\0\\1\end{pmatrix}$$

Parametric equations:

$$\begin{cases} x = 2+\lambda+\mu & ① \\ y = 2-3\lambda & ② \\ z = 1-5\lambda+\mu & ③ \end{cases}$$

Eliminate λ from ① and ③:

$$\begin{cases} 3x+y = 8+3\mu & (3\times①+②)\rightarrow④ \\ 5y-3z = 7-3\mu & (5\times②-3\times③)\rightarrow⑤ \end{cases}$$

Now eliminate μ and simplify:

$$\begin{aligned} 3x+6y-3z &= 15 & (④+⑤) \\ x+2y-z &= 5 \end{aligned}$$

In scalar product form:

$$\begin{aligned} \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-1\end{pmatrix} &= 5 \\ \mathbf{r}\cdot\begin{pmatrix}1\\2\\-1\end{pmatrix} &= 5 \end{aligned}$$

17. $$\begin{aligned} (2\mathbf{i}+13\mathbf{j}+\mathbf{k}+\lambda(-\mathbf{i}+3\mathbf{j}-2\mathbf{k}))\cdot(2\mathbf{i}-\mathbf{j}-\mathbf{k}) &= 11 \\ 4-13-1+\lambda(-2-3+2) &= 11 \\ -10-3\lambda &= 11 \\ \lambda &= -7 \end{aligned}$$

$$\begin{aligned} \mathbf{r} &= 2\mathbf{i}+13\mathbf{j}+\mathbf{k}-7(-\mathbf{i}+3\mathbf{j}-2\mathbf{k}) \\ &= 9\mathbf{i}-8\mathbf{j}+15\mathbf{k} \end{aligned}$$

18. Let D represent the position of the debris and S the position of the spacecraft, then for a collision to occur

$$\begin{aligned} \overrightarrow{\mathrm{DS}} &= t_{\mathrm{debris}}\mathbf{v}_{\mathrm{spacecraft}} \\ \begin{pmatrix}5750\\-13250\\3370\end{pmatrix}-\begin{pmatrix}1200\\3000\\900\end{pmatrix} &= t\left(\begin{pmatrix}2000\\-3600\\1000\end{pmatrix}-\begin{pmatrix}600\\1400\\240\end{pmatrix}\right) \\ \begin{pmatrix}4550\\-16250\\2470\end{pmatrix} &= t\begin{pmatrix}1400\\-5000\\760\end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{i}: \quad t &= \frac{4550}{1400} = 3.25 \\ \mathbf{j}: \quad t &= \frac{-16250}{-5000} = 3.25 \\ \mathbf{k}: \quad t &= \frac{2470}{760} = 3.25 \end{aligned}$$

The spacecraft and debris will collide at time $t = 3.25$ hours.

19. Position of the fighter at the time of interception is

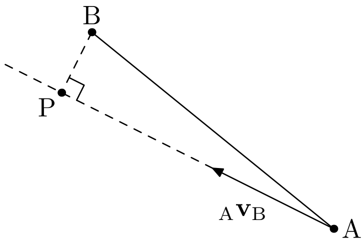
$$\begin{aligned}\mathbf{r} &= (150\mathbf{i} + 470\mathbf{j} + 2\mathbf{k}) + \frac{10}{60}(300\mathbf{i} + 180\mathbf{j}) \\ &= (200\mathbf{i} + 500\mathbf{j} + 2\mathbf{k})\,\text{km}\end{aligned}$$

Call this point P, and call the initial position of the fighter point A, then

$$\begin{aligned}\overrightarrow{\text{AP}} &= (200\mathbf{i} + 500\mathbf{j} + 2\mathbf{k}) - (80\mathbf{i} + 400\mathbf{j} + 3\mathbf{k}) \\ &= (120\mathbf{i} + 100\mathbf{j} - \mathbf{k})\,\text{km}\end{aligned}$$

$$\begin{aligned}\overrightarrow{\text{AP}} &= t\mathbf{v}_\text{A} \\ 120\mathbf{i} + 100\mathbf{j} - \mathbf{k} &= \frac{10}{60}\mathbf{v}_\text{A} \\ \mathbf{v}_\text{A} &= 6(120\mathbf{i} + 100\mathbf{j} - \mathbf{k}) \\ &= (720\mathbf{i} + 600\mathbf{j} - 6\mathbf{k})\ \text{km/h}\end{aligned}$$

20. Let P be the point of minimum separation.



$$\begin{aligned}\overrightarrow{\text{AB}} &= \mathbf{r}_\text{B} - \mathbf{r}_\text{A} \\ &= \begin{pmatrix}2\\40\\26\end{pmatrix} - \begin{pmatrix}30\\-37\\-30\end{pmatrix} \\ &= \begin{pmatrix}-28\\77\\56\end{pmatrix}\end{aligned}$$

$$\begin{aligned}{}_\text{A}\mathbf{v}_\text{B} &= \mathbf{v}_\text{A} - \mathbf{v}_\text{B} \\ &= \begin{pmatrix}5\\8\\3\end{pmatrix} - \begin{pmatrix}8\\0\\-2\end{pmatrix} \\ &= \begin{pmatrix}-3\\8\\5\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\overrightarrow{\text{BP}} &= -\overrightarrow{\text{AB}} + t\,{}_\text{A}\mathbf{v}_\text{B} \\ &= -\begin{pmatrix}-28\\77\\56\end{pmatrix} + t\begin{pmatrix}-3\\8\\5\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\overrightarrow{\text{BP}} \cdot {}_\text{A}\mathbf{v}_\text{B} &= 0 \\ \left(-\begin{pmatrix}-28\\77\\56\end{pmatrix} + t\begin{pmatrix}-3\\8\\5\end{pmatrix}\right)\cdot\begin{pmatrix}-3\\8\\5\end{pmatrix} &= 0 \\ (-84 - 616 - 280) + t(9 + 64 + 25) &= 0 \\ -980 + 98t &= 0 \\ t &= 10\text{s}\end{aligned}$$

$$\begin{aligned}\overrightarrow{\text{BP}} &= -\begin{pmatrix}-28\\77\\56\end{pmatrix} + 10\begin{pmatrix}-3\\8\\5\end{pmatrix} \\ &= \begin{pmatrix}-2\\-3\\-6\end{pmatrix} \\ |\overrightarrow{\text{BP}}| &= \sqrt{(-2)^2 + (-3)^2 + (-6)^2} \\ &= 7\text{m}\end{aligned}$$

21. $\begin{pmatrix}2\\-2\\1\end{pmatrix}$ is normal to plane Π_1.

$\begin{pmatrix}-2\\2\\-1\end{pmatrix}$ is normal to plane Π_2.

$\begin{pmatrix}-2\\2\\-1\end{pmatrix} = -1\begin{pmatrix}2\\-2\\1\end{pmatrix}$.

$\therefore\quad \begin{pmatrix}-2\\2\\-1\end{pmatrix} \parallel \begin{pmatrix}2\\-2\\1\end{pmatrix}$

$\therefore\quad \begin{pmatrix}2\\-2\\1\end{pmatrix}$ is normal to plane Π_2.

$\therefore\quad \Pi_2 \parallel \Pi_1$ □

Another approach: proof by contradiction.

Suppose Π_1 and Π_2 are not parallel. If that is the case, there exists a line of intersection between the planes, i.e. a set of points $\mathbf{r}$ that simultaneously satisfies

$$\mathbf{r}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix} = 12 \qquad (①)$$

and

$$\mathbf{r}\cdot\begin{pmatrix}-2\\2\\-1\end{pmatrix} = 15 \qquad (②)$$

Starting with ②:

$$\begin{aligned}\mathbf{r}\cdot\begin{pmatrix}-2\\2\\-1\end{pmatrix} &= 15 \\ \mathbf{r}\cdot\left(-1\begin{pmatrix}2\\-2\\1\end{pmatrix}\right) &= 15 \\ -\mathbf{r}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix} &= 15 \\ \mathbf{r}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix} &= -15\end{aligned}$$

substituting ①

$$12 = -15$$

which means that our original supposition leads to a contradiction, hence Π_1 and Π_2 are parallel. □

To find the distance the planes are apart, consider the line

$$\mathbf{r} = \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

We know that this line is perpendicular to both planes, so the distance along this line between the point where it intercepts Π_1 and where it intercepts Π_2 will represent the perpendicular (and hence minimum) distance between the planes. Call these points P_1 and P_2.

$$\begin{aligned} \lambda_1 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} &= 12 \\ \lambda_1(4+4+1) &= 12 \\ \lambda_1 &= \frac{4}{3} \\ P_1 &= \frac{4}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \lambda_2 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} &= 15 \\ \lambda_2(-4-4-1) &= 15 \\ \lambda_2 &= -\frac{5}{3} \\ P_2 &= -\frac{5}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \overrightarrow{P_2P_1} &= \frac{4}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} - -\frac{5}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \\ &= 3\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \\ |\overrightarrow{P_2P_1}| &= 3\sqrt{2^2 + (-2)^2 + 1^2} \\ &= 9 \end{aligned}$$

The planes are 8 units apart.

22. (a) $\mathbf{r} = (-10000\mathbf{i} - 5000\mathbf{j} + 500\mathbf{k})$
$\quad + 60\,(80\mathbf{i} + 50\mathbf{j} + 5\mathbf{k})$
$\quad = (-5200\mathbf{i} - 2000\mathbf{j} + 800\mathbf{k})$

(b) When it is due west, the $\mathbf{j}$ component will be zero.
$-5\,000 + 50t = 0$
$t = 100\text{s}$
i.e. at 1 minute and 40 seconds after 1pm.

(c) When it is due north, the $\mathbf{i}$ component will be zero.
$-10\,000 + 80t = 0$
$t = 125\text{s}$
i.e. at 2 minutes and 5 seconds after 1pm.

The altitude at that time is given by the $\mathbf{k}$ component:
$\mathbf{500} + 5 \times 125t = 1\,125\text{m}$

(d) Five minutes is 300 seconds:

$$\begin{aligned} \mathbf{r} &= (-10000\mathbf{i} - 5000\mathbf{j} + 500\mathbf{k}) \\ &\quad + 300\,(80\mathbf{i} + 50\mathbf{j} + 5\mathbf{k}) \\ &= (14000\mathbf{i} + 10000\mathbf{j} + 2000\mathbf{k}) \\ \text{distance} &= \sqrt{14000^2 + 10000^2 + 2000^2} \\ &= 2000\sqrt{7^2 + 5^2 + 1^2} \\ &= 10000\sqrt{3}\text{m} \\ &= 10\sqrt{3}\text{km} \end{aligned}$$

(e) Let P be the position nearest O.

$$\overrightarrow{OP} = \begin{pmatrix} -10000 \\ -5000 \\ 500 \end{pmatrix} + t\begin{pmatrix} 80 \\ 50 \\ 5 \end{pmatrix}$$

$$\begin{aligned} \overrightarrow{OP} \cdot \begin{pmatrix} 80 \\ 50 \\ 5 \end{pmatrix} &= 0 \\ \left(\begin{pmatrix} -10000 \\ -5000 \\ 500 \end{pmatrix} + t\begin{pmatrix} 80 \\ 50 \\ 5 \end{pmatrix}\right) \cdot \begin{pmatrix} 80 \\ 50 \\ 5 \end{pmatrix} &= 0 \\ (-800000 - 250000 + 2500) & \\ +t(6400 + 2500 + 25) &= 0 \\ -1052500 + 8925t &= 0 \\ t &= 118\text{s} \end{aligned}$$

$$\begin{aligned} \overrightarrow{OP} &= \begin{pmatrix} -10000 \\ -5000 \\ 500 \end{pmatrix} + 118\begin{pmatrix} 80 \\ 50 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} -566 \\ 896 \\ 1090 \end{pmatrix} \\ |\overrightarrow{OP}| &= \sqrt{(-566)^2 + (896)^2 + (1090)^2} \\ &= 1520\text{m} \\ &= 1.52\text{km} \end{aligned}$$

Miscellaneous Exercise 3

1. (a) No working needed. Refer to the answer in Sadler.

   (b) No working needed. Refer to the answer in Sadler.

   (c) $$\begin{aligned}\cos\theta &= \frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{p}||\mathbf{q}|}\\ &= \frac{2\times -2+3\times 4+-2\times 3}{\sqrt{2^2+3^2+(-2)^2}\sqrt{(-2)^2+4^2+3^2}}\\ &= \frac{2}{\sqrt{17}\sqrt{29}}\\ \theta &= 85^\circ\end{aligned}$$

   (d) $$\begin{aligned}\cos\theta &= \frac{\mathbf{p}\cdot\mathbf{i}}{|\mathbf{p}||\mathbf{i}|}\\ &= \frac{2}{\sqrt{17}}\\ \theta &= 61^\circ\end{aligned}$$

   (e) $$\begin{aligned}\cos\theta &= \frac{\mathbf{q}\cdot\mathbf{j}}{|\mathbf{q}||\mathbf{j}|}\\ &= \frac{4}{\sqrt{29}}\\ \theta &= 42^\circ\end{aligned}$$

2. $$\begin{pmatrix}2\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\-2\end{pmatrix}=\begin{pmatrix}-2\\1\\6\end{pmatrix}+\mu\begin{pmatrix}-2\\-1\\2\end{pmatrix}$$
$$\lambda\begin{pmatrix}2\\3\\-2\end{pmatrix}-\mu\begin{pmatrix}-2\\-1\\2\end{pmatrix}=\begin{pmatrix}-4\\1\\4\end{pmatrix}$$
$$\begin{aligned}2\lambda+2\mu &= -4 && \text{①}\\ 3\lambda+\mu &= 1 && \text{②}\\ -2\lambda-2\mu &= 4 && \text{③}\\ 4\lambda &= 6 && 2\times\text{②}+\text{③}\\ \lambda &= \frac{3}{2}\\ -2\left(\frac{3}{2}\right)-2\mu &= 4 && \text{subst. ③}\\ \mu &= -\frac{7}{2}\end{aligned}$$

   We would normally need to confirm that these values work in the third equation (i.e. ①) but in this case ① and ③ are redundant (i.e. they can be rearranged to be identical equations) so any solution of ③ must also be a solution of ①. (This also means that we didn't really need to find μ.)

   The point of intersection is given by:
$$\begin{aligned}\mathbf{r} &= 2\mathbf{i}+2\mathbf{k}+\lambda(2\mathbf{i}+3\mathbf{j}-2\mathbf{k})\\ &= 2\mathbf{i}+2\mathbf{k}+\frac{3}{2}(2\mathbf{i}+3\mathbf{j}-2\mathbf{k})\\ &= 5\mathbf{i}+4.5\mathbf{j}-\mathbf{k}\end{aligned}$$

3. The resultant is
$$(\mathbf{i}+5\mathbf{j}-4\mathbf{k})+(\mathbf{i}-3\mathbf{j}+3\mathbf{k}) = 2\mathbf{i}+2\mathbf{j}-\mathbf{k}$$
   This has magnitude
$$\sqrt{2^2+2^2+(-1)^2}=3$$
   so a unit vector parallel to the resultant is
$$\frac{1}{3}(2\mathbf{i}+2\mathbf{j}-\mathbf{k})$$

4. $\mathrm{f}(x) = x^2+3x$
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{h\to 0}\frac{\mathrm{f}(x+h)-\mathrm{f}(x)}{h}\\ &= \lim_{h\to 0}\frac{\big((x+h)^2+3(x+h)\big)-\big(x^2+3x\big)}{h}\\ &= \lim_{h\to 0}\frac{x^2+2xh+h^2+3x+3h-x^2-3x}{h}\\ &= \lim_{h\to 0}\frac{2xh+h^2+3h}{h}\\ &= \lim_{h\to 0}(2x+h+3)\\ &= 2x+3\end{aligned}$$

5. (a) $$\begin{aligned}z &= \frac{3+5\sqrt{3}\mathrm{i}}{-3+2\sqrt{3}\mathrm{i}}\\ &= \frac{3+5\sqrt{3}\mathrm{i}}{-3+2\sqrt{3}\mathrm{i}}\,\frac{-3-2\sqrt{3}\mathrm{i}}{-3-2\sqrt{3}\mathrm{i}}\\ &= \frac{-9-6\sqrt{3}\mathrm{i}-15\sqrt{3}\mathrm{i}+30}{9+6\sqrt{3}\mathrm{i}-6\sqrt{3}\mathrm{i}+12}\\ &= \frac{21-21\sqrt{3}\mathrm{i}}{21}\\ &= 1-\sqrt{3}\mathrm{i}\end{aligned}$$

   (b) $$\begin{aligned}r &= \sqrt{1^2+(-\sqrt{3})^2}\\ &= 2\end{aligned}$$
   θ is in the 4th quadrant (positive real component, negative imaginary component) and
$$\begin{aligned}\tan\theta &= \frac{-\sqrt{3}}{1}\\ \theta &= -\frac{\pi}{3}\\ \therefore\quad z &= 2\,\mathrm{cis}\left(-\frac{\pi}{3}\right)\end{aligned}$$

6. $$\begin{aligned}\cos\theta &= \frac{\begin{pmatrix}2\\3\\-2\end{pmatrix}\cdot\begin{pmatrix}3\\-2\\1\end{pmatrix}}{\left|\begin{pmatrix}2\\3\\-2\end{pmatrix}\right|\left|\begin{pmatrix}3\\-2\\1\end{pmatrix}\right|}\\ &= \frac{-2}{\sqrt{17}\sqrt{14}}\\ \theta &= 97.4^\circ\end{aligned}$$

7. $$\begin{aligned}4\operatorname{cis}\frac{-\pi}{6} &= 4\cos\frac{-\pi}{6} + \left(4\sin\frac{-\pi}{6}\right)\mathrm{i}\\ &= 2\sqrt{3} - 2\mathrm{i}\\ \frac{1}{4\operatorname{cis}\frac{-\pi}{6}} &= \frac{1}{2\sqrt{3}-2\mathrm{i}}\\ &= \frac{1}{2\sqrt{3}-2\mathrm{i}}\,\frac{2\sqrt{3}+2\mathrm{i}}{2\sqrt{3}+2\mathrm{i}}\\ &= \frac{2\sqrt{3}+2\mathrm{i}}{12+4\sqrt{3}\mathrm{i}-4\sqrt{3}\mathrm{i}+4}\\ &= \frac{2\sqrt{3}+2\mathrm{i}}{16}\\ &= \frac{\sqrt{3}}{8}+\frac{1}{8}\mathrm{i}\end{aligned}$$

   Alternatively:

   $$\begin{aligned}\frac{1}{4\operatorname{cis}\frac{-\pi}{6}} &= \frac{1}{4}\operatorname{cis}\left(0-\frac{-\pi}{6}\right)\\ &= \frac{1}{4}\operatorname{cis}\frac{\pi}{6}\\ &= \frac{1}{4}\left(\cos\frac{\pi}{6}\right)+\frac{1}{4}\left(\sin\frac{\pi}{6}\right)\mathrm{i}\\ &= \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)+\frac{1}{4}\left(\frac{1}{2}\right)\mathrm{i}\\ &= \frac{\sqrt{3}}{8}+\frac{1}{8}\mathrm{i}\end{aligned}$$

8. No working required. If you're having trouble understanding this question, think about what *operation* a 90° rotation in the Argand plane represents.

9. (a) The first line is parallel to $(3\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ and the second parallel to $(6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$. Since these are scalar multiples, the lines are parallel.

   (b) First observe that the lines are not parallel. Next equate the lines and rearrange to make three equations from the three components:

   $$\begin{aligned}-\lambda - 2\mu &= -7\\ 3\lambda - \mu &= 0\\ \lambda - 2\mu &= -15\end{aligned}$$

   Solve the first two simultaneously

   $$\begin{aligned}\lambda &= 1\\ \mu &= 3\end{aligned}$$

   Does this solution satisfy the third equation?

   $$(1) - 2(3) \neq -15$$

   No: they are skew lines.

   (c) First observe that the lines are not parallel. Next equate the lines and rearrange to make three equations from the three components:

   $$\begin{aligned}-\lambda - \mu &= 1\\ \lambda &= -3\\ \lambda - 2\mu &= -7\end{aligned}$$

   Solve the first two simultaneously

   $$\begin{aligned}\lambda &= -3\\ \mu &= 2\end{aligned}$$

   Does this solution satisfy the third equation?

   $$(-3) - 2(2) = -7$$

   Yes: the lines intersect.

   (d) The first line is parallel to $(\mathbf{i} + \mathbf{j} - \mathbf{k})$ and the second parallel to $(-\mathbf{i} - \mathbf{j} + \mathbf{k})$. Since these are scalar multiples, the lines are parallel.

10. The height corresponds to the $\mathbf{k}$ component, so

   $$\begin{aligned}3\lambda &= 180\\ \lambda &= 60\end{aligned}$$

   and the initial position is

   $$\begin{aligned}\mathbf{r} &= 60\,(10\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})\\ &= (600\mathbf{i} + 240\mathbf{j} + 180\mathbf{k})\,\mathrm{m}\end{aligned}$$

   The displacement from there to the touchdown point at $(0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k})$ m is

   $$\begin{aligned}\mathbf{s} &= (\mathbf{0}) - (600\mathbf{i} + 240\mathbf{j} + 180\mathbf{k})\\ &= (-600\mathbf{i} - 240\mathbf{j} - 180\mathbf{k})\,\mathrm{m}\end{aligned}$$

   giving a velocity of

   $$\begin{aligned}\mathbf{v} &= \frac{1}{15}(-600\mathbf{i} - 240\mathbf{j} - 180\mathbf{k})\\ &= (-40\mathbf{i} - 16\mathbf{j} - 12\mathbf{k})\,\mathrm{m/s}\end{aligned}$$

   The distance travelled is

   $$\begin{aligned}d &= |\,(-600\mathbf{i} - 240\mathbf{j} - 180\mathbf{k})\,|\\ &= 60\sqrt{10^2 + 4^2 + 3^2}\\ &= 670\mathrm{m}\end{aligned}$$

11. (a) $$\begin{aligned}zw &= (2)(1)\operatorname{cis}\left(\frac{\pi}{4}+\frac{\pi}{6}\right)\\ &= 2\operatorname{cis}\frac{5\pi}{12}\end{aligned}$$

   (b) $$\begin{aligned}\frac{z}{w} &= \frac{2}{1}\operatorname{cis}\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\\ &= 2\operatorname{cis}\frac{\pi}{12}\end{aligned}$$

   (c) $$\begin{aligned}w^2 &= 1^2\operatorname{cis}\left(2\times\frac{\pi}{6}\right)\\ &= \operatorname{cis}\frac{\pi}{3}\end{aligned}$$

(d) $$\begin{aligned} z^3 &= 2^3 \operatorname{cis}\left(3 \times \frac{\pi}{4}\right) \\ &= 8 \operatorname{cis} \frac{3\pi}{4} \end{aligned}$$

(e) $$\begin{aligned} w^9 &= 1^9 \operatorname{cis}\left(9 \times \frac{\pi}{6}\right) \\ &= \operatorname{cis} \frac{3\pi}{2} \\ &= \operatorname{cis}\left(-\frac{\pi}{2}\right) \end{aligned}$$

(f) $$\begin{aligned} z^9 &= 2^9 \operatorname{cis}\left(9 \times \frac{\pi}{4}\right) \\ &= 512 \operatorname{cis} \frac{9\pi}{4} \\ &= 512 \operatorname{cis} \frac{\pi}{4} \end{aligned}$$

12. The first part of this question is really a 2D question, since "passes directly over" means we are not concerned with depth, so ignore the **k** component until it comes to finding the depth of the submarine.

$$\begin{aligned} (1150\mathbf{i} + 827\mathbf{j}) + t\,(10\mathbf{i} - 2\mathbf{j}) &= (1345\mathbf{i} + 970\mathbf{j}) + t\,(-5\mathbf{i} - 13\mathbf{j}) \\ t\,((10\mathbf{i} - 2\mathbf{j}) - (-5\mathbf{i} - 13\mathbf{j})) &= (1345\mathbf{i} + 970\mathbf{j}) - (1150\mathbf{i} + 827\mathbf{j}) \\ t\,(15\mathbf{i} + 11\mathbf{j}) &= (195\mathbf{i} + 143\mathbf{j}) \\ t &= 13 \end{aligned}$$

satisfies both components, so the tanker passes over the submarine at $t = 13$ seconds. The depth of the submarine at that time is

$$\begin{aligned} d &= 4 \times 13 \\ &= 52 \text{ m below the surface.} \end{aligned}$$

13. The plane $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = -14$ is perpendicular to the line $\mathbf{r} = \lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$.

The plane $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = 42$ is perpendicular to the line $\mathbf{r} = \lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$.

Two planes perpendicular to the same line are be parallel to each other. Therefore the two planes are parallel.

The perpendicular line $\mathbf{r} = \lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$ intersects plane $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = -14$ at point A:

$$\begin{aligned} \left(\lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}\right) \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} &= -14 \\ \lambda(4 + 9 + 36) &= -14 \\ 49\lambda &= -14 \\ \lambda &= -\frac{2}{7} \\ \mathbf{r}_\mathrm{A} &= -\frac{2}{7} \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \end{aligned}$$

The same line intersects plane $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = 42$ at point B:

$$\begin{aligned} \left(\lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}\right) \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} &= 42 \\ \lambda(4 + 9 + 36) &= 42 \\ 49\lambda &= 42 \\ \lambda &= \frac{6}{7} \\ \mathbf{r}_\mathrm{B} &= \frac{6}{7} \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \overrightarrow{\mathrm{AB}} &= \frac{6}{7} \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} - -\frac{2}{7} \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \\ &= \frac{8}{7} \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \\ |\overrightarrow{\mathrm{AB}}| &= \frac{8}{7}\sqrt{2^2 + (-3)^2 + 6^2} \\ &= 8 \end{aligned}$$

The distance between the planes is 8 units.

We can generalise this result. Suppose we have a pair of parallel planes expressed as

$$\begin{aligned} &\mathbf{r} \cdot \mathbf{n} = a \\ \text{and} \quad &\mathbf{r} \cdot \mathbf{n} = b \end{aligned}$$

These planes are perpendicular to the line

$$\mathbf{r} = \lambda\mathbf{n}$$

The points of intersection between the planes and this line are given by

$$\begin{aligned} (\lambda_\mathrm{A}\mathbf{n}) \cdot \mathbf{n} &= a \\ \lambda_\mathrm{A} &= \frac{a}{\mathbf{n} \cdot \mathbf{n}} \\ \mathbf{r}_\mathrm{A} &= \lambda_\mathrm{A}\mathbf{n} \\ &= \frac{a}{\mathbf{n} \cdot \mathbf{n}}\mathbf{n} \\ &= \frac{a}{|\mathbf{n}|^2}\mathbf{n} \end{aligned}$$

similarly

$$\begin{aligned}\mathbf{r}_\mathrm{B} &= \frac{b}{|\mathbf{n}|^2}\mathbf{n}\\ \overrightarrow{\mathrm{AB}} &= \mathbf{r}_\mathrm{B} - \mathbf{r}_\mathrm{A}\\ &= \frac{b-a}{|\mathbf{n}|^2}\mathbf{n}\\ |\overrightarrow{\mathrm{AB}}| &= \frac{|b-a|}{|\mathbf{n}|^2}|\mathbf{n}|\\ &= \frac{|b-a|}{|\mathbf{n}|}\end{aligned}$$

14. Let P be the point on the line nearest the origin. $\overrightarrow{\mathrm{OP}}$ is perpendicular to the line.

$$\begin{aligned}\left(\begin{pmatrix}2\\3\\-1\end{pmatrix} + \lambda_\mathrm{P}\begin{pmatrix}5\\2\\1\end{pmatrix}\right)\cdot\begin{pmatrix}5\\2\\1\end{pmatrix} &= 0\\ (10+6-1)+\lambda_\mathrm{P}(25+4+1) &= 0\\ 15+30\lambda_\mathrm{P} &= 0\\ \lambda_\mathrm{P} &= -0.5\end{aligned}$$

$$\begin{aligned}\overrightarrow{\mathrm{OP}} &= \begin{pmatrix}2\\3\\-1\end{pmatrix} - 0.5\begin{pmatrix}5\\2\\1\end{pmatrix}\\ &= \begin{pmatrix}-0.5\\2\\-1.5\end{pmatrix}\\ |\overrightarrow{\mathrm{OP}}| &= \sqrt{(-0.5)^2+2^2+(-1.5)^2}\\ &= \frac{1}{2}\sqrt{(-1)^2+4^2+(-3)^2}\\ &= \frac{\sqrt{26}}{2}\end{aligned}$$

15. Let A be the point $\begin{pmatrix}1\\0\\-4\end{pmatrix}$. Let P be the point on the line nearest to A. $\overrightarrow{\mathrm{AP}}$ is perpendicular to the line.

$$\begin{aligned}\overrightarrow{\mathrm{OP}} &= \begin{pmatrix}-3\\-7\\8\end{pmatrix} + \lambda_\mathrm{P}\begin{pmatrix}3\\4\\-5\end{pmatrix}\\ \overrightarrow{\mathrm{AP}} &= \begin{pmatrix}-3\\-7\\8\end{pmatrix} + \lambda_\mathrm{P}\begin{pmatrix}3\\4\\-5\end{pmatrix} - \begin{pmatrix}1\\0\\-4\end{pmatrix}\\ &= \begin{pmatrix}-4\\-7\\12\end{pmatrix} + \lambda_\mathrm{P}\begin{pmatrix}3\\4\\-5\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\left(\begin{pmatrix}-4\\-7\\12\end{pmatrix} + \lambda_\mathrm{P}\begin{pmatrix}3\\4\\-5\end{pmatrix}\right)\cdot\begin{pmatrix}3\\4\\-5\end{pmatrix} &= 0\\ (-12-28-60)+\lambda_\mathrm{P}(9+16+25) &= 0\\ -100+50\lambda_\mathrm{P} &= 0\\ \lambda_\mathrm{P} &= 2\end{aligned}$$

$$\begin{aligned}\overrightarrow{\mathrm{AP}} &= \begin{pmatrix}-4\\-7\\12\end{pmatrix} + 2\begin{pmatrix}3\\4\\-5\end{pmatrix}\\ &= \begin{pmatrix}2\\1\\2\end{pmatrix}\\ \mathrm{AP} &= \sqrt{2^2+1^2+2^2}\\ &= 3 \text{ units}\end{aligned}$$

Chapter 4

Exercise 4A

Proofs can be particularly challenging because it is often not clear what path will lead to success. Even the best mathematician will take several attempts at some problems before arriving at a successful proof. Indeed, some conjectures have remained unproven for centuries before a proof is finally found[1]. Because of this you should be particularly slow to reach for these worked solutions when you find the work difficult. Make sure you try several different approaches before taking even a peek here. The more you can do on your own or together with other students, the more satisfying it will be for you.

1. (a)

$$\begin{aligned} e+f &= 180^\circ && \text{(angles in a st. line)}\\ f+g &= 180^\circ && \text{(angles in a st. line)}\\ \therefore\quad e+f &= f+g\\ \therefore\quad e &= g\\ c &= e && \text{(Alternate angles)}\\ \therefore\quad c &= g \end{aligned}$$

□

(b)

$$\begin{aligned} e+f &= 180^\circ && \text{(angles in a st. line)}\\ c &= e && \text{(Alternate angles)}\\ \therefore\quad c+f &= 180^\circ \end{aligned}$$

□

2. (a)

$$\begin{aligned} c+f &= 180^\circ && \text{(cointerior angles)}\\ e+f &= 180^\circ && \text{(angles in a st. line)}\\ \therefore\quad c+f &= e+f\\ \therefore\quad c &= e \end{aligned}$$

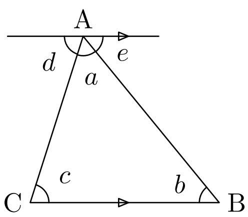
□

(b)

$$\begin{aligned} d+e &= 180^\circ && \text{(cointerior angles)}\\ a+d &= 180^\circ && \text{(angles in a st. line)}\\ \therefore\quad a+d &= d+e\\ \therefore\quad a &= e \end{aligned}$$

□

3.

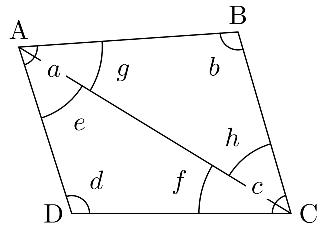


$$\begin{aligned} d+a+e &= 180^\circ && \text{(angles in a st. line)}\\ d &= c && \text{(alternate angles)}\\ \therefore\quad c+a+e &= 180^\circ\\ e &= b && \text{(alternate angles)}\\ \therefore\quad c+a+b &= 180^\circ \end{aligned}$$

□

Now that this is proved, it can be used as a theorem in other proofs.

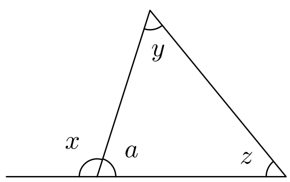
4.



$$\begin{aligned} a &= e+g\\ c &= f+h\\ d+e+f &= 180^\circ && \text{(angles in }\triangle)\\ b+g+h &= 180^\circ && \text{(angles in }\triangle)\\ \therefore\quad \begin{matrix}(d+e+f)\\+(b+g+h)\end{matrix} &= 180^\circ+180^\circ\\ \therefore\quad \begin{matrix}d+(e+g)\\+b+(f+h)\end{matrix} &= 360^\circ\\ \therefore\quad d+a+b+c &= 360^\circ \end{aligned}$$

□

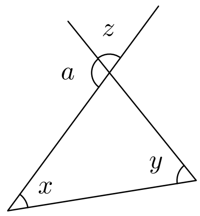
5.



$$\begin{aligned} a+y+z &= 180^\circ && \text{(angles in }\triangle)\\ a+x &= 180^\circ && \text{(angles in a st. line)}\\ \therefore\quad a+x &= a+y+z\\ \therefore\quad x &= y+z \end{aligned}$$

□

6.



$$\begin{aligned} a &= x+y && \text{(ext. angle of }\triangle)\\ a+z &= 180^\circ && \text{(angle in a st. line)}\\ \therefore\quad x+y+z &= 180^\circ \end{aligned}$$

□

[1] Fermat's Last Theorem states that there are no positive integers a, b and c such that $a^n + b^n = c^n$ for n an integer greater than two. Originally proposed in 1637 it was finally proven in 1995 by British mathematician Andrew Wiles (although it had been proven for a number of special cases by various different people previously, including Fermat himself). The proof runs to well over 100 typewritten pages.

7. There are often many possible counter examples; citing any one will do to disprove a statement. The following are representative.

   (a) 2 is both prime and even.

   (b) A rectangle with length 2cm and width 1cm has four right angles but is not a square.

   (c) Rhombus ABCD has angles A and C 45° and angles B and D 135° and all four sides equal but is not a square.

   (d) If $n = 11$ then $n^2 - n + 11$ is a multiple of 11 and not prime.

   (e) 2^2 can not be expressed as the sum of two other square numbers. (The only possible candidates to add to 4 would be 0^2 and 1^2 and none of $0+0$, $0+1$ or $1+1$ yield 4.)

   (f) $x^2 + 4 = 0$ (i.e. $a = 1$, $b = 0$, $c = 4$) has no real solutions.

8. Any odd number can be represented as $2n+1$ for a suitably chosen integer n. Then

$$\begin{aligned}(2n+1)^2 + 3 &= 4n^2 + 2(2n) + 1 + 3\\ &= 4n^2 + 4n + 4\\ &= 4(n^2 + n + 1)\\ &= 4 \times \text{an integer}\\ &= \text{a multiple of } 4\end{aligned}$$

□

9. Any odd number can be represented as $2n+1$ for a suitably chosen integer n. Then

$$\begin{aligned}&1 + (2n+1) + (2n+1)^2 + (2n+1)^3\\ &= 2n + 2 + 4n^2 + 4n + 1 + 8n^3 + 12n^2 + 6n + 1\\ &= 8n^3 + 16n^2 + 12n + 4\\ &= 4(2n^3 + 4n^2 + 3n + 1)\\ &= 4 \times \text{an integer}\\ &= \text{a multiple of } 4\end{aligned}$$

□

10. Any odd number can be represented as $2n-1$ for a suitably chosen integer n. The next consecutive odd number is

$$2n - 1 + 2 = 2n + 1$$

Then

$$\begin{aligned}(2n-1)(2n+1) &= 4n^2 - 1\\ &= (2n)^2 - 1\\ &= (\text{an even number})^2 - 1\\ &= \text{square of an an even no.} - 1\end{aligned}$$

□

11. Given

$$\mathrm{T}_n = 3n - 1$$

then the next consecutive term is

$$\mathrm{T}_{n+1} = 3(n+1) - 1$$

then

$$\begin{aligned}\mathrm{T}_n + \mathrm{T}_{n+1} &= 3n - 1 + 3(n+1) - 1\\ &= 3n - 1 + 3n + 3 - 1\\ &= 6n + 1\\ &= 2(3n) + 1\\ &= 2(\text{an integer}) + 1\\ &= \text{an even number} + 1\\ &= \text{an odd number}\end{aligned}$$

□

12. $$\begin{aligned}\mathrm{F}_{n+5} &= \mathrm{F}_{n+4} + \mathrm{F}_{n+3}\\ &= \mathrm{F}_{n+3} + \mathrm{F}_{n+2} + \mathrm{F}_{n+3}\\ &= 2\mathrm{F}_{n+3} + \mathrm{F}_{n+2}\\ &= 2(\mathrm{F}_{n+2} + \mathrm{F}_{n+1}) + \mathrm{F}_{n+2}\\ &= 3\mathrm{F}_{n+2} + 2\mathrm{F}_{n+1}\\ &= 3(\mathrm{F}_{n+1} + \mathrm{F}_n) + 2\mathrm{F}_{n+1}\\ &= 5\mathrm{F}_{n+1} + 3\mathrm{F}_n\end{aligned}$$

□

Exercise 4B

1.
$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= -\overrightarrow{OA} + \overrightarrow{OB}\\ \overrightarrow{CD} &= \overrightarrow{CO} + \overrightarrow{OD}\\ &= \overrightarrow{OC} + \overrightarrow{OD}\\ &= -h\overrightarrow{OA} + h\overrightarrow{OB}\\ &= h(-\overrightarrow{OA} + \overrightarrow{OB})\\ &= h\overrightarrow{AB}\end{aligned}$$

Since $\overrightarrow{CD}$ is a scalar multiple of $\overrightarrow{AB}$, CD is parallel to AB. □

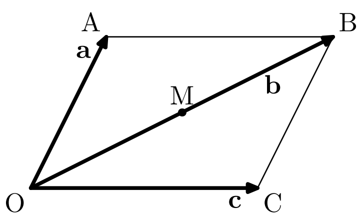
2.
$$\begin{aligned}\overrightarrow{PQ} &= 0.5\overrightarrow{OA} + 0.5\overrightarrow{AB}\\ &= 0.5\mathbf{a} + 0.5(-\overrightarrow{OA} + \overrightarrow{OB})\\ &= 0.5\mathbf{a} - 0.5\mathbf{a} + 0.5\mathbf{b}\\ &= 0.5\mathbf{b}\\ \overrightarrow{SR} &= 0.5\overrightarrow{OC} + 0.5\overrightarrow{CB}\\ &= 0.5\mathbf{c} + 0.5(-\overrightarrow{OC} + \overrightarrow{OB})\\ &= 0.5\mathbf{c} - 0.5\mathbf{c} + 0.5\mathbf{b}\\ &= 0.5\mathbf{b}\\ \therefore\quad \overrightarrow{SR} &= \overrightarrow{PQ}\end{aligned}$$

□

This is sufficient to prove that PQRS is a parallelogram, since it shows that SR and PQ are both parallel and equal in length. However, since the question also asks us to find expressions for $\overrightarrow{QR}$ and $\overrightarrow{PS}$ . . .

$$\begin{aligned}\overrightarrow{QR} &= 0.5\overrightarrow{AB} + 0.5\overrightarrow{BC}\\ &= 0.5(-\overrightarrow{OA} + \overrightarrow{OB}) + 0.5(-\overrightarrow{OB} + \overrightarrow{OC})\\ &= -0.5\mathbf{a} + 0.5\mathbf{b} - 0.5\mathbf{b} + 0.5\mathbf{c}\\ &= 0.5\mathbf{c} - 0.5\mathbf{a}\\ \overrightarrow{PS} &= 0.5\overrightarrow{AO} + 0.5\overrightarrow{OC}\\ &= -0.5\mathbf{a} + 0.5\mathbf{c}\\ &= 0.5\mathbf{c} - 0.5\mathbf{a}\\ \therefore\quad \overrightarrow{PS} &= \overrightarrow{QR}\end{aligned}$$

3.



$$\begin{aligned}\overrightarrow{CM} &= \overrightarrow{CB} - 0.5\overrightarrow{OB}\\ &= \mathbf{a} - 0.5\mathbf{b}\\ \overrightarrow{CA} &= \overrightarrow{CB} - \overrightarrow{OB} + \overrightarrow{OA}\\ &= \mathbf{a} - \mathbf{b} + \mathbf{a}\\ &= 2\mathbf{a} - \mathbf{b}\\ \therefore\quad \overrightarrow{CA} &= 2\overrightarrow{CM}\end{aligned}$$

Hence M lies on CA and is the midpoint of CA. □

4.
$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OA} + \overrightarrow{AM}\\ k\overrightarrow{OB} &= \overrightarrow{OA} + h\overrightarrow{AC}\\ \overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{OC}\\ \overrightarrow{AC} &= -\overrightarrow{OA} + \overrightarrow{OC}\\ \therefore\quad k(\overrightarrow{OA} + \overrightarrow{OC}) &= \overrightarrow{OA} + h(-\overrightarrow{OA} + \overrightarrow{OC})\\ k\overrightarrow{OA} + k\overrightarrow{OC} &= \overrightarrow{OA} - h\overrightarrow{OA} + h\overrightarrow{OC}\\ (h + k - 1)\overrightarrow{OA} &= (h - k)\overrightarrow{OC}\\ h - k &= 0\\ h + k - 1 &= 0\\ h &= k\\ 2h - 1 &= 0\\ h = k &= 0.5\end{aligned}$$

□

5. Refer to the diagram provided for question 4.

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AM} + \overrightarrow{MB}\\ \overrightarrow{OC} &= \overrightarrow{OM} + \overrightarrow{MC}\\ \overrightarrow{OM} &= \overrightarrow{MB} && \text{(since M bisects OB)}\\ \overrightarrow{AM} &= \overrightarrow{MC} && \text{(since M bisects AC)}\\ \therefore\quad \overrightarrow{OC} &= \overrightarrow{MB} + \overrightarrow{AM}\\ &= \overrightarrow{AB}\end{aligned}$$

Similarly

$$\begin{aligned}\overrightarrow{OA} &= \overrightarrow{OM} + \overrightarrow{MA}\\ \overrightarrow{CB} &= \overrightarrow{CM} + \overrightarrow{MB}\\ \therefore\quad \overrightarrow{OA} &= \overrightarrow{MB} + \overrightarrow{CM}\\ &= \overrightarrow{CB}\end{aligned}$$

$\therefore$ OABC is a parallelogram. □

6. (a)
   i. $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b}$
   ii. $\overrightarrow{AC} = \overrightarrow{AO} + 0.5\overrightarrow{OB} = -\mathbf{a} + 0.5\mathbf{b}$
   iii. $\overrightarrow{AD} = 0.5\overrightarrow{AB} = -0.5\mathbf{a} + 0.5\mathbf{b}$
   iv. $$\begin{aligned}\overrightarrow{OD} &= \overrightarrow{OA} + \overrightarrow{AD}\\ &= \mathbf{a} - 0.5\mathbf{a} + 0.5\mathbf{b}\\ &= 0.5\mathbf{a} + 0.5\mathbf{b}\end{aligned}$$
   v. $\overrightarrow{OM} = \frac{2}{3}\overrightarrow{OD} = \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$
   vi. $$\begin{aligned}\overrightarrow{AM} &= \overrightarrow{AO} + \overrightarrow{OM}\\ &= -\mathbf{a} + \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}\\ &= -\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}\end{aligned}$$

   (b) M lies on AC such that AM:MC=2:1 if and only if $\overrightarrow{AM} = 2\overrightarrow{MC}$.
   To prove: $\overrightarrow{AM} = 2\overrightarrow{MC}$
   Proof:

R.H.S.:

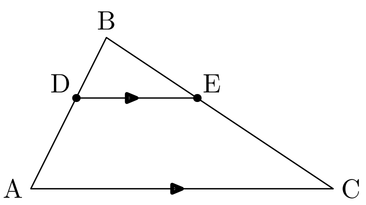
$$\begin{aligned}2\overrightarrow{\text{MC}} &= 2(\overrightarrow{\text{AC}} - \overrightarrow{\text{AM}})\\ &= 2\left(-\mathbf{a} + \frac{1}{2}\mathbf{b} - \left(-\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}\right)\right)\\ &= 2\left(-\frac{1}{3}\mathbf{a} + \frac{1}{6}\mathbf{b}\right)\\ &= -\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}\\ &= \overrightarrow{\text{AM}}\\ &= \text{L.H.S.}\end{aligned}$$

□

(c) M lies on BE such that BM:ME=2:1 if and only if $\overrightarrow{\text{BM}} = 2\overrightarrow{\text{ME}}$.
To prove: $\overrightarrow{\text{BM}} = 2\overrightarrow{\text{ME}}$
Proof:
R.H.S.:

$$\begin{aligned}2\overrightarrow{\text{ME}} &= 2(\overrightarrow{\text{BE}} - \overrightarrow{\text{BM}})\\ \overrightarrow{\text{BE}} &= \overrightarrow{\text{BO}} + \overrightarrow{\text{OE}}\\ &= \frac{1}{2}\mathbf{a} - \mathbf{b}\\ \overrightarrow{\text{BM}} &= \overrightarrow{\text{BO}} + \overrightarrow{\text{OM}}\\ &= -\mathbf{b} + \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}\\ &= \frac{1}{3}\mathbf{a} - \frac{2}{3}\mathbf{b}\\ 2\overrightarrow{\text{ME}} &= 2\left(\frac{1}{2}\mathbf{a} - \mathbf{b} - \left(\frac{1}{3}\mathbf{a} - \frac{2}{3}\mathbf{b}\right)\right)\\ &= 2\left(\frac{1}{6}\mathbf{a} - \frac{1}{3}\mathbf{b}\right)\\ &= \frac{1}{3}\mathbf{a} - \frac{2}{3}\mathbf{b}\\ &= \overrightarrow{\text{BM}}\\ &= \text{L.H.S.}\end{aligned}$$

□

7. 

$$\begin{aligned}\overrightarrow{\text{AB}} &= \mathbf{a}\\ \overrightarrow{\text{AC}} &= \mathbf{c}\\ \overrightarrow{\text{AD}} &= h\overrightarrow{\text{AB}}\\ &= h\mathbf{a}\\ \text{Let } \overrightarrow{\text{CE}} &= k\overrightarrow{\text{CE}}\\ \overrightarrow{\text{CB}} &= -\overrightarrow{\text{AC}} + \overrightarrow{\text{AB}}\\ &= \mathbf{a} - \mathbf{b}\end{aligned}$$

$$\begin{aligned}\therefore\quad \overrightarrow{\text{CE}} &= k(\mathbf{a} - \mathbf{b})\\ &= k\mathbf{a} - k\mathbf{b}\\ \overrightarrow{\text{DE}} &= -\overrightarrow{\text{AD}} + \overrightarrow{\text{AC}} + \overrightarrow{\text{CE}}\\ &= -h\mathbf{a} + \mathbf{b} + k\mathbf{a} - k\mathbf{b}\\ &= (k-h)\mathbf{a} + (1-k)\mathbf{b}\end{aligned}$$

But $\overrightarrow{\text{DE}}$ is parallel to $\overrightarrow{\text{AC}}$, so it must be a scalar multiple of **b**, so the component that is not parallel to **b** must be zero:

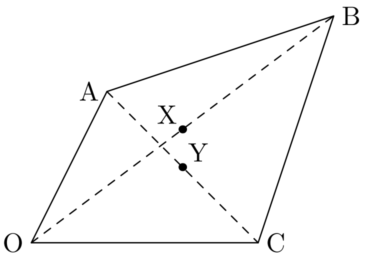
$$\begin{aligned}k - h &= 0\\ k &= h\\ \therefore\quad \overrightarrow{\text{CE}} &= h\overrightarrow{\text{CE}}\end{aligned}$$

□

8.

$$\begin{aligned}\overrightarrow{\text{OA}} + \overrightarrow{\text{AM}} &= \overrightarrow{\text{OM}}\\ \mathbf{a} + k\overrightarrow{\text{AC}} &= h\overrightarrow{\text{OD}}\\ \overrightarrow{\text{OD}} &= \overrightarrow{\text{OA}} + 0.5\overrightarrow{\text{AB}}\\ &= \mathbf{a} + 0.5(-\overrightarrow{\text{OA}} + \overrightarrow{\text{OB}})\\ &= \mathbf{a} + 0.5(\mathbf{b} - \mathbf{a})\\ &= 0.5\mathbf{a} + 0.5\mathbf{b}\\ \overrightarrow{\text{AC}} &= -\overrightarrow{\text{OA}} + 0.5\overrightarrow{\text{OB}}\\ &= 0.5\mathbf{b} - \mathbf{a}\\ \therefore\quad \mathbf{a} + k(0.5\mathbf{b} - \mathbf{a}) &= h(0.5\mathbf{a} + 0.5\mathbf{b}))\\ (1-k)\mathbf{a} + \frac{k}{2}\mathbf{b} &= \frac{h}{2}\mathbf{a} + \frac{h}{2}\mathbf{b}\\ (1-k)\mathbf{a} - \frac{h}{2}\mathbf{a} &= \frac{h}{2}\mathbf{b} - \frac{k}{2}\mathbf{b}\\ 1 - k - \frac{h}{2} &= 0\\ \text{and } \frac{h}{2} &= \frac{k}{2}\\ \therefore\quad h &= k\\ \therefore\quad 1 - k - \frac{k}{2} &= 0\\ \frac{3k}{2} &= 1\\ \therefore\quad h = k &= \frac{2}{3}\\ \overrightarrow{\text{BE}} &= -\overrightarrow{\text{OB}} + 0.5\overrightarrow{\text{OA}}\\ &= 0.5\mathbf{a} - \mathbf{b}\\ \overrightarrow{\text{BM}} &= -\overrightarrow{\text{OB}} + \frac{2}{3}\overrightarrow{\text{OD}}\\ &= -\mathbf{b} + \frac{2}{3}(0.5\mathbf{a} + 0.5\mathbf{b})\\ &= -\frac{2}{3}\mathbf{b} + \frac{1}{3}\mathbf{a}\\ &= \frac{2}{3}(0.5\mathbf{a} - \mathbf{b})\\ &= \frac{2}{3}\overrightarrow{\text{BE}}\end{aligned}$$

∴ the medians intersect at a point two thirds of the way along their length measured from the vertex. □

9. 

To prove:

$$\overrightarrow{OA} + \overrightarrow{BA} + \overrightarrow{OC} + \overrightarrow{BC} = 4\overrightarrow{XY}$$

R.H.S.

$$\begin{aligned} 4\overrightarrow{XY} &= \overrightarrow{XY} + \overrightarrow{XY} + \overrightarrow{XY} + \overrightarrow{XY} \\ &= 0.5\overrightarrow{BO} + \overrightarrow{OA} + 0.5\overrightarrow{AC} \\ &\quad - 0.5\overrightarrow{BO} + \overrightarrow{BA} + 0.5\overrightarrow{AC} \\ &\quad + 0.5\overrightarrow{BO} + \overrightarrow{OC} - 0.5\overrightarrow{AC} \\ &\quad - 0.5\overrightarrow{BO} + \overrightarrow{BC} - 0.5\overrightarrow{AC} \\ &= \overrightarrow{OA} + \overrightarrow{BA} + \overrightarrow{OC} + \overrightarrow{BC} \\ &= \text{L.H.S.} \end{aligned}$$

□

This proof depends on expressing $\overrightarrow{XY}$ in four different ways. It may not be obvious at first that this is going to work, but after you've done enough of these you sometimes get a hunch about an approach that will pay off. Practice is the key.

10. (a) $\mathbf{a} \cdot \mathbf{c} = 0$ (because they are perpendicular).

(b) $\overrightarrow{AC} = -\mathbf{a} + \mathbf{c}$
$\overrightarrow{OB} = \mathbf{a} + \mathbf{c}$

(c) To prove: $|\overrightarrow{AC}| = |\overrightarrow{OB}|$ First consider the L.H.S.:

$$\begin{aligned} |\overrightarrow{AC}| &= \sqrt{(AC)^2} \\ &= \sqrt{\overrightarrow{AC} \cdot \overrightarrow{AC}} \\ &= \sqrt{(-\mathbf{a} + \mathbf{c}) \cdot (-\mathbf{a} + \mathbf{c})} \\ &= \sqrt{\mathbf{a} \cdot \mathbf{a} - 2\mathbf{a} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{c}} \\ &= \sqrt{\mathbf{a} \cdot \mathbf{a} - 2 \times 0 + \mathbf{c} \cdot \mathbf{c}} \\ &= \sqrt{\mathbf{a} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{c}} \end{aligned}$$

Now consider the R.H.S.:

$$\begin{aligned} |\overrightarrow{OB}| &= \sqrt{(OB)^2} \\ &= \sqrt{\overrightarrow{OB} \cdot \overrightarrow{OB}} \\ &= \sqrt{(\mathbf{a} + \mathbf{c}) \cdot (\mathbf{a} + \mathbf{c})} \\ &= \sqrt{\mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{c}} \\ &= \sqrt{\mathbf{a} \cdot \mathbf{a} + 2 \times 0 + \mathbf{c} \cdot \mathbf{c}} \\ &= \sqrt{\mathbf{a} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{c}} \\ &= \text{L.H.S.} \end{aligned}$$

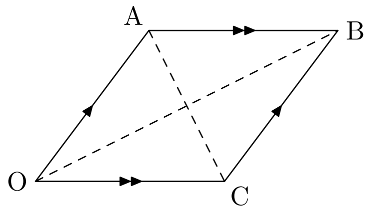
□

11. (a) $\overrightarrow{AB} = -\mathbf{a} + \mathbf{b}$

(b) To prove: $(AB)^2 = (OA)^2 + (OB)^2$

$$\begin{aligned} (OA)^2 &= \mathbf{a} \cdot \mathbf{a} \\ (OB)^2 &= \mathbf{b} \cdot \mathbf{b} \\ (AB)^2 &= (-\mathbf{a} + \mathbf{b}) \cdot (-\mathbf{a} + \mathbf{b}) \\ &= \mathbf{a} \cdot \mathbf{a} - 2\mathbf{a} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{c} \\ &= \mathbf{a} \cdot \mathbf{a} + 2 \times 0 + \mathbf{c} \cdot \mathbf{c} \\ &= \mathbf{a} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{c} \\ &= (OA)^2 + (OB)^2 \end{aligned}$$

□

12. 

$$\begin{aligned} \text{Let } \overrightarrow{OA} &= \mathbf{a} \\ \text{and } \overrightarrow{OC} &= \mathbf{c} \\ \overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{AB} \\ &= \mathbf{a} + \mathbf{c} \\ \overrightarrow{CA} &= \overrightarrow{OA} - \overrightarrow{OC} \\ &= \mathbf{a} - \mathbf{c} \\ |\mathbf{a}| - |\mathbf{c}| &= \mathbf{a} \cdot \mathbf{a} - \mathbf{c} \cdot \mathbf{c} \\ &= (\mathbf{a} + \mathbf{c}) \cdot (\mathbf{a} - \mathbf{c}) \\ &= \overrightarrow{OB} \cdot \overrightarrow{CA} \end{aligned}$$

If the perpendiculars are parallel, then

$$\begin{aligned} \overrightarrow{OB} \cdot \overrightarrow{CA} &= 0 \\ \therefore \quad |\mathbf{a}| - |\mathbf{c}| &= 0 \\ |\mathbf{a}| &= |\mathbf{c}| \end{aligned}$$

□

13. (a) $\overrightarrow{AC} = -\mathbf{a} + \mathbf{c}$

$$\begin{aligned} \overrightarrow{BD} &= \mathbf{a} + \frac{1}{2}\overrightarrow{AC} \\ &= \mathbf{a} + \frac{1}{2}(-\mathbf{a} + \mathbf{c}) \\ &= \frac{1}{2}(\mathbf{a} + \mathbf{c}) \end{aligned}$$

(b)

$$\begin{aligned} \overrightarrow{AC} \cdot \overrightarrow{BD} &= (-\mathbf{a} + \mathbf{c}) \cdot \frac{1}{2}(\mathbf{a} + \mathbf{c}) \\ &= \frac{1}{2}(-\mathbf{a} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{c}) \\ &= \frac{1}{2}(-(|\mathbf{a}|)^2 + (|\mathbf{c}|)^2) \end{aligned}$$

but $|\mathbf{a}| = |\mathbf{c}|$ because the triangle is isosceles, so

$$\begin{aligned} \overrightarrow{AC} \cdot \overrightarrow{BD} &= \frac{1}{2}(-(|\mathbf{c}|)^2 + (|\mathbf{c}|)^2) \\ &= 0 \end{aligned}$$

$\therefore$ AC and BD are perpendicular and $\angle$BDA is a right angle. □

14. (a) $\overrightarrow{CB} = \overrightarrow{CO} + \overrightarrow{OB}$
$= -\mathbf{c} + \mathbf{b}$
$\overrightarrow{AO} = \overrightarrow{OB}$
$= \mathbf{b}$
$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$
$= \mathbf{b} + \mathbf{c}$

(b) $\overrightarrow{AC} \cdot \overrightarrow{CB} = (\mathbf{b} + \mathbf{c}) \cdot (\mathbf{b} - \mathbf{c})$
$= \mathbf{b} \cdot \mathbf{b} - \mathbf{c} \cdot \mathbf{c}$
$= (|\mathbf{b}|)^2 - (|\mathbf{c}|)^2$
but $|\mathbf{b}| = |\mathbf{c}|$ because they are both radii of the circle, so
$\overrightarrow{AC} \cdot \overrightarrow{CB} = (|\mathbf{c}|)^2 - (|\mathbf{c}|)^2$
$= 0$
$\therefore$ AC and CB are perpendicular and ∠ACB is a right angle. □

15.
$$\begin{aligned}
\overrightarrow{BC} &= -\mathbf{b} + \mathbf{c} \\
\mathbf{a} \cdot \overrightarrow{BC} &= 0 \\
\therefore \quad \mathbf{a} \cdot (-\mathbf{b} + \mathbf{c}) &= 0 \\
-\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} &= 0 \\
\mathbf{a} \cdot \mathbf{c} &= \mathbf{a} \cdot \mathbf{b}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{AC} &= -\mathbf{a} + \mathbf{c} \\
\mathbf{b} \cdot \overrightarrow{AC} &= 0 \\
\therefore \quad \mathbf{b} \cdot (-\mathbf{a} + \mathbf{c}) &= 0 \\
-\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} &= 0 \\
\mathbf{b} \cdot \mathbf{c} &= \mathbf{a} \cdot \mathbf{b}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{AB} &= -\mathbf{a} + \mathbf{b} \\
\mathbf{c} \cdot \overrightarrow{AB} &= \mathbf{c} \cdot (-\mathbf{a} + \mathbf{b}) \\
&= -\mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c} \\
&= -\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b} \\
&= 0
\end{aligned}$$

$\therefore \overrightarrow{CF}$ is perpendicular to $\overrightarrow{AB}$. □

Exercise 4C

1. If the number is even, we can represent it as $2n$, for a suitably chosen integer value n.

$$(2n)^2 = 4n^2$$

which is a multiple of 4, and hence is even.

If the number is odd, we can represent it as $2n+1$ for a suitably chosen integer value n.

$$\begin{aligned}
(2n + 1)^2 &= 4n^2 + 4n + 1 \\
&= 4(n^2 + n) + 1
\end{aligned}$$

which is one more than a multiple of 4, and hence is odd. □

2. There are five possible cases. The number is

- a multiple of 5;
- one more than a multiple of 5;
- two more than a multiple of 5;
- three more than a multiple of 5; or
- four more than a multiple of 5.

Considering each of these exhaustively:

- If the number is a multiple of 5:

$$\begin{aligned}
(5n)^2 &= 25n^2 \\
&= 5(5n^2)
\end{aligned}$$

The square is a multiple of 5.

- If the number is one more than a multiple of 5:

$$\begin{aligned}
(5n + 1)^2 &= 25n^2 + 10n + 1 \\
&= 5(5n^2 + 2n) + 1
\end{aligned}$$

The square is 1 more than a multiple of 5.

- If the number is two more than a multiple of 5:

$$\begin{aligned}
(5n + 2)^2 &= 25n^2 + 20n + 4 \\
&= 5(5n^2 + 4n) + 4
\end{aligned}$$

The square is 4 more than a multiple of 5.

- If the number is three more than a multiple of 5:

$$\begin{aligned}
(5n + 3)^2 &= 25n^2 + 30n + 9 \\
&= 5(5n^2 + 4n + 1) + 4
\end{aligned}$$

The square is 4 more than a multiple of 5.

- If the number is four more than a multiple of 5:

$$\begin{aligned}
(5n + 4)^2 &= 25n^2 + 80n + 16 \\
&= 5(5n^2 + 4n + 3) + 1
\end{aligned}$$

The square is 1 more than a multiple of 5.

□

3. The the number is

   - a multiple of 3;
   - one more than a multiple of 3; or
   - two more than a multiple of 3.

   Considering each of these exhaustively:

   - If the number is a multiple of 3:

   $$\begin{aligned}(3n)^3 &= 27n^3\\ &= 9(3n^3)\end{aligned}$$

   The cube is a multiple of 9.

   - If the number is one more than a multiple of 9:

   $$\begin{aligned}(3n+1)^3 &= (9n^2+6n+1)(3n+1)\\ &= 27n^3+9n^2+18n^2+6n+3n+1\\ &= 27n^3+27n^2+9n+1\\ &= 9(3n^3+3n^3+n)+1\end{aligned}$$

   The cube is 1 more than a multiple of 9.

   - If the number is two more than a multiple of 3:

   $$\begin{aligned}(3n+2)^3 &= (9n^2+12n+4)(3n+2)\\ &= 27n^3+18n^2+36n^2+24n+12n+8\\ &= 27n^3+54n^2+36n+9-1\\ &= 9(3n^3+6n^3+4n+1)-1\end{aligned}$$

   The cube is 1 less than a multiple of 9.

   □

4. - Suppose T_n is even, i.e. $T_n = 2x$ for some integer x, then

   $$\begin{aligned}T_{n+1} &= 3T_n+2\\ &= 3(2x)+2\\ &= 2(3x+1)\end{aligned}$$

   Hence T_{n+1} is also even.

   - Suppose T_n is odd, i.e. $T_n = 2x+1$ for some integer x, then

   $$\begin{aligned}T_{n+1} &= 3T_n+2\\ &= 3(2x+1)+2\\ &= 6x+5\\ &= 2(3x+2)+1\end{aligned}$$

   Hence T_{n+1} is also odd.

   $\therefore T_{n+1}$ has the same parity as T_n. □

5. Consider $x^5 - x = x(x-1)(x+1)(x^2+1)$

   - If $x = 5n$ then x^5-x has $x = 5n$ as a factor, so it is a multiple of 5.
   - If $x = 5n+1$ then $x^5 - x$ has $(x-1) = (5n+1)-1 = 5n$ as a factor, so it is a multiple of 5.
   - If $x = 5n+2$ then

   $$\begin{aligned}x^2+1 &= (5n+2)^2+1\\ &= 25n^2+20n+5\\ &= 5(5n^2+4n+1)\end{aligned}$$

   Hence as x^5-x has $(x^2+1) = 5(5n^2+4n+1)$ as a factor, it is a multiple of 5.

   - If $x = 5n+3$ then

   $$\begin{aligned}x^2+1 &= (5n+3)^2+1\\ &= 25n^2+30n+10\\ &= 5(5n^2+6n+2)\end{aligned}$$

   Hence as x^5-x has $(x^2+1) = 5(5n^2+6n+2)$ as a factor, it is a multiple of 5.

   - If $x = 5n+4$ then $x^5 - x$ has $(x+1) = (5n+4)+1 = 5n+5 = 5(n+1)$ as a factor, so it is a multiple of 5.

   Hence, $x^5 - x$ for $x > 1$ is always a multiple of 5. □

   As one or other of x and $x-1$ is even, $x^5 - x$ always has 2 as a factor. Since it has both 2 and 5 as factors, it is always a multiple of 10.

   If x is odd, both $x-1$ and $x+1$ are even, so $x^5 - x$ is a multiple of $2 \times 2 \times 5 = 20$.

   If x is even, $x-1$ and $x+1$ are both odd. For x^2+1:

   $$\begin{aligned}x^2+1 &= (2n)^2+1\\ &= 4n^2+1\end{aligned}$$

   which is also odd, so $x^5 - x$ has only one factor of 2, and so is not a multiple of 20. We can check this with an example. If $x = 2$,

   $$\begin{aligned}x^5-x &= 2^5-2\\ &= 32-2\\ &= 30\end{aligned}$$

   which is not a multiple of 20.

6. - If $x = 7n$ then x^7-x has $x = 7n$ as a factor, so it is a multiple of 7.
   - If $x = 7n+1$ then $x^7 - x$ has $(x-1) = (7n+1)-1 = 7n$ as a factor, so it is a multiple of 7.
   - If $x = 7n+2$ then

   $$\begin{aligned}x^2+x+1 &= (7n+2)^2+(7n+2)+1\\ &= 49n^2+28n+4+7n+2+1\\ &= 49n^2+35n+7\\ &= 7(7n^2+5n+1)\end{aligned}$$

   Hence as $x^7 - x$ has $(x^2+x+1) = 7(7n^2+5n+1)$ as a factor, it is a multiple of 7.

- If $x = 7n + 3$ then

$$\begin{aligned} x^2 - x + 1 &= (7n+3)^2 - (7n+3) + 1 \\ &= 49n^2 + 42n + 9 - 7n - 3 + 1 \\ &= 49n^2 + 35n + 7 \\ &= 7(7n^2 + 5n + 1) \end{aligned}$$

  Hence as $x^7 - x$ has $(x^2 - x + 1) = 7(7n^2 + 5n + 1)$ as a factor, it is a multiple of 7.

- If $x = 7n + 4$ then

$$\begin{aligned} x^2 + x + 1 &= (7n+4)^2 + (7n+4) + 1 \\ &= 49n^2 + 56n + 16 + 7n + 4 + 1 \\ &= 49n^2 + 56n + 21 \\ &= 7(7n^2 + 8n + 3) \end{aligned}$$

  Hence as $x^7 - x$ has $(x^2 + x + 1) = 7(7n^2 + 8n + 3)$ as a factor, it is a multiple of 7.

- If $x = 7n + 5$ then

$$\begin{aligned} x^2 - x + 1 &= (7n+5)^2 - (7n+5) + 1 \\ &= 49n^2 + 70n + 25 - 7n - 5 + 1 \\ &= 49n^2 + 63n + 21 \\ &= 7(7n^2 + 9n + 3) \end{aligned}$$

  Hence as $x^7 - x$ has $(x^2 - x + 1) = 7(7n^2 + 9n + 3)$ as a factor, it is a multiple of 7.

- If $x = 7n + 6$ then $x^7 - x$ has $(x + 1) = (7n+6)+1 = 7n+7 = 7(n+1)$ as a factor, so it is a multiple of 7.

Hence, $x^7 - x$ for $x > 1$ is always a multiple of 7. □

Exercise 4D

1. Suppose the triangle is right angled. Then the hypotenuse is the longest side, 10cm. By Pythagoras's theorem, the length of the hypotenuse is given by

$$h^2 = 8^2 + 9^2 = 145$$

   but $10^2 = 100$: a contradiction.

   Therefore a triangle with sides 8cm, 9cm and 10cm is not right angled. □

2. Suppose that the lines intersect at point P.

$$\begin{aligned} \text{P} &= 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda\,(4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \\ \text{and P} &= 8\mathbf{i} + \mu\,(2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \end{aligned}$$

   Equating corresponding components:

$$\begin{aligned} 2 + 4\lambda &= 8 + 2\mu \\ 3 - 2\lambda &= \mu \\ -1 + 3\lambda &= -3\mu \end{aligned}$$

$$\begin{aligned} 2 + 4\lambda &= 8 + 2\mu \\ 6 - 4\lambda &= 2\mu \\ 8 &= 8 + 4\mu \\ \mu &= 0 \\ 3 - 2\lambda &= 0 \\ \lambda &= 1.5 \end{aligned}$$

   From $\mu = 0$ we get

$$\begin{aligned} \text{P} &= 8\mathbf{i} + 0\,(2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \\ &= 8\mathbf{i} \end{aligned}$$

   From $\lambda = 1.5$ we get

$$\begin{aligned} \text{P} &= 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + 1.5\,(4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \\ &= 8\mathbf{i} + 3.5\mathbf{k} \end{aligned}$$

   resulting in a contradiction.

   Therefore the lines do not intersect. □

3. Suppose that there exist integers p and q such that $6p + 10q = 151$.

   $6p + 10q = 2(3p + 5q)$ is even. But 151 is odd: a contradiction.

   Therefore no such integer p and q exist. □

4. Suppose that for some positive real a and b

$$\begin{aligned} \frac{a}{b} + \frac{b}{a} &< 2 \\ \frac{a^2 + b^2}{ab} &< 2 \\ a^2 + b^2 &< 2ab \\ a^2 - 2ab + b^2 &< 0 \\ (a - b)^2 &< 0 \end{aligned}$$

   But this requires the square of a real number to be negative: a contradiction.

Therefore no such positive real a and b exist. $\square$

(Note that the step taken in the third line above is only valid because we know ab is positive. If ab was negative then the inequality would change direction.)

5. Suppose the triangle is right angled. Then

$$\begin{aligned}(3x)^2 + (4x+5)^2 &= (5x+4)^2\\ 9x^2 + 16x^2 + 40x + 25 &= 25x^2 + 40x + 16\\ 25x^2 + 40x + 25 &= 25x^2 + 40x + 16\\ 25 &= 16\end{aligned}$$

Therefore the triangle can not be right angled for any value of x. $\square$

6. Suppose that $\log_2 5$ is rational. Then

$$\log_2 5 = \frac{a}{b}$$

for integer a and b with no common factors.

$$\begin{aligned}2^{\frac{a}{b}} &= 5\\ 2^a &= 5^b\end{aligned}$$

2^a is an integer power of 2, and hence is even. But 5^b is an integer power of an odd number, and hence is odd: a contradiction.

Therefore $\log_2 5$ is irrational. $\square$

7. Suppose there exists some smallest positive rational number $\frac{a}{b}$ where a and b are positive integers.

Because this is the smallest rational number, every other positive rational is greater than it.

$$\begin{aligned}\text{Then}\quad \frac{a}{b+1} &> \frac{a}{b}\\ ab &> a(b+1)\\ ab &> ab + a\\ 0 &> a\end{aligned}$$

But this is a contradiction, since we specified a and b both positive.

Therefore there is no smallest rational number greater than zero. $\square$

8. Suppose that there are a finite number of primes. Since every finite set of numbers must have a largest member, there is a largest prime, a, (a very large number, since it is larger than all known primes).

Now consider the number b obtained from the product of all the primes:

$$b = 2 \times 3 \times 5 \times 7 \times 11 \times \ldots \times a$$

Now consider the number $b+1$. This is clearly larger than a. Because b has every prime as a factor, $b+1$ will have a remainder of 1 when divided by every prime number. This means that the only factors of $b+1$ will be 1 and itself. Therefore b is a prime number larger than a: a contradiction.

Therefore there is an infinite number of primes. $\square$

Miscellaneous Exercise 4

1. $$\begin{aligned}\cos(-x) &= \cos(0-x)\\ &= \cos 0 \cos x + \sin 0 \sin x\\ &= 1\cos x + 0 \sin x\\ &= \cos x\end{aligned}$$ $\square$

2. $$\begin{aligned}\cos\left(\frac{\pi}{2} - x\right) &= \sin\frac{\pi}{2}\cos x - \cos\frac{\pi}{2}\sin x\\ &= 1\cos x - 0\sin x\\ &= \cos x\end{aligned}$$ $\square$

3. (a) $$\begin{aligned}z &= 2\operatorname{cis}\frac{\pi}{6}\\ &= 2\cos\frac{\pi}{6} + 2\mathrm{i}\sin\frac{\pi}{6}\\ &= \sqrt{3} + \mathrm{i}\end{aligned}$$

(b) $$\begin{aligned}w &= -1 - \sqrt{3}\mathrm{i}\\ &= \sqrt{(-1)^2 + (-\sqrt{3})^2}\operatorname{cis}\left(\tan^{-1}\frac{-\sqrt{3}}{-1}\right)\\ &= 2\operatorname{cis}\frac{-2\pi}{3} \quad \text{(3rd quadrant)}\end{aligned}$$

(c) $$\begin{aligned}zw &= \left(2\operatorname{cis}\frac{\pi}{6}\right)\left(2\operatorname{cis}\frac{-2\pi}{3}\right)\\ &= 4\operatorname{cis}\left(\frac{\pi}{6} - \frac{2\pi}{3}\right)\\ &= 4\operatorname{cis}\frac{-\pi}{2}\\ &= 4\cos\frac{-\pi}{2} + 4\mathrm{i}\sin\frac{-\pi}{2}\\ &= -4\mathrm{i}\end{aligned}$$

(d)
$$\begin{aligned}\frac{z}{w} &= \frac{2\operatorname{cis}\frac{\pi}{6}}{2\operatorname{cis}\frac{-2\pi}{3}}\\ &= \operatorname{cis}\left(\frac{\pi}{6}+\frac{2\pi}{3}\right)\\ &= \operatorname{cis}\frac{5\pi}{6}\\ &= \cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}\\ &= -\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}\end{aligned}$$

4. (a)
$$\begin{aligned}\overrightarrow{\mathrm{OB}} &= \mathbf{a}+\mathbf{c}\\ \overrightarrow{\mathrm{OB}}\cdot\overrightarrow{\mathrm{OA}} &= (\mathbf{a}+\mathbf{c})\cdot\mathbf{a}\\ &= a^2+\mathbf{a}\cdot\mathbf{c}\\ \overrightarrow{\mathrm{OB}}\cdot\overrightarrow{\mathrm{OC}} &= (\mathbf{a}+\mathbf{c})\cdot\mathbf{c}\\ &= c^2+\mathbf{a}\cdot\mathbf{c}\end{aligned}$$
□

(b) Since OA=OC then $a^2 = c^2$ and it follows from the above that $\overrightarrow{\mathrm{OB}}\cdot\overrightarrow{\mathrm{OA}} = \overrightarrow{\mathrm{OB}}\cdot\overrightarrow{\mathrm{OC}}$
$$\begin{aligned}(\mathrm{OB})(\mathrm{OA})\cos\alpha &= (\mathrm{OB})(\mathrm{OC})\cos\beta\\ \cos\alpha &= \cos\beta\\ \alpha &= \beta\end{aligned}$$
(since both α and β are between 0 and 180°) □

5. L.H.S.:
$$\begin{aligned}&(1+\sin\mathrm{A}\cos\mathrm{A})(\sin\mathrm{A}-\cos\mathrm{A})\\ &= \sin\mathrm{A}-\cos\mathrm{A}+\sin^2\mathrm{A}\cos\mathrm{A}-\sin\mathrm{A}\cos^2\mathrm{A}\\ &= \sin\mathrm{A}-\cos\mathrm{A}+(1-\cos^2\mathrm{A})\cos\mathrm{A}\\ &\quad-\sin\mathrm{A}(1-\sin^2\mathrm{A})\\ &= \sin\mathrm{A}-\cos\mathrm{A}+\cos\mathrm{A}-\cos^3\mathrm{A}-\sin\mathrm{A}+\sin^3\mathrm{A}\\ &= -\cos^3\mathrm{A}+\sin^3\mathrm{A}\\ &= \sin^3\mathrm{A}-\cos^3\mathrm{A}\\ &= \text{R.H.S.}\end{aligned}$$
□

6. L.H.S.:
$$\begin{aligned}\cos^4\theta-\sin^4\theta &= (\cos^2\theta+\sin^2\theta)(\cos^2\theta-\sin^2\theta)\\ &= 1(\cos^2\theta-\sin^2\theta)\\ &= (1-\sin^2\theta)-\sin^2\theta\\ &= 1-2\sin^2\theta\\ &= \text{R.H.S.}\end{aligned}$$
□

7.
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{h\to0}\frac{(x+h)^3+(x+h)-(x^3+x)}{h}\\ &= \lim_{h\to0}\frac{x^3+3x^2h+3xh^2+h^3+x+h-x^3-x}{h}\\ &= \lim_{h\to0}\frac{3x^2h+3xh^2+h^3+h}{h}\\ &= \lim_{h\to0}(3x^2+3xh+h^2+1)\\ &= 3x^2+1\end{aligned}$$

8. There are infinite possible correct solutions. First find any vector perpendicular to the one given (for example by setting one component to zero and then swapping the other two, changing the sign of one) such as:
$$\begin{pmatrix}0\\1\\-4\end{pmatrix}$$
then scale this vector so that it has unit magnitude:
$$\frac{1}{\sqrt{0^2+1^2+(-4)^2}}\begin{pmatrix}0\\1\\-4\end{pmatrix} = \frac{1}{\sqrt{17}}\begin{pmatrix}0\\1\\-4\end{pmatrix}$$

9. The set describes the locus of a circle radius 3 centred $(4+4\mathrm{i})$ on the Argand plane.

(a) The minimum possible value of $\mathrm{Im}(z)$ is $4-3=1$.

(b) The maximum possible value of $\mathrm{Re}(z)$ is $4+3=7$.

(c) The distance between the origin and the circle's centre is $|4+4\mathrm{i}| = 4\sqrt{2}$. Since this is greater than 3, the minimum possible value of $|z|$ is $4\sqrt{2}-3$.

(d) Similarly the maximum possible value of $|z|$ is $4\sqrt{2}+3$.

(e) Since $\bar{z}$ is just the reflection of z in the real axis, the maximum value of $|\bar{z}|$ is equal to the maximum value of $|z|$, i.e. $4\sqrt{2}+3$.

10.
$$\begin{aligned}\mathbf{r}_\mathrm{F} &= \frac{1}{2}(\mathbf{r}_\mathrm{A}+\mathbf{r}_\mathrm{B})\\ &= \frac{1}{2}(2\mathbf{i}+6\mathbf{j}-4\mathbf{k})\\ &= \mathbf{i}+3\mathbf{j}-2\mathbf{k}\end{aligned}$$
The radius of the sphere is AF:
$$\begin{aligned}\mathrm{AF} &= |\mathbf{r}_\mathrm{F}-\mathbf{r}_\mathrm{A}|\\ &= |\,(\mathbf{i}+3\mathbf{j}-2\mathbf{k})-(-\mathbf{i}+6\mathbf{j}-8\mathbf{k})\,|\\ &= |\,(2\mathbf{i}-3\mathbf{j}+6\mathbf{k})\,|\\ &= \sqrt{2^2+(-3)^2+6^2}\\ &= 7\end{aligned}$$

CF, DF and EF are all also equal to the radius of the sphere, i.e. 7.
$$\begin{aligned}\mathrm{CF} &= 7\\ |\mathbf{r}_\mathrm{F}-\mathbf{r}_\mathrm{C}| &= 7\\ |\,(\mathbf{i}+3\mathbf{j}-2\mathbf{k})-(7\mathbf{i}+c\mathbf{k})| &= 7\\ |-6\mathbf{i}+3\mathbf{j}+(-2-c)\mathbf{k}| &= 7\\ (-6)^2+3^2+(-2-c)^2 &= 49\\ (-2-c)^2 &= 4\\ -2-c &= \pm2\\ 2+c &= \mp2\\ c &= 0\end{aligned}$$

(rejecting the solution $c = -4$ because we are given that c is non-negative.)

$$\begin{aligned}
\text{DF} &= 7 \\
|\mathbf{r}_\text{F} - \mathbf{r}_\text{D}| &= 7 \\
|(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) - (-\mathbf{i} + d\mathbf{j} - 5\mathbf{k})| &= 7 \\
|2\mathbf{i} + (3-d)\mathbf{j} + 3\mathbf{k}| &= 7 \\
2^2 + (3-d)^2 + 3^2 &= 49 \\
(3-d)^2 &= 36 \\
3 - d &= \pm 6 \\
d &= 3 \mp 6 \\
d &= 9
\end{aligned}$$

(rejecting the negative solution again.)

$$\begin{aligned}
\text{DE} &= 7 \\
|\mathbf{r}_\text{F} - \mathbf{r}_\text{E}| &= 7 \\
|(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) - (4\mathbf{i} + \mathbf{j} + e\mathbf{k})| &= 7 \\
|-3\mathbf{i} + 2\mathbf{j} + (-2-e)\mathbf{k}| &= 7 \\
(-3)^2 + 2^2 + (-2-e)^2 &= 49 \\
(-2-e)^2 &= 36 \\
-2 - e &= \pm 6 \\
e &= -2 \mp 6 \\
e &= 4
\end{aligned}$$

(rejecting the negative solution again.)

11.
$$\begin{aligned}
\overrightarrow{\text{AB}} &= \mathbf{r}_\text{B} - \mathbf{r}_\text{A} \\
&= \begin{pmatrix} 2200 \\ 4000 \\ 600 \end{pmatrix} - \begin{pmatrix} 600 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1600 \\ 4000 \\ 600 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
{}_\text{A}\mathbf{v}_\text{B} &= \mathbf{v}_\text{A} - \mathbf{v}_\text{B} \\
&= \begin{pmatrix} -196 \\ 213 \\ 18 \end{pmatrix} - \begin{pmatrix} -240 \\ 100 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 44 \\ 113 \\ 18 \end{pmatrix}
\end{aligned}$$

For the missiles to intercept,

$$\begin{aligned}
\overrightarrow{\text{AB}} &= t\,{}_\text{A}\mathbf{v}_\text{B} \\
\begin{pmatrix} 1600 \\ 4000 \\ 600 \end{pmatrix} &= t \begin{pmatrix} 44 \\ 113 \\ 18 \end{pmatrix}
\end{aligned}$$

i components:

$$\begin{aligned}
1600 &= 44t \\
t &= \frac{400}{11}
\end{aligned}$$

k components:

$$\begin{aligned}
600 &= 18t \\
t &= \frac{100}{3}
\end{aligned}$$

Because $\overrightarrow{\text{AB}}$ is not a scalar multiple of ${}_\text{A}\mathbf{v}_\text{B}$ the missiles will not intercept.

To find how much it misses by (i.e. the minimum distance), let P be the point of closest approach.

$$\begin{aligned}
\overrightarrow{\text{BP}} &= \overrightarrow{\text{BA}} + \overrightarrow{\text{AP}} \\
&= -\overrightarrow{\text{AB}} + t\,{}_\text{A}\mathbf{v}_\text{B} \\
&= -\begin{pmatrix} 1600 \\ 4000 \\ 600 \end{pmatrix} + t \begin{pmatrix} 44 \\ 113 \\ 18 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{\text{BP}} \cdot {}_\text{A}\mathbf{v}_\text{B} &= 0 \\
\left(-\begin{pmatrix} 1600 \\ 4000 \\ 600 \end{pmatrix} + t \begin{pmatrix} 44 \\ 113 \\ 18 \end{pmatrix}\right) \cdot \begin{pmatrix} 44 \\ 113 \\ 18 \end{pmatrix} &= 0 \\
-1600 \times 44 - 4000 \times 113 - 600 \times 18 & \\
+t(44^2 + 113^2 + 18^2) &= 0 \\
-533200 + 15029t &= 0 \\
t &= 35.478
\end{aligned}$$

$$\begin{aligned}
|\overrightarrow{\text{BP}}| &= \left| -\begin{pmatrix} 1600 \\ 4000 \\ 600 \end{pmatrix} + t \begin{pmatrix} 44 \\ 113 \\ 18 \end{pmatrix} \right| \\
|\overrightarrow{\text{BP}}| &= \left| \begin{pmatrix} -38.96 \\ 9.02 \\ 38.61 \end{pmatrix} \right| \\
&= 55.59\text{m}
\end{aligned}$$

The missile misses by about 56m.

After 20 seconds the positions are now

$$\begin{aligned}
\mathbf{r}_\text{A} &= \begin{pmatrix} 600 \\ 0 \\ 0 \end{pmatrix} + 20 \begin{pmatrix} -196 \\ 213 \end{pmatrix} 18 \\
&= \begin{pmatrix} -3320 \\ 4260 \\ 360 \end{pmatrix} \\
\mathbf{r}_\text{B} &= \begin{pmatrix} 2200 \\ 4000 \\ 600 \end{pmatrix} + 20 \begin{pmatrix} -240 \\ 100 \end{pmatrix} 0 \\
&= \begin{pmatrix} -2600 \\ 6000 \\ 600 \end{pmatrix} \\
\overrightarrow{\text{AB}} &= \mathbf{r}_\text{B} - \mathbf{r}_\text{A} \\
&= \begin{pmatrix} -2600 \\ 6000 \\ 600 \end{pmatrix} - \begin{pmatrix} -3320 \\ 4260 \\ 360 \end{pmatrix} \\
&= \begin{pmatrix} 720 \\ 1740 \\ 240 \end{pmatrix}
\end{aligned}$$

In order to intercept 15 seconds later:

$$\begin{aligned}
15\,{}_{\mathrm{A}}\mathbf{v}_{\mathrm{B}} &= \overrightarrow{\mathrm{AB}}\\
{}_{\mathrm{A}}\mathbf{v}_{\mathrm{B}} &= \frac{1}{15}\begin{pmatrix}720\\1740\\240\end{pmatrix}\\
&= \begin{pmatrix}48\\116\\16\end{pmatrix}\\
\mathbf{v}_{\mathrm{A}} - \mathbf{v}_{\mathrm{B}} &= \begin{pmatrix}48\\116\\16\end{pmatrix}\\
\mathbf{v}_{\mathrm{A}} &= \begin{pmatrix}48\\116\\16\end{pmatrix} + \mathbf{v}_{\mathrm{B}}\\
&= \begin{pmatrix}48\\116\\16\end{pmatrix} + \begin{pmatrix}-240\\100\\0\end{pmatrix}\\
&= \begin{pmatrix}-192\\216\\16\end{pmatrix}\ \mathrm{m/s}
\end{aligned}$$

Chapter 5

Exercise 5A

1. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^5 - x^2\right) = 5x^4 - 2x$

2. $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(3 + x^3\right) &= 0 + 3x^2\\ &= 3x^2\end{aligned}$$

3. $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(5 - \cos x) &= 0 - -\sin x\\ &= \sin x\end{aligned}$$

4. $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\sin x - \cos x) &= \cos x - -\sin x\\ &= \cos x + \sin x\end{aligned}$$

5. $\dfrac{\mathrm{d}}{\mathrm{d}x}(\cos x - \sin x) = -\sin x - \cos x$

6. $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(x - \tan x) &= 1 - (1 + \tan^2 x)\\ &= -\tan^2 x\end{aligned}$$

7. $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}((x+1)(2x-3)) &= 1(2x-3) + 2(x+1)\\ &= 2x - 3 + 2x + 2\\ &= 4x - 1\end{aligned}$$

8. $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(5x^2(1-5x)\right) &= 10x(1-5x) - 5(5x^2)\\ &= 10x - 50x^2 - 25x^2\\ &= 10x - 75x^2\end{aligned}$$

9. $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(6\sin x) &= (0)(\sin x) + (6)(\cos x)\\ &= 6\cos x\end{aligned}$$

10. $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(4\cos x) &= (0)(\cos x) + (4)(-\sin x)\\ &= -4\sin x\end{aligned}$$

11. $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(x\sin x) &= (1)(\sin x) + (x)(\cos x)\\ &= \sin x + x\cos x\end{aligned}$$

12. $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\cos x\right) &= (2x)(\cos x) + (x^2)(-\sin x)\\ &= 2x\cos x - x^2\sin x\end{aligned}$$

13. $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{3x^2-1}\right) &= \frac{(1)(3x^2-1) - (x)(6x)}{(3x^2-1)^2}\\ &= \frac{3x^2 - 1 - 6x^2}{(3x^2-1)^2}\\ &= \frac{-3x^2-1}{(3x^2-1)^2}\\ &= -\frac{3x^2+1}{(3x^2-1)^2}\end{aligned}$$

14. $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2+1}{x^2-1}\right) &= \frac{(2x)(x^2-1) - (x^2+1)(2x)}{(x^2-1)^2}\\ &= \frac{(2x)(x^2-1-x^2-1)}{(x^2-1)^2}\\ &= \frac{(2x)(-2)}{(x^2-1)^2}\\ &= -\frac{4x}{(x^2-1)^2}\end{aligned}$$

15. $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\cos x}{x}\right) &= \frac{(-\sin x)(x) - (\cos x)(1)}{x^2}\\ &= \frac{-x\sin x - \cos x}{x^2}\\ &= -\frac{\sin x}{x} - \frac{\cos x}{x^2}\end{aligned}$$

16. $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{x}\right) &= \frac{(\cos x)(x) - (\sin x)(1)}{x^2}\\ &= \frac{x\cos x - \sin x}{x^2}\\ &= \frac{\cos x}{x} - \frac{\sin x}{x^2}\end{aligned}$$

17. $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{\sin x}\right) &= \frac{(1)(\sin x) - (x)(\cos x)}{\sin^2 x}\\ &= \frac{\sin x - x\cos x}{\sin^2 x}\end{aligned}$$

18. $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{\cos x}\right) &= \frac{(1)(\cos x) - (x)(-\sin x)}{\cos^2 x}\\ &= \frac{\cos x + x\sin x}{\cos^2 x}\end{aligned}$$

19. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 6u\frac{\mathrm{d}u}{\mathrm{d}x}\\ &= 6(x^2+1)(2x)\\ &= 12x(x^2+1)\end{aligned}$$

20. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2\sqrt{u}}\frac{\mathrm{d}u}{\mathrm{d}x}\\ &= \frac{1}{2\sqrt{x^2-1}}(2x)\\ &= \frac{2x}{2\sqrt{x^2-1}}\\ &= \frac{x}{\sqrt{x^2-1}}\end{aligned}$$

21. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (\cos u)\frac{\mathrm{d}u}{\mathrm{d}x}\\ &= 6\cos(6x)\end{aligned}$$

22. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (-\sin u)\frac{\mathrm{d}u}{\mathrm{d}x}\\ &= (-\sin(2x+3))\,(2)\\ &= -2\sin(2x+3)\end{aligned}$$

23. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\sin x\cos x$

24. $\frac{dy}{dx} = 3\sin^2 x \cos x$

25. $\frac{dy}{dx} = (5\cos^4 x)(-\sin x)$
$= -5\cos^4 x \sin x$

26. $\frac{dy}{dx} = (-\sin 3x)(3)$
$= -3\sin 3x$

27. $\frac{dy}{dx} = (\cos(3x-7))(3)$
$= 3\cos(3x-7)$

28. $\frac{dy}{dx} = \left(\cos(x^2-3)\right)(2x)$
$= 2x\cos(x^2-3)$

29. $\frac{dy}{dx} = -3(-\sin x)$
$= 3\sin x$

30. $\frac{dy}{dx} = 3 + 2(-\sin x)$
$= 3 - 2\sin x$

31. $\frac{dy}{dx} = 2\cos 2x$
(By this stage you should be getting to the point of being able to do these in a single step.)

32. $\frac{d}{dx}\tan 2x = \frac{2}{\cos^2 2x}$

33. $\frac{dy}{dx} = 2x + \sin x$

34. $$\begin{aligned}\frac{d}{dx}\frac{1+\sin x}{x^2} &= \frac{(\cos x)(x^2) - (1+\sin x)(2x)}{x^4}\\ &= \frac{x^2\cos x - 2x - 2x\sin x}{x^4}\\ &= \frac{x\cos x - 2 - 2\sin x}{x^3}\end{aligned}$$

35. $$\begin{aligned}\frac{d}{dx}\tan^2 x &= (2\tan x)(1+\tan^2 x)\\ &= 2\tan x + 2\tan^3 x\end{aligned}$$

36. $\frac{dy}{dx} = 3\cos x + 2\sin x$

37. $\frac{dy}{dx} = -3\sin 3x$

38. $\frac{dy}{dx} = -9\sin 9x$

39. $\frac{d}{dx}\tan 3x = \frac{3}{\cos^2 3x}$

40. $\frac{d}{dx}(\tan x + \tan 2x) = \frac{1}{\cos^2 x} + \frac{2}{\cos^2 2x}$

41. $$\begin{aligned}\frac{d}{dx}(3\cos 2x) &= 3(-2\sin 2x)\\ &= -6\sin 2x\end{aligned}$$

42. $$\begin{aligned}\frac{d}{dx}(5\sin 3x) &= 5(3\cos 3x)\\ &= 15\cos 3x\end{aligned}$$

43. $\frac{d}{dx}(2\sin 3x + 3\cos 2x) = 6\cos 3x - 6\sin 2x$

44. $$\begin{aligned}\frac{d}{dx}(\sin^5 x) &= (5\sin^4 x)(\cos x)\\ &= 5\sin^4 x\cos x\end{aligned}$$

45. $$\begin{aligned}\frac{d}{dx}(5\cos^2 x) &= (10\cos x)(-\sin x)\\ &= -10\cos x\sin x\end{aligned}$$

46. $$\begin{aligned}\frac{d}{dx}(-3\cos^4 x) &= (-12\cos^3 x)(-\sin x)\\ &= 12\cos^3 x\sin x\end{aligned}$$

47. $$\begin{aligned}\frac{d}{dx}(\cos^{0.5} x) &= (0.5\cos^{-0.5} x)(-\sin x)\\ &= -0.5\cos^{-0.5} x\sin x\\ &= -\frac{\sin x}{2\cos^{0.5} x}\end{aligned}$$

48. $$\begin{aligned}\frac{d}{dx}\sqrt{\sin x} &= (0.5\sin^{-0.5} x)(\cos x)\\ &= 0.5\sin^{-0.5} x\cos x\\ &= \frac{\cos x}{2\sqrt{\sin x}}\end{aligned}$$

49–57 You should be able to do these in a single step . . . no working required.

58. $$\begin{aligned}f'(x) &= 1\cos x + x(-\sin x)\\ &= \cos x - x\sin x\end{aligned}$$

59. $$\begin{aligned}f'(x) &= 2x\cos x + x^2(-\sin x)\\ &= 2x\cos x - x^2\sin x\end{aligned}$$

60. $$\begin{aligned}f'(x) &= 2\sin x + 2x(\cos x)\\ &= 2\sin x + 2x\cos x\end{aligned}$$

61. $$\begin{aligned}f'(x) &= (2\sin 3x)(3\cos 3x)\\ &= 6\sin 3x\cos 3x\end{aligned}$$

62. $$\begin{aligned}f'(x) &= (3\cos^2 2x)(-2\sin 2x)\\ &= -6\cos^2 2x\sin 2x\end{aligned}$$

63. (a) L.H.S:
$$\begin{aligned}\frac{d}{dx}\sec x &= \frac{d}{dx}\frac{1}{\cos x}\\ &= \frac{d}{dx}\cos^{-1} x\\ &= (-\cos^{-2} x)(-\sin x)\\ &= \frac{\sin x}{\cos^2 x}\\ &= \frac{1}{\cos x}\frac{\sin x}{\cos x}\\ &= \sec x\tan x\\ &= \text{R.H.S.}\end{aligned}$$

□

(b)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\csc x &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\sin x}\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1} x\\ &= (-\sin^{-2} x)(\cos x)\\ &= -\frac{\cos x}{\sin^2 x}\\ &= -\frac{1}{\sin x}\frac{\cos x}{\sin x}\\ &= -\csc x \cot x\\ &= \text{R.H.S.}\end{aligned}$$

□

(c)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\cot x &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\cos x}{\sin x}\\ &= \frac{(-\sin x)(\sin x) - (\cos x)(\cos x)}{\sin^2 x}\\ &= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}\\ &= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x}\\ &= -\frac{1}{\sin^2 x}\\ &= -\csc^2 x\\ &= \text{R.H.S.}\end{aligned}$$

□

64. $$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos x$$
$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

65. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -2\sin 2x\\ -2\sin\left(2\times\frac{\pi}{6}\right) &= -2\sin\frac{\pi}{3}\\ &= -2\times\frac{\sqrt{3}}{2}\\ &= -\sqrt{3}\end{aligned}$$

66. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 2\cos x\cos x + 2\sin x(-\sin x)\\ &= 2\cos^2 x - 2\sin^2 x\\ &= 2\cos 2x\end{aligned}$$
$2\cos(2\times 0) = 2$
alternatively:
$$\begin{aligned}2\sin x\cos x &= \sin 2x\\ \frac{\mathrm{d}}{\mathrm{d}x}\sin 2x &= 2\cos 2x\\ 2\cos(2\times 0) &= 2\end{aligned}$$

67. $$\frac{\mathrm{d}y}{\mathrm{d}x} = 6\sin x\cos x$$
$6\sin\pi\cos\pi = 0$

68. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \cos x\\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\cos x\\ &= -\sin x\end{aligned}$$

69. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -5\sin 5x\\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}(-5\sin 5x)\\ &= -25\cos 5x\end{aligned}$$

70. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 6\cos 2x\\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}(6\cos 2x)\\ &= -12\sin 2x\end{aligned}$$

71. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \cos x - \sin x\\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}(\cos x - \sin x)\\ &= -\sin x - \cos x\end{aligned}$$

72. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 3\sin^2 x\cos x\\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}(3\sin^2 x\cos x)\\ &= (6\sin x\cos x)(\cos x) + (3\sin^2 x)(-\sin x)\\ &= 6\sin x\cos^2 x - 3\sin^3 x\\ &= 6\sin x(1-\sin^2 x) - 3\sin^3 x\\ &= 6\sin x - 6\sin^3 x - 3\sin^3 x\\ &= 6\sin x - 9\sin^3 x\end{aligned}$$

73. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -4\cos x\sin x\\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}(-4\cos x\sin x)\\ &= (4\sin x)(\sin x) + (-4\cos x)(\cos x)\\ &= 4\sin^2 x - 4\cos^2 x\\ &= -4(\cos^2 x - \sin^2 x)\\ &= -4\cos 2x\end{aligned}$$

74. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \sin x + x\cos x\\ \sin\frac{\pi}{2} + \frac{\pi}{2}\cos\frac{\pi}{2} &= 1 + 0\\ &= 1\\ y - y_1 &= m(x - x_1)\\ y - \frac{\pi}{2} &= 1(x - \frac{\pi}{2})\\ y &= x\end{aligned}$$

75. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 1 - 6\sin 2x\\ 1 - 6\sin(2\times 0) &= 1\\ y - y_1 &= m(x - x_1)\\ y - 3 &= 1(x - 0)\\ y &= x + 3\end{aligned}$$

76.
$$\begin{aligned}\frac{dy}{dx} &= 3(1+\tan^2 2x)(2)\\ &= 6+6\tan^2 2x\\ 6+6\tan^2(2\times\frac{\pi}{8}) &= 6+6\tan^2\frac{\pi}{4}\\ &= 6+6(1^2)\\ &= 12\\ y-y_1 &= m(x-x_1)\\ y-3 &= 12(x-\frac{\pi}{8})\\ y &= 12x-\frac{3\pi}{2}+3\end{aligned}$$

77. (a)
$$\begin{aligned}f'(x) &= 2\cos 2x\\ f'(\pi/6) &= 2\cos(\pi/3)\\ &= 1\end{aligned}$$

(b)
$$\begin{aligned}f''(x) &= -4\sin 2x\\ f''(\pi/6) &= -4\sin(\pi/3)\\ &= -2\sqrt{3}\end{aligned}$$

78. (a)
$$\begin{aligned}f'(x) &= 2\sin x\cos x\\ &= \sin 2x\\ f'(\pi/6) &= \sin(\pi/3)\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

(b)
$$\begin{aligned}f''(x) &= 2\cos 2x\\ f''(\pi/6) &= 2\cos(\pi/3)\\ &= 1\end{aligned}$$

79. (a)
$$\begin{aligned}f'(x) &= 12\sin^2 x\cos x\\ f'(2) &= 12\sin^2(2)\cos(2)\\ &= -4.13 \text{ (2d.p.)}\end{aligned}$$

(b)
$$\begin{aligned}f''(x) &= (24\sin x\cos x)(\cos x)\\ &\quad + (12\sin^2 x)(-\sin x)\\ &= 24\sin x\cos^2 x - 12\sin^3 x\\ &= 24\sin x(1-\sin^2 x) - 12\sin^3 x\\ &= 24\sin x - 24\sin^3 x - 12\sin^3 x\\ &= 24\sin x - 36\sin^3 x\\ f''(2) &= 24\sin(2) - 36\sin^3(2)\\ &= -5.24 \text{ (2d.p.)}\end{aligned}$$

80. Because our limits are defined in terms of radians, it is necessary to do a conversion when working in degrees:

$$\begin{aligned}\sin x^\circ &= \sin\left(\frac{\pi}{180}x\right)\\ \frac{dy}{dx} &= \cos\left(\frac{\pi}{180}x\right)\left(\frac{\pi}{180}\right)\end{aligned}$$

converting back to degrees

$$= \frac{\pi}{180}\cos x^\circ$$

81. The length of the rectangle as drawn is $20\cos\theta$ and the breadth is $20\sin\theta$ so the area is given by

$$\begin{aligned}A &= (20\cos\theta)(20\sin\theta)\\ &= 400\sin\theta\cos\theta\\ &= 200(2\sin\theta\cos\theta)\\ &= 200\sin 2\theta\\ \frac{dA}{d\theta} &= 400\cos 2\theta\end{aligned}$$

Set the derivative to zero to find the maximum:

$$\begin{aligned}400\cos 2\theta &= 0\\ 2\theta &= \frac{\pi}{2}\end{aligned}$$

which implies that the diagonals are perpendicular, hence the rectangle is a square. □

$$\begin{aligned}A &= 200\sin\frac{\pi}{2}\\ &= 200\text{cm}^2\end{aligned}$$

For side length, just take the square root:

$$\begin{aligned}l &= \sqrt{200}\\ &= 10\sqrt{2}\end{aligned}$$

82.
$$\begin{aligned}A &= \frac{1}{2}(8)(10)\sin 0.1t\\ &= 40\sin 0.1t\\ \frac{dA}{dt} &= 4\cos 0.1t\end{aligned}$$

(a)
$$\begin{aligned}\frac{dA}{dt} &= 4\cos(0.1\times 1)\\ &= 3.98\text{cm}^2/\text{s}\end{aligned}$$

(b)
$$\begin{aligned}\frac{dA}{dt} &= 4\cos(0.1\times 5)\\ &= 3.51\text{cm}^2/\text{s}\end{aligned}$$

(c)
$$\begin{aligned}\frac{dA}{dt} &= 4\cos(0.1\times 10)\\ &= 2.16\text{cm}^2/\text{s}\end{aligned}$$

(d)
$$\begin{aligned}\frac{dA}{dt} &= 4\cos(0.1\times 20)\\ &= -1.66\text{cm}^2/\text{s}\end{aligned}$$

83. (a) The maximum value of x is 5 and occurs when $3t = \frac{\pi}{2}$, i.e. $t = \frac{\pi}{6}$.

(b)
$$\begin{aligned}5\sin 3t &= 2.5\\ \sin 3t &= 0.5\\ 3t &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}\\ t &= \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}\end{aligned}$$

(c)
$$\begin{aligned}\frac{dx}{dt} &= 15\cos 3t\\ &= 15\cos(3\times 0.6)\\ &= -3.4\end{aligned}$$

(d) $$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= \frac{\mathrm{d}}{\mathrm{d}t}15\cos 3t \\ &= -45\sin 3t \\ &= -9(5\sin 3t) \\ &= -9x\end{aligned}$$

$\square$ $k = -9$.

84.
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}3\sin\theta + 4\cos\theta &= 3\cos\theta - 4\sin\theta \\ 3\cos\theta - 4\sin\theta &= 0 \\ 4\sin\theta &= 3\cos\theta \\ \tan\theta &= \frac{3}{4} \\ \theta &= 0.64353 \quad \sin\theta + 4\cos\theta = 5\end{aligned}$$

(This can be confirmed as a maximum, if necessary, either graphically or by evaluating points on either side, or using the second derivative test.)

Miscellaneous Exercise 5

1. Draw and shade a circle centred at $3 + \mathrm{i}$ having radius 3.

2. Proof by exhaustion:
Because $(-n)^2 = n^2$ it will be sufficient to prove this for non-negative integers.

   Any non-negatve integer can be represented as $n = 10t + u$ where t is a natural number and u is a single digit.

   Hence any square can be represented as

$$\begin{aligned}n^2 &= (10t + u)^2 \\ &= 100t^2 + 20ut + u^2 \\ &= 10(10t^2 + 2ut) + u^2\end{aligned}$$

   Because $10(10t^2 + 2ut)$ is a multiple of 10, it has a zero in the units digit, so the units digit of n^2 is determined solely by the units digit of u^2.

   The possible units digit of any square number can hence be determined exhaustively:

u	0	1	2	3	4	5	6	7	8	9
u^2	0	1	4	9	6	5	6	9	4	1

   (where the tens digit of u^2 has been discarded).

   Thus the only possible last digits of any square number are 0, 1, 4, 5, 6 and 9. It is not possible to obtain a last digit of 2, 3, 7 or 8. $\square$

3.
$$\begin{aligned}r &= \sqrt{(-\sqrt{3})^2 + 1^2} \\ &= 2 \\ \tan\theta &= \frac{1}{-\sqrt{3}} \\ \theta &= \frac{5\pi}{6} \text{ (second quadrant)} \\ (-\sqrt{3} + \mathrm{i}) &= 2\operatorname{cis}\frac{5\pi}{6}\end{aligned}$$

4. (a)
$$\begin{aligned}\sin x &= \frac{\cos x}{\sqrt{3}} \\ \tan x &= \frac{1}{\sqrt{3}} \\ x &= \frac{\pi}{6}, \frac{7\pi}{6}\end{aligned}$$

(b)
$$\begin{aligned}\sin^2 x + (1 + \cos x)(\cos x) &= 0.5 \\ \sin^2 x + \cos x + \cos^2 x &= 0.5 \\ 1 + \cos x &= 0.5 \\ \cos x &= -0.5 \\ x &= \pm\frac{2\pi}{3}\end{aligned}$$

(c)
$$\begin{aligned}\sin x(2\cos x - \sin x) &= \cos^2 x \\ 2\sin x\cos x - \sin^2 x &= \cos^2 x \\ \sin 2x &= \sin^2 x + \cos^2 x \\ &= 1 \\ 2x &= \frac{\pi}{2}, -\frac{3\pi}{2} \\ x &= \frac{\pi}{4}, -\frac{3\pi}{4}\end{aligned}$$

5.
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(-\sin x)(x) - (\cos x)(1)}{x^2} \\ &= -\frac{x\sin x + \cos x}{x^2}\end{aligned}$$

At $x = \frac{\pi}{2}$,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}}{\left(\frac{\pi}{2}\right)^2}\\ &= -\frac{\frac{\pi}{2} + 0}{\left(\frac{\pi}{2}\right)^2}\\ &= -\frac{1}{\frac{\pi}{2}}\\ &= -\frac{2}{\pi}\\ y - y_1 &= m(x - x_1)\\ y - 0 &= -\frac{2}{\pi}\left(x - \frac{\pi}{2}\right)\\ y &= -\frac{2}{\pi}x + 1\end{aligned}$$

6. Equate **i** and **j** components and solve for μ and λ:

$$\begin{aligned}3 + 4\lambda &= 2 + 3\mu\\ 2 + \lambda &= 1 + \mu\\ -6 - 3\lambda &= -3 - 3\mu\\ -3 + \lambda &= -1\\ \lambda &= 2\\ \mu &= 3\end{aligned}$$

Now see whether this solution works for the **k** components:

$$\begin{aligned}-1 + 3\lambda &= 1 + 2\mu\\ -1 + 3(2) &\neq 1 + 2(3)\end{aligned}$$

The lines do not intersect.

7. Point A:

$$(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) + 1(-\mathbf{i} + 3\mathbf{j} + \mathbf{k}) = \mathbf{i} + 6\mathbf{j}$$

Point B:

$$(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) + 5(-\mathbf{i} + 3\mathbf{j} + \mathbf{k}) = -3\mathbf{i} + 18\mathbf{j} + 4\mathbf{k}$$

$\overrightarrow{\mathrm{AC}} : \overrightarrow{\mathrm{BA}} = -1 : 4$ means $\overrightarrow{\mathrm{AC}} : \overrightarrow{\mathrm{AB}} = -1 : -4 = 1 : 4$ so the position vector of C is

$$\begin{aligned}\overrightarrow{\mathrm{OC}} &= \overrightarrow{\mathrm{OA}} + \frac{1}{4}\overrightarrow{\mathrm{AB}}\\ &= \mathbf{i} + 6\mathbf{j} + \frac{1}{4}\left((-3\mathbf{i} + 18\mathbf{j} + 4\mathbf{k}) - (\mathbf{i} + 6\mathbf{j})\right)\\ &= \mathbf{i} + 6\mathbf{j} + \frac{1}{4}(-4\mathbf{i} + 12\mathbf{j} + 4\mathbf{k})\\ &= \mathbf{i} + 6\mathbf{j} + (-\mathbf{i} + 3\mathbf{j} + \mathbf{k})\\ &= 9\mathbf{j} + \mathbf{k}\end{aligned}$$

8. To prove:

$$z_1\overline{z_1} = |z_1|^2$$

Proof:
LHS:

$$\begin{aligned}z_1 + \overline{z_1} &= (a + b\mathrm{i})(a - b\mathrm{i})\\ &= a^2 - ab\mathrm{i} + ab\mathrm{i} - b^2\mathrm{i}^2\\ &= a^2 + b^2\\ &= |a + b\mathrm{i}|^2\\ &= |z_1|^2\\ &= \text{RHS}\end{aligned}$$

□

To prove:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

Proof:
LHS:

$$\begin{aligned}\overline{z_1 + z_2} &= \overline{a + b\mathrm{i} + c + d\mathrm{i}}\\ &= \overline{a + c + (b + d)\mathrm{i}}\\ &= a + c - (b + d)\mathrm{i}\\ &= a - b\mathrm{i} + c - d\mathrm{i}\\ &= \overline{a + b\mathrm{i}} + \overline{c + d\mathrm{i}}\\ &= \overline{z_1} + \overline{z_2}\\ &= \text{RHS}\end{aligned}$$

□

To prove:

$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$$

Proof:
LHS:

$$\begin{aligned}\overline{z_1 z_2} &= \overline{(a + b\mathrm{i})(c + d\mathrm{i})}\\ &= \overline{ac + ad\mathrm{i} + bc\mathrm{i} - bd}\\ &= \overline{ac - bd + (ad + bc)\mathrm{i}}\\ &= ac - bd - (ad + bc)\mathrm{i}\\ &= ac - bd - ad\mathrm{i} - bc\mathrm{i}\\ &= a(c - d\mathrm{i}) - b\mathrm{i}(c - d\mathrm{i})\\ &= (a - b\mathrm{i})(c - d\mathrm{i})\\ &= \overline{(a + b\mathrm{i})}\,\overline{(c + d\mathrm{i})}\\ &= \overline{z_1}\,\overline{z_2}\\ &= \text{RHS}\end{aligned}$$

□

To prove:

$$|z_1 z_2| = |z_1||z_2|$$

Proof:

LHS:

$$\begin{aligned}
|z_1 z_2| &= |(a+b\mathrm{i})(c+d\mathrm{i})| \\
&= |ac + ad\mathrm{i} + bc\mathrm{i} - bd| \\
&= |ac - bd + (ad+bc)\mathrm{i}| \\
&= \sqrt{(ac-bd)^2 + (ad+bc)^2} \\
&= \sqrt{\begin{matrix} a^2c^2 - 2abcd + b^2d^2 \\ +a^2d^2 + 2abcd + b^2c^2 \end{matrix}} \\
&= \sqrt{a^2(c^2+d^2) + b^2(d^2+c^2)} \\
&= \sqrt{(a^2+b^2)(c^2+d^2)} \\
&= \sqrt{(a^2+b^2)}\sqrt{(c^2+d^2)} \\
&= |z_1||z_2| \\
&= \text{RHS}
\end{aligned}$$

□

To prove:

$$|z_1 \div z_2| = |z_1| \div |z_2|$$

Proof:
LHS:

$$\begin{aligned}
|z_1 \div z_2| &= \left| \frac{a+b\mathrm{i}}{c+d\mathrm{i}} \frac{c-d\mathrm{i}}{c-d\mathrm{i}} \right| \\
&= \left| \frac{ac+bd+(-ad+bc)\mathrm{i}}{c^2+d^2} \right| \\
&= \frac{|ac+bd+(-ad+bc)\mathrm{i}|}{c^2+d^2} \\
&= \frac{\sqrt{(ac+bd)^2+(-ad+bc)^2}}{c^2+d^2} \\
&= \frac{\sqrt{\begin{matrix} a^2c^2 + 2abcd + b^2d^2 \\ +a^2d^2 - 2abcd + b^2c^2 \end{matrix}}}{c^2+d^2} \\
&= \frac{\sqrt{a^2(c^2+d^2)+b^2(d^2+c^2)}}{c^2+d^2} \\
&= \frac{\sqrt{(a^2+b^2)(c^2+d^2)}}{c^2+d^2} \\
&= \frac{\sqrt{(a^2+b^2)}\sqrt{(c^2+d^2)}}{c^2+d^2} \\
&= \frac{\sqrt{(a^2+b^2)}}{\sqrt{(c^2+d^2)}} \\
&= |z_1| \div |z_2| \\
&= \text{RHS}
\end{aligned}$$

□

9. To prove:

$$\sin \mathrm{A} \sin 2\mathrm{A} = 2\cos \mathrm{A} - 2\cos^3 \mathrm{A}$$

Proof:

LHS:

$$\begin{aligned}
\sin \mathrm{A} \sin 2\mathrm{A} &= \sin \mathrm{A}(2 \sin \mathrm{A} \cos \mathrm{A}) \\
&= 2\sin^2 \mathrm{A} \cos \mathrm{A} \\
&= 2(1-\cos^2 \mathrm{A}) \cos \mathrm{A} \\
&= 2\cos \mathrm{A} - 2\cos^3 \mathrm{A} \\
&= \text{RHS}
\end{aligned}$$

□

10. To prove:

$$\sin 3x = 3\sin x - 4\sin^3 x$$

Proof:
LHS:

$$\begin{aligned}
\sin 3x &= \sin(2x+x) \\
&= \sin 2x \cos x + \cos 2x \sin x \\
&= (2\sin x \cos)x \cos x + (1-2\sin^2 x)\sin x \\
&= 2\sin x \cos^2 x + \sin x - 2\sin^3 x \\
&= 2\sin x(1-\sin^2 x) + \sin x - 2\sin^3 x \\
&= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x \\
&= 3\sin x - 4\sin^3 x \\
&= \text{RHS}
\end{aligned}$$

□

11. (a)
$$\begin{aligned}
\overrightarrow{\mathrm{ED}} &= \overrightarrow{\mathrm{EA}} + \frac{1}{2}\overrightarrow{\mathrm{AC}} \\
&= -h\mathbf{c} + \frac{1}{2}(-\mathbf{a}+\mathbf{c}) \\
&= \frac{1}{2}\mathbf{c} - h\mathbf{c} - \frac{1}{2}\mathbf{a}
\end{aligned}$$

(b)
$$\begin{aligned}
\overrightarrow{\mathrm{DF}} &= \overrightarrow{\mathrm{CF}} + \frac{1}{2}\overrightarrow{\mathrm{AC}} \\
&= -k\mathbf{c} + \frac{1}{2}(-\mathbf{a}+\mathbf{c}) \\
&= \frac{1}{2}\mathbf{c} - k\mathbf{c} - \frac{1}{2}\mathbf{a}
\end{aligned}$$

(c) i.
$$\begin{aligned}
\overrightarrow{\mathrm{ED}} &= m\overrightarrow{\mathrm{DF}} \\
\frac{1}{2}\mathbf{c} - h\mathbf{c} - \frac{1}{2}\mathbf{a} &= m\left(\frac{1}{2}\mathbf{c} - k\mathbf{c} - \frac{1}{2}\mathbf{a}\right) \\
\left(\frac{m}{2} - \frac{1}{2}\right)\mathbf{a} &= \left(\frac{m}{2} - mk - \frac{1}{2} + h\right)\mathbf{c} \\
\frac{m}{2} - \frac{1}{2} &= 0 \\
m &= 1
\end{aligned}$$

□

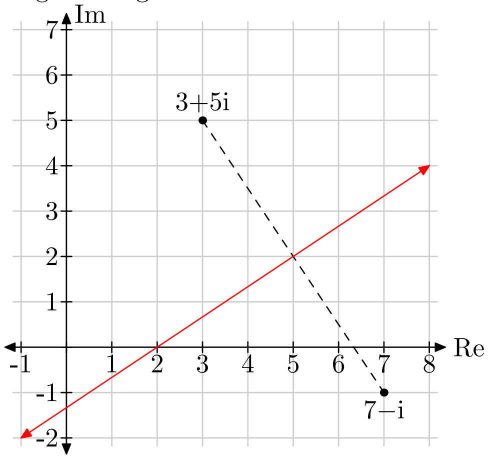
ii. Taking the right hand side of the third line above and substuting for m:

$$\begin{aligned}
\frac{m}{2} - mk - \frac{1}{2} + h &= 0 \\
\frac{1}{2} - k - \frac{1}{2} + h &= 0 \\
-k + h &= 0 \\
h &= k
\end{aligned}$$

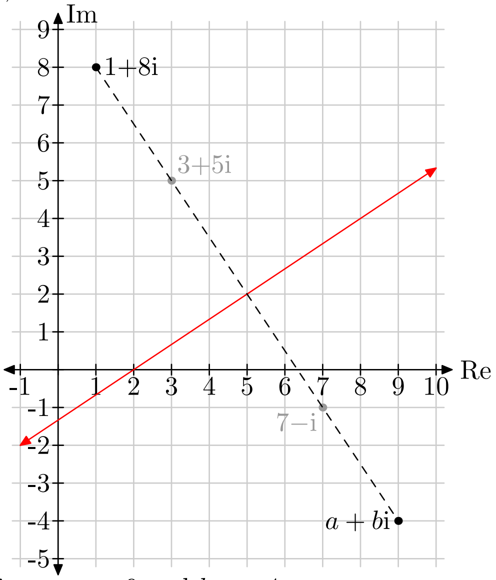
□

12.

13. From the first set of points given we can obtain the Argand diagram:

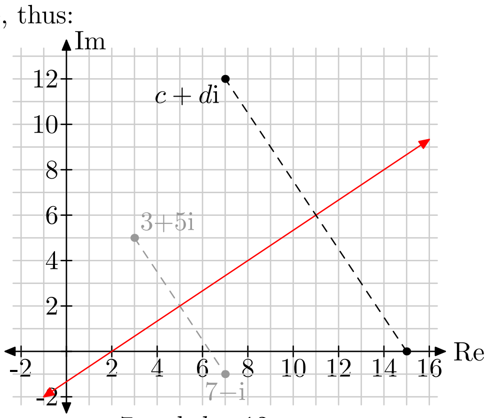


From the second set we see that the line is the set of points equidistant from $(1+8\text{i})$ and $(a+b\text{i})$, hence $(a+b\text{i})$ is the reflection of $(1+8\text{i})$ in the line, thus:



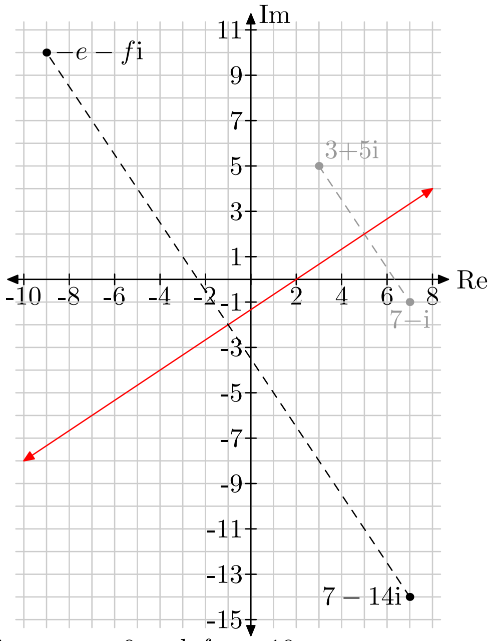
giving us $a = 9$ and $b = -4$.

From the third set we see that the line is the set of points equidistant from $(15+0\text{i})$ and $(c+d\text{i})$, hence $(c+d\text{i})$ is the reflection of $(-2+0\text{i})$ in the line, thus:



giving us $c = 7$ and $d = 12$.

From the fourth set we see that the line is the set of points equidistant from $(7-14\text{i})$ and $(-e-f\text{i})$, hence $(-e-f\text{i})$ is the reflection of $(7-14\text{i})$ in the line, thus:



giving us $e = 9$ and $f = -10$.

14. The dog will cause the light to switch on if the line along which it is walking intersects a sphere of radius 6m centred at the light. The equation of such a sphere is

$$\left|\mathbf{r} - \begin{pmatrix} -1 \\ 8 \\ 5 \end{pmatrix}\right| = 6$$

and the dog will trigger the light if there exists a real solution to this when we substitute the expression for $\mathbf{r}$ from the line into this equation.

$$\left|\begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} + \lambda -111 - \begin{pmatrix} -1 \\ 8 \\ 5 \end{pmatrix}\right| = 6$$

$$\left|\begin{pmatrix} -1 \\ -8 \\ -5 \end{pmatrix} + \lambda -111\right| = 6$$

$$(-1-\lambda)^2 + (-8+\lambda)^2 + (-5+\lambda)^2 = 6^2$$

$$1 + 2\lambda + \lambda^2 + 64 - 16\lambda + \lambda^2 + 25 - 10\lambda + \lambda^2 = 36$$

$$3\lambda^2 - 24\lambda + 54 = 0$$

$$\lambda^2 - 8\lambda + 18 = 0$$

We know this will have real solutions only if the discriminant (the bit in the square root in the quadratic formula) is not negative, i.e.

$$(-8)^2 - 4 \times 1 \times 18 \geq 0$$

but in fact

$$(-8)^2 - 4 \times 1 \times 18 = -8$$

so the quadratic has no solution and the dog will not trigger the light.

There are at least two other ways you might have approached this problem.

- using scalar product ideas and vectors to find the minimum distance
- by finding an expression for the distance between the dog and the light as a function of lambda and determining its mimimum using calculus or other methods.

Chapter 6

Exercise 6A

1. $$\begin{aligned} y + x\frac{\mathrm{d}y}{\mathrm{d}x} + 8 &= -2\frac{\mathrm{d}y}{\mathrm{d}x} \\ x\frac{\mathrm{d}y}{\mathrm{d}x} + 2\frac{\mathrm{d}y}{\mathrm{d}x} &= -y - 8 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{y+8}{x+2} \end{aligned}$$

2. $$\begin{aligned} y + x\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x} - 4 &= 6x \\ \frac{\mathrm{d}y}{\mathrm{d}x}(x+1) &= 6x - y + 4 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{6x - y + 4}{x+1} \end{aligned}$$

3. $$\begin{aligned} 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} - 2 &= 6xy + 3x^2\frac{\mathrm{d}y}{\mathrm{d}x} \\ 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} - 3x^2\frac{\mathrm{d}y}{\mathrm{d}x} &= 6xy + 2 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{6xy+2}{3y^2 - 3x^2} \end{aligned}$$

4. $$\begin{aligned} 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 6x^2y + 2x^3\frac{\mathrm{d}y}{\mathrm{d}x} + 5 \\ 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 2x^3\frac{\mathrm{d}y}{\mathrm{d}x} &= 6x^2y + 5 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{6x^2y+5}{2y - 2x^3} \end{aligned}$$

5. $$\begin{aligned} 10y\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x + 2y + 2x\frac{\mathrm{d}y}{\mathrm{d}x} - 3 \\ 10y\frac{\mathrm{d}y}{\mathrm{d}x} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x + 2y - 3 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x+2y-3}{10y - 2x} \end{aligned}$$

6. $$\begin{aligned} 1 + 6y\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x + 2y + 2x\frac{\mathrm{d}y}{\mathrm{d}x} \\ 6y\frac{\mathrm{d}y}{\mathrm{d}x} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x + 2y - 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x+2y-1}{6y-2x} \end{aligned}$$

7. $$\begin{aligned} 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 9 \\ 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 9 - 2x \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{9-2x}{2y} \end{aligned}$$

8. $$\begin{aligned} 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 9\frac{\mathrm{d}y}{\mathrm{d}x} \\ 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 9\frac{\mathrm{d}y}{\mathrm{d}x} &= -2x \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-2x}{2y-9} \\ &= \frac{2x}{9-2y} \end{aligned}$$

9. $$\begin{aligned} 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 9y + 9x\frac{\mathrm{d}y}{\mathrm{d}x} \\ 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 9x\frac{\mathrm{d}y}{\mathrm{d}x} &= 9y - 2x \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{9y-2x}{2y-9x} \end{aligned}$$

10. $$\begin{aligned} 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 9y + 9x\frac{\mathrm{d}y}{\mathrm{d}x} + 1 + \frac{\mathrm{d}y}{\mathrm{d}x} \\ 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 9x\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{\mathrm{d}y}{\mathrm{d}x} &= 9y - 2x + 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{9y-2x+1}{2y-9x-1} \end{aligned}$$

11. $$\begin{aligned} \cos x - (\sin y)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (\sin y)\frac{\mathrm{d}y}{\mathrm{d}x} &= \cos x \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\cos x}{\sin x} \\ &= \frac{1}{\tan x} \\ &= \cot x \end{aligned}$$

12. $$\begin{aligned} 2x\cos y + x^2(-\sin y)\frac{\mathrm{d}y}{\mathrm{d}x} &= 10y + 10x\frac{\mathrm{d}y}{\mathrm{d}x} \\ -x^2\sin y\frac{\mathrm{d}y}{\mathrm{d}x} - 10x\frac{\mathrm{d}y}{\mathrm{d}x} &= 10y - 2x\cos y \\ \frac{\mathrm{d}y}{\mathrm{d}x}(x^2\sin y + 10x) &= 2x\cos y - 10y \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x\cos y - 10y}{x^2\sin y + 10x} \end{aligned}$$

13. $$\begin{aligned} 6 + y + x\frac{\mathrm{d}y}{\mathrm{d}x} + 2\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{6+y}{x+2} \\ &= -\frac{6+2}{-3+2} \\ &= 8 \end{aligned}$$

14. $$\begin{aligned} 6\frac{\mathrm{d}y}{\mathrm{d}x} + y + x\frac{\mathrm{d}y}{\mathrm{d}x} &= 3 \\ 6\frac{\mathrm{d}y}{\mathrm{d}x} + x\frac{\mathrm{d}y}{\mathrm{d}x} &= 3 - y \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3-y}{6+x} \\ &= \frac{3-2}{6+2} \\ &= \frac{1}{8} \end{aligned}$$

15.
$$\begin{aligned}3x^2 &= y + x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x}\\ 3x^2 - y &= \frac{\mathrm{d}y}{\mathrm{d}x}(x+2y)\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3x^2-y}{x+2y}\\ &= \frac{3(1)^2-(-3)}{1+2(-3)}\\ &= \frac{6}{-5}\\ &= -1.2\end{aligned}$$

16.
$$\begin{aligned}2y\frac{\mathrm{d}y}{\mathrm{d}x} + 3y + 3x\frac{\mathrm{d}y}{\mathrm{d}x} &= 4\\ 2y\frac{\mathrm{d}y}{\mathrm{d}x} + 3x\frac{\mathrm{d}y}{\mathrm{d}x} &= 4-3y\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4-3y}{2y+3x}\\ &= \frac{4-3(-4)}{2(-4)+3(1)}\\ &= \frac{16}{-5}\\ &= -3.2\end{aligned}$$

17.
$$\begin{aligned}2x + \frac{x\frac{\mathrm{d}y}{\mathrm{d}x} - y}{x^2} &= 2\frac{\mathrm{d}y}{\mathrm{d}x}\\ 2(1) + \frac{1\frac{\mathrm{d}y}{\mathrm{d}x} - 1}{1^2} &= 2\frac{\mathrm{d}y}{\mathrm{d}x}\\ 2 + \frac{\mathrm{d}y}{\mathrm{d}x} - 1 &= 2\frac{\mathrm{d}y}{\mathrm{d}x}\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= 1\\ y - y_1 &= m(x-x_1)\\ y - 1 &= 1(x-1)\\ y &= x\end{aligned}$$

18.
$$\begin{aligned}10x + \frac{1}{2\sqrt{xy}}(y + x\frac{\mathrm{d}y}{\mathrm{d}x}) &= 2y\frac{\mathrm{d}y}{\mathrm{d}x}\\ 10(4) + \frac{1}{2\sqrt{(4)(9)}}(9 + 4\frac{\mathrm{d}y}{\mathrm{d}x}) &= 2(9)\frac{\mathrm{d}y}{\mathrm{d}x}\\ 40 + \frac{1}{12}(9 + 4\frac{\mathrm{d}y}{\mathrm{d}x}) &= 18\frac{\mathrm{d}y}{\mathrm{d}x}\\ 480 + 9 + 4\frac{\mathrm{d}y}{\mathrm{d}x} &= 216\frac{\mathrm{d}y}{\mathrm{d}x}\\ 212\frac{\mathrm{d}y}{\mathrm{d}x} &= 489\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{489}{212}\end{aligned}$$

19.
$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 2xy + x^2\frac{\mathrm{d}y}{\mathrm{d}x}\\ &= 2xy + x^2(x^2y)\\ &= 2xy + x^4y\end{aligned}$$

20. Solve for $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$:

$$\begin{aligned}2x + 8y\frac{\mathrm{d}y}{\mathrm{d}x} - 2 + 6\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ 2x - 2 &= 0\\ x &= 1\\ (1)^2 + 4y^2 - 2(1) + 6y &= 17\\ 4y^2 + 6y + 1 - 2 - 17 &= 0\\ 4y^2 + 6y - 18 &= 0\\ 2y^2 + 3y - 9 &= 0\\ (2y-3)(y+3) &= 0\\ y &= 1.5\\ \text{or } y &= -3\end{aligned}$$

The points are $(1, 1.5)$ and $(1, -3)$.

21. Where the tangent is vertical, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined. We could find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ and then identify the points where this is undefined, but it may be simpler to instead find an expression for $\frac{\mathrm{d}x}{\mathrm{d}y}$. Where the tangent is vertical $\frac{\mathrm{d}x}{\mathrm{d}y} = 0$.

$$\begin{aligned}2x\frac{\mathrm{d}x}{\mathrm{d}y} + 2y - 4\frac{\mathrm{d}x}{\mathrm{d}y} + 6 &= 0\\ 2y + 6 &= 0\\ y &= -3\\ x^2 + (-3)^2 - 4x + 6(-3) + 12 &= 0\\ x^2 - 4x + 9 - 18 + 12 &= 0\\ x^2 - 4x + 3 &= 0\\ (x-3)(x-1) &= 0\\ x &= 3\\ \text{or } x &= 1\end{aligned}$$

The points are $(3, -3)$ and $(1, -3)$.

22.
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} - 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x+1\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x+1}{1-3y^2}\end{aligned}$$

at $(1, 0)$:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2(1)+1}{1-3(0)^2}\\ &= 3\\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{2(1-3y^2) - (2x+1)(-6y\frac{\mathrm{d}y}{\mathrm{d}x})}{(1-3y^2)^2}\\ &= \frac{2(1-3y^2) + 6y(2x+1)\frac{2x+1}{1-3y^2}}{(1-3y^2)^2}\\ &= \frac{2(1-3y^2)^2 + 6y(2x+1)^2}{(1-3y^2)^3}\end{aligned}$$

at $(1, 0)$:

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{2(1-3(0)^2)^2 + 6(0)(2(1)+1)^2}{(1-3(0)^2)^3}\\ &= \frac{2(1)^2}{(1)^3}\\ &= 2\end{aligned}$$

23.
$$2x = 2(\cos y)\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{x}{\cos y}$$

at $\left(1, \frac{\pi}{6}\right)$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\cos\frac{\pi}{6}}\\ &= \frac{1}{\frac{\sqrt{3}}{2}}\\ &= \frac{2}{\sqrt{3}}\\ y - \frac{\pi}{6} &= \frac{2}{\sqrt{3}}(x-1)\\ 6y - \pi &= \frac{12}{\sqrt{3}}(x-1)\\ &= 4\sqrt{3}(x-1)\\ 6y &= 4\sqrt{3}x - 4\sqrt{3} + \pi\end{aligned}$$

24. $2y\frac{dy}{dx} - \sin x = 3\frac{dy}{dx}$

$$\begin{aligned}2y\frac{dy}{dx} - 3\frac{dy}{dx} &= \sin x\\ \frac{dy}{dx} &= \frac{\sin x}{2y-3}\\ \frac{d^2y}{dx^2} &= \frac{(\cos x)(2y-3) - (\sin x)(2\frac{dy}{dx})}{(2y-3)^2}\\ &= \frac{(\cos x)(2y-3) - (2\sin x)\frac{\sin x}{2y-3}}{(2y-3)^2}\\ &= \frac{(2y-3)^2\cos x - 2\sin^2 x}{(2y-3)^3}\\ &= \frac{\cos x}{2y-3} - \frac{2\sin^2 x}{(2y-3)^3}\end{aligned}$$

25. $(2\cos y)\frac{dy}{dx} - 2x = 2$

$$\begin{aligned}\frac{dy}{dx} &= \frac{2+2x}{2\cos y}\\ &= \frac{1+x}{\cos y}\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{\cos y - (1+x)(-\sin y)\frac{dy}{dx}}{\cos^2 y}\\ &= \frac{\cos y - (1+x)(-\sin y)\frac{1+x}{\cos y}}{\cos^2 y}\\ &= \frac{\cos^2 y - (1+x)(-\sin y)(1+x)}{\cos^3 y}\\ &= \frac{\cos^2 y + (1+x)^2\sin y}{\cos^3 y}\end{aligned}$$

At $\left(-2, \frac{\pi}{6}\right)$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1+(-2)}{\cos\frac{\pi}{6}}\\ &= \frac{-1}{\frac{\sqrt{3}}{2}}\\ &= -\frac{2}{\sqrt{3}}\\ &= -\frac{2\sqrt{3}}{3}\\ \frac{d^2y}{dx^2} &= \frac{\cos^2\frac{\pi}{6} + (1+(-2))^2\sin\frac{\pi}{6}}{\cos^3\frac{\pi}{6}}\\ &= \frac{\frac{3}{4} + (1)\frac{1}{2}}{\frac{3\sqrt{3}}{8}}\\ &= \frac{\frac{5}{4}}{\frac{3\sqrt{3}}{8}}\\ &= \frac{10}{3\sqrt{3}}\\ &= \frac{10\sqrt{3}}{9}\end{aligned}$$

26. $6x + 2y\frac{dy}{dx} = 0$

$$\begin{aligned}6x + 2y(-1) &= 0\\ 6x - 2y &= 0\\ 2y &= 6x\\ y &= 3x\end{aligned}$$

We need to find the points that satisfy this equation and also lie on the ellipse. You could solve this graphically or with technology or algebraically as follows:

$$\begin{aligned}3x^2 + (3x)^2 &= 9\\ 3x^2 + 9x^2 &= 9\\ 12x^2 &= 9\\ x^2 &= \frac{3}{4}\\ x &= \pm\frac{\sqrt{3}}{2}\\ y &= 3x\\ &= \pm\frac{3\sqrt{3}}{2}\end{aligned}$$

The points having gradient -1 are $\left(\frac{\sqrt{3}}{2}, \frac{3\sqrt{3}}{2}\right)$ and $\left(-\frac{\sqrt{3}}{2}, -\frac{3\sqrt{3}}{2}\right)$.

Exercise 6B

1. (a) $\frac{dx}{dt} = 6\cos 2t$

   (b) $\frac{dy}{dt} = -10\sin 5t$

   (c)
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\ &= -10\sin 5t\frac{1}{6\cos 2t}\\ &= -\frac{5\sin 5t}{3\cos 2t}\end{aligned}$$

2. (a)
$$\begin{aligned}\frac{dx}{dt} &= 2\sin t\cos t\\ &= \sin 2t\end{aligned}$$

   (b) $\frac{dy}{dt} = -3\sin 3t$

   (c)
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\ &= -3\sin 3t\frac{1}{\sin 2t}\\ &= -\frac{3\sin 3t}{\sin 2t}\end{aligned}$$

3.
$$\begin{aligned}\frac{dx}{dt} &= 3\\ \frac{dy}{dt} &= 2t\\ \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\ &= \frac{2t}{3}\end{aligned}$$

4.
$$\begin{aligned}\frac{dx}{dt} &= 2t\\ \frac{dy}{dt} &= 3\\ \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\ &= \frac{3}{2t}\end{aligned}$$

5.
$$\begin{aligned}\frac{dx}{dt} &= 15t^2\\ \frac{dy}{dt} &= 2t+2\\ \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\ &= \frac{2t+2}{15t^2}\end{aligned}$$

6.
$$\begin{aligned}\frac{dx}{dt} &= 6t+6\\ \frac{dy}{dt} &= -\frac{1}{(t+1)^2}\\ \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\ &= -\frac{1}{(t+1)^2}\frac{1}{6(t+1)}\\ &= -\frac{1}{6(t+1)^3}\end{aligned}$$

7.
$$\begin{aligned}\frac{dx}{dt} &= 2t\\ \frac{dy}{dt} &= 2(t-1)\\ \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\ &= \frac{2(t-1)}{2t}\\ &= \frac{t-1}{t}\\ &= 1-\frac{1}{t}\end{aligned}$$

8.
$$\begin{aligned}\frac{dx}{dt} &= \frac{(t-1)-t}{(t-1)^2}\\ &= -\frac{1}{(t-1)^2}\\ \frac{dy}{dt} &= -2\frac{1}{(t+1)^2}\\ \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\ &= -2\frac{1}{(t+1)^2}(-(t-1)^2)\\ &= \frac{2(t-1)^2}{(t+1)^2}\end{aligned}$$

9.
$$\begin{aligned}\frac{dx}{dt} &= 2t\\ \frac{dy}{dt} &= 3t^2\\ \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\ &= \frac{3t^2}{2t}\\ &= \frac{3t}{2}\end{aligned}$$

   at $t=-1$:
$$\frac{dy}{dx} = -\frac{3}{2}$$

10.
$$\begin{aligned}\frac{dx}{dt} &= -\frac{1}{(t+1)^2}\\ \frac{dy}{dt} &= 2t\\ \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\ &= -2t(t+1)^2 \text{at } t=2:\\ \frac{dy}{dx} &= -2\times 2(2+1)^2\\ &= -36\end{aligned}$$

11.
$$\begin{aligned}
\frac{dx}{dt} &= 4t+3\\
\frac{dy}{dt} &= 3t^2-12\\
\frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\
&= \frac{3t^2-12}{4t+3}\\
\frac{dy}{dx} &= 0\\
\frac{3t^2-12}{4t+3} &= 0\\
3t^2-12 &= 0\\
t^2-4 &= 0\\
t^2 &= 4\\
t &= \pm 2
\end{aligned}$$

for $t=+2$:
$$\begin{aligned}
x &= 2(2)^2+3(2)\\
&= 14\\
y &= (2)^3-12(2)\\
&= -16
\end{aligned}$$

for $t=-2$:
$$\begin{aligned}
x &= 2(-2)^2+3(-2)\\
&= 2\\
y &= (-2)^3-12(-2)\\
&= 16
\end{aligned}$$

The points on the curve where $\frac{dy}{dx}=0$ are $(14,-16)$ and $(2,16)$.

12.
$$\begin{aligned}
\frac{dx}{dt} &= -3\sin t\\
\frac{dy}{dt} &= 5\cos t\\
\frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\
&= \frac{5\cos t}{-3\sin t} \text{at } t=\frac{\pi}{6}:\\
\frac{dy}{dx} &= \frac{5\cos\frac{\pi}{6}}{-3\sin\frac{\pi}{6}}\\
&= \frac{\frac{5\sqrt{3}}{2}}{\frac{-3}{2}}\\
&= -\frac{5\sqrt{3}}{3}\\
y-5\sin\frac{\pi}{6} &= -\frac{5\sqrt{3}}{3}\left(x-3\cos\frac{\pi}{6}\right)\\
y-\frac{5}{2} &= -\frac{5\sqrt{3}}{3}\left(x-\frac{3\sqrt{3}}{2}\right)\\
y &= -\frac{5\sqrt{3}}{3}\left(x-\frac{3\sqrt{3}}{2}\right)+\frac{5}{2}\\
&= -\frac{5\sqrt{3}x}{3}+\frac{15}{2}+\frac{5}{2}\\
&= -\frac{5\sqrt{3}x}{3}+10
\end{aligned}$$

13. (a)
$$\begin{aligned}
\frac{dx}{dt} &= 4\cos t\\
\frac{dy}{dt} &= 4\cos 2t\\
\frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\
&= \frac{4\cos 2t}{4\sin t}\\
&= \frac{\cos 2t}{\cos t}
\end{aligned}$$

(b) At $t=\frac{\pi}{6}$:
$$\begin{aligned}
x &= 4\sin\frac{\pi}{6}\\
&= 2\\
y &= 2\sin\frac{\pi}{3}\\
&= \sqrt{3}\\
\frac{dy}{dx} &= \frac{\cos\frac{\pi}{3}}{\cos\frac{\pi}{6}}\\
&= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\\
&= \frac{1}{\sqrt{3}}\\
&= \frac{\sqrt{3}}{3}
\end{aligned}$$

(c)
$$\begin{aligned}
\frac{\cos 2t}{\cos t} &= 0\\
\cos 2t &= 0\\
2t &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}\\
t &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}
\end{aligned}$$

14. (a)
$$\begin{aligned}
\frac{dy}{dt} &= 1-\frac{2}{t^2}\\
\frac{dx}{dt} &= 2+\frac{1}{t^2}\\
\frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\
&= \frac{1-\frac{2}{t^2}}{2+\frac{1}{t^2}}\\
&= \frac{t^2-2}{2t^2+1}
\end{aligned}$$

(b)
$$\begin{aligned}
\frac{d^2y}{dx^2} &= \dot{d}t\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}\\
&= \left(\frac{2t(2t^2+1)-(t^2-2)4t}{(2t^2+1)^2}\right)\frac{1}{2+\frac{1}{t^2}}\\
&= \left(\frac{2t(2t^2+1)-2t(2t^2-4)}{(2t^2+1)^2}\right)\frac{t^2}{2t^2+1}\\
&= \left(\frac{2t(2t^2+1-2t^2+4)}{(2t^2+1)^2}\right)\frac{t^2}{2t^2+1}\\
&= \frac{2t^3(5)}{(2t^2+1)^3}\\
&= \frac{10t^3}{(2t^2+1)^3}
\end{aligned}$$

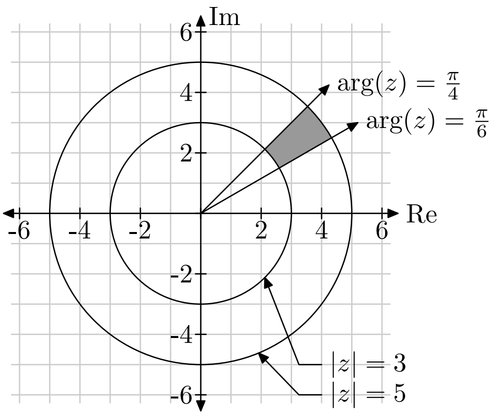
Miscellaneous Exercise 6

1. $$6\operatorname{cis}\frac{3\pi}{4} = 6\cos\frac{3\pi}{4} + 6\mathrm{i}\sin\frac{3\pi}{4}$$
$$= -3\sqrt{2} + 3\sqrt{2}\mathrm{i}$$

2. $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x(1-\sin x) - (-\cos x)(1+\sin x)}{(1-\sin x)^2}$$
$$= \frac{\cos x(1-\sin x+1+\sin x}{(1-\sin x)^2}$$
$$= \frac{2\cos x}{(1-\sin x)^2}$$

3. $a = 3$ (read from the radius of the inner circle)
$b = 5$ (read from the radius of the outer circle)
$c = 3$ (when $\theta = \pi$, $r \approx 9.5 \approx 3\pi$)
At point A:
$$3\theta = 3$$
coordinates are $(3, 1)$
At point B:
$$3\theta = 5$$
coordinates are $\left(5, \frac{5}{3}\right)$

4.



5. (a) $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2(3-2x) - (2x+1)(-2)}{(3-2x)^2}$$
$$= \frac{6-4x+4x+2}{(3-2x)^2}$$
$$= \frac{8}{(3-2x)^2}$$

(b) $$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\sin^2(2x+1)\cos(2x+1)(2)$$
$$= 6\sin^2(2x+1)\cos(2x+1)$$

(c) $$6xy + 3x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} = 5$$
$$3x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} = 5 - 6xy$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5-6xy}{3(x^2+y^2)}$$

(d) $$\frac{\mathrm{d}x}{\mathrm{d}t} = 2t+3$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 4t^3$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}$$
$$= \frac{4t^3}{2t+3}$$

6. $$\mathrm{f}'(x) = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$
$$= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$
$$= \lim_{h\to 0}\frac{(x+h)-(x)}{h(\sqrt{x+h}+\sqrt{x})}$$
$$= \lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$
$$= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}$$
$$= \frac{1}{\sqrt{x+0}+\sqrt{x}}$$
$$= \frac{1}{2\sqrt{x}}$$

7. $$\left(\begin{pmatrix}-1\\-10\\4\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\-1\end{pmatrix}\right)\cdot\begin{pmatrix}2\\4\\-1\end{pmatrix} = 5$$
$$2(-1+2\lambda) + 4(-10+3\lambda) - 1(4-\lambda) = 5$$
$$-2+4\lambda-40+12\lambda-4+\lambda = 5$$
$$17\lambda - 46 = 5$$
$$17\lambda = 51$$
$$\lambda = 3$$

$$\mathbf{r} = \begin{pmatrix}-1\\-10\\4\end{pmatrix} + 3\begin{pmatrix}2\\3\\-1\end{pmatrix}$$
$$\mathbf{r} = \begin{pmatrix}5\\-1\\1\end{pmatrix}$$

8. $$\frac{\mathrm{d}x}{\mathrm{d}t} = -5\sin t$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 5\cos t$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}$$
$$= \frac{5\cos t}{-5\sin t}$$
$$= -\frac{\cos t}{\sin t}$$
At $t = \frac{2\pi}{3}$:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{\cos\frac{2\pi}{3}}{\sin\frac{2\pi}{3}}\\
&= -\frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\\
&= \frac{1}{\sqrt{3}}\\
y - 5\sin\frac{2\pi}{3} &= \frac{1}{\sqrt{3}}(x - 5\cos\frac{2\pi}{3})\\
\sqrt{3}y - \frac{15}{2} &= x + \frac{5}{2}\\
\sqrt{3}y &= x + \frac{5}{2} + \frac{15}{2}\\
\sqrt{3}y &= x + 10
\end{aligned}$$

9. (a) $$\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} = \frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$$

(b) $$\lim_{h\to 0}\frac{\cos 3(x+h)-\cos 3(x)}{h} = \frac{\mathrm{d}}{\mathrm{d}x}\cos 3x = -3\sin 3x$$

(c) $$\lim_{h\to 0}\frac{\tan(x+h)-\tan(x)}{h} = \frac{\mathrm{d}}{\mathrm{d}x}\tan x = 1-\tan^2 x$$

(d) $$\lim_{h\to 0}\frac{(x+h)^2-x^2}{h} = \frac{\mathrm{d}}{\mathrm{d}x}x^2 = 2x$$

(e) $$\lim_{h\to 0}\frac{(x+h)^3-x^3}{h} = \frac{\mathrm{d}}{\mathrm{d}x}x^3 = 3x^2$$

(f) $$\lim_{h\to 0}\frac{\sin^2(x+h)-\sin^2(x)}{h} = \frac{\mathrm{d}}{\mathrm{d}x}\sin^2 x = 2\sin x\cos x = \sin 2x$$

10. (a)
$$\begin{aligned}
4y^3\frac{\mathrm{d}y}{\mathrm{d}x} &= 4x^3 - 4y - 4x\frac{\mathrm{d}y}{\mathrm{d}x}\\
4y^3\frac{\mathrm{d}y}{\mathrm{d}x} + 4x\frac{\mathrm{d}y}{\mathrm{d}x} &= 4x^3 - 4y\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4x^3-4y}{4y^3+4x}\\
&= \frac{x^3-y}{y^3+1}
\end{aligned}$$

(b)
$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(2)^3-(1)}{(1)^3+(2)}\\
&= \frac{7}{3}
\end{aligned}$$

11. (a) Any positive or negative scalar multiple of $(2\mathbf{i}+\mathbf{j}-3\mathbf{k})$ is a correct answer.

(b) Any vector that has a zero dot product with $(2\mathbf{i}+\mathbf{j}-3\mathbf{k})$ is a correct answer. One simple way to find an example perpendicular is to make one of the components zero, then swap the other two and change the sign of one of them. For example, if we zero the $\mathbf{k}$ component, swap the $\mathbf{i}$ and $\mathbf{j}$ components and negate the new $\mathbf{i}$ component we get $(-\mathbf{i}+2\mathbf{j}+0\mathbf{k})$ and $(2\mathbf{i}+1\mathbf{j})-3\cdot(-\mathbf{i}+2\mathbf{j}+0\mathbf{k}) = -2+2+0 = 0$.

12. Start with defining P and Q in terms of A and B and vice versa:

$$\begin{aligned}
\mathrm{P} &= \mathrm{A}+\mathrm{B}\\
\mathrm{Q} &= \mathrm{A}-\mathrm{B}\\
\mathrm{P}+\mathrm{Q} &= \mathrm{A}+\mathrm{B}+\mathrm{A}-\mathrm{B}\\
&= 2\mathrm{A}\\
\therefore\quad \mathrm{A} &= \frac{\mathrm{P}+\mathrm{Q}}{2}\\
\mathrm{P}-\mathrm{Q} &= \mathrm{A}+\mathrm{B}-\mathrm{A}+\mathrm{B}\\
&= 2\mathrm{B}\\
\therefore\quad \mathrm{B} &= \frac{\mathrm{P}-\mathrm{Q}}{2}
\end{aligned}$$

$$\begin{aligned}
\sin\mathrm{P}+\sin\mathrm{Q} &= \sin(\mathrm{A}+\mathrm{B})+\sin(\mathrm{A}-\mathrm{B})\\
&= \sin\mathrm{A}\cos\mathrm{B}-\cos\mathrm{A}\sin\mathrm{B}\\
&\quad +\sin\mathrm{A}\cos\mathrm{B}+\cos\mathrm{A}\sin\mathrm{B}\\
&= 2\sin\mathrm{A}\cos\mathrm{B}\\
&= 2\sin\frac{\mathrm{P}+\mathrm{Q}}{2}\cos\frac{\mathrm{P}-\mathrm{Q}}{2}\\
\sin\mathrm{P}-\sin\mathrm{Q} &= \sin(\mathrm{A}+\mathrm{B})-\sin(\mathrm{A}-\mathrm{B})\\
&= \sin\mathrm{A}\cos\mathrm{B}-\cos\mathrm{A}\sin\mathrm{B}\\
&\quad -\sin\mathrm{A}\cos\mathrm{B}-\cos\mathrm{A}\sin\mathrm{B}\\
&= 2\cos\mathrm{A}\sin\mathrm{B}\\
&= 2\sin\frac{\mathrm{P}+\mathrm{Q}}{2}\cos\frac{\mathrm{P}-\mathrm{Q}}{2}
\end{aligned}$$

□

13. $$\lambda\begin{pmatrix}2\\3\\1\end{pmatrix}+\mu\begin{pmatrix}1\\4\\-1\end{pmatrix}+\eta\begin{pmatrix}2\\0\\0\end{pmatrix}=\begin{pmatrix}-6\\5\\-3\end{pmatrix}$$

If we look at the $\mathbf{j}$ and $\mathbf{k}$ components, we can solve for λ and μ, then solve for η using the $\mathbf{i}$ component.

$$\begin{aligned}
3\lambda+4\mu &= 5\\
\lambda-\mu &= -3\\
4\lambda-4\mu &= -12\\
7\lambda &= -7\\
\lambda &= -1\\
-1-\mu &= -3\\
\mu &= 2\\
2(-1)+1(2)+2\eta &= -6\\
\eta &= -3\\
\begin{pmatrix}-6\\5\\-3\end{pmatrix} &= -\mathbf{a}+2\mathbf{b}-3\mathbf{c}
\end{aligned}$$

14. $$\begin{aligned}\overrightarrow{AB} &= (3\mathbf{p}-\mathbf{q})-(2\mathbf{p}+\mathbf{q})\\ &= \mathbf{p}-2\mathbf{q}\\ \overrightarrow{AC} &= (6\mathbf{p}-7\mathbf{q})-(2\mathbf{p}+\mathbf{q})\\ &= 4\mathbf{p}-8\mathbf{q}\\ &= 4\overrightarrow{AB}\end{aligned}$$

$\therefore$ A, B and C are collinear. □

Since $\overrightarrow{AB}$ is one quarter of $\overrightarrow{AC}$ it follows that

$$\begin{aligned}\overrightarrow{AB} : \overrightarrow{BC} &= 1:3\\ \text{and } \overrightarrow{AB} : \overrightarrow{AC} &= 1:4\end{aligned}$$

15. (a) $$\begin{aligned}\cos\theta &= \frac{(\mathbf{i}-2\mathbf{k})\cdot(2\mathbf{i}+3\mathbf{j}-\mathbf{k})}{|(\mathbf{i}-2\mathbf{k})||(2\mathbf{i}+3\mathbf{j}-\mathbf{k})|}\\ \cos\theta &= \frac{2+2}{\sqrt{5}\sqrt{14}}\\ \theta &= 1.07\end{aligned}$$

(b) $$\begin{aligned}-2\mathbf{i}+3\mathbf{j}+8\mathbf{k}+\lambda(\mathbf{i}-2\mathbf{k})\\ = 5\mathbf{i}+6\mathbf{j}-3\mathbf{k}+\mu(2\mathbf{i}+3\mathbf{j}-\mathbf{k})\end{aligned}$$

j components:

$$\begin{aligned}3 &= 6+3\mu\\ \mu &= -1\end{aligned}$$

i components:

$$\begin{aligned}-2+\lambda &= 5+2\mu\\ \lambda &= 7+2(-1)\\ \lambda &= 5\end{aligned}$$

Lines intersect if these values also satisfy the **k** components:

$$\begin{aligned}8-2(5) &= -3-(-1)\\ -2 &= -2\end{aligned}$$

Therefore the lines intersect.

$$\begin{aligned}\text{P} &= -2\mathbf{i}+3\mathbf{j}+8\mathbf{k}+5(\mathbf{i}-2\mathbf{k})\\ &= 3\mathbf{i}+3\mathbf{j}-2\mathbf{k}\end{aligned}$$

(c) $$\begin{aligned}\overrightarrow{AB} &= (3\mathbf{i}+\mathbf{j}+\mathbf{k})-(2\mathbf{i}-\mathbf{j}+3\mathbf{k})\\ &= \mathbf{i}+2\mathbf{j}-2\mathbf{k}\\ \mathbf{r}\cdot(\mathbf{i}+2\mathbf{j}-2\mathbf{k}) &= (3\mathbf{i}+3\mathbf{j}-2\mathbf{k})\cdot(\mathbf{i}+2\mathbf{j}-2\mathbf{k})\\ &= 3+6+4\\ \mathbf{r}\cdot(\mathbf{i}+2\mathbf{j}-2\mathbf{k}) &= 13\end{aligned}$$

16. (a) $$\begin{aligned}\frac{dx}{dt} &= 1-\frac{1}{t^2}\\ \frac{dy}{dt} &= 2t+2\\ \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\ &= \frac{2t+2}{1-\frac{1}{t^2}}\\ &= \frac{2t^2(t+1)}{t^2-1}\\ &= \frac{2t^2(t+1)}{(t+1)(t-1)}\\ &= \frac{2t^2}{t-1}\end{aligned}$$

(b) $$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{4t(t-1)\frac{dt}{dx}-(2t^2)\frac{dt}{dx}}{(t-1)^2}\\ &= \frac{4t^2-4t-2t^2}{(t-1)^2}\frac{dt}{dx}\\ &= \frac{2t^2-4t}{(t-1)^2}\frac{1}{1-\frac{1}{t^2}}\\ &= \frac{2t(t-2)}{(t-1)^2}\frac{t^2}{t^2-1}\\ &= \frac{2t^3(t-2)}{(t-1)^2(t+1)(t-1)}\\ &= \frac{2t^3(t-2)}{(t-1)^3(t+1)}\end{aligned}$$

17. (a) $$\begin{aligned}\overrightarrow{AC} &= -\mathbf{a}+\mathbf{c}\\ \overrightarrow{GE} &= \mathbf{a}+0.5\overrightarrow{AC}\\ &= \mathbf{a}+0.5(-\mathbf{a}+\mathbf{c})\\ &= 0.5(\mathbf{a}+\mathbf{c})\\ \overrightarrow{GE}\cdot\overrightarrow{AC} &= 0\\ 0.5(\mathbf{a}+\mathbf{c})\cdot 0.5(-\mathbf{a}+\mathbf{c}) &= 0\\ (\mathbf{a}+\mathbf{c})\cdot(-\mathbf{a}+\mathbf{c}) &= 0\\ -\mathbf{a}^2+\mathbf{c}^2 &= 0\\ \mathbf{a}^2 &= \mathbf{c}^2\end{aligned}$$

□

(b) $$\begin{aligned}\overrightarrow{BC} &= -\mathbf{b}+\mathbf{c}\\ \overrightarrow{GD} &= \mathbf{b}+0.5\overrightarrow{BC}\\ &= \mathbf{b}+0.5(-\mathbf{b}+\mathbf{c})\\ &= 0.5(\mathbf{b}+\mathbf{c})\\ \overrightarrow{GD}\cdot\overrightarrow{BC} &= 0\\ 0.5(\mathbf{b}+\mathbf{c})\cdot 0.5(-\mathbf{b}+\mathbf{c}) &= 0\\ (\mathbf{b}+\mathbf{c})\cdot(-\mathbf{b}+\mathbf{c}) &= 0\\ -\mathbf{b}^2+\mathbf{c}^2 &= 0\\ \mathbf{b}^2 &= \mathbf{c}^2\end{aligned}$$

□

(c)
$$\begin{aligned}
\overrightarrow{AB} &= -\mathbf{a}+\mathbf{b}\\
\overrightarrow{GF} &= \mathbf{a}+0.5\overrightarrow{AB}\\
&= \mathbf{a}+0.5(-\mathbf{a}+\mathbf{b}\\
&= 0.5(\mathbf{a}+\mathbf{b})\\
\overrightarrow{GF}\cdot\overrightarrow{AB} &= 0.5(\mathbf{a}+\mathbf{b})\cdot 0.5(-\mathbf{a}+\mathbf{b})\\
&= 0.25(\mathbf{a}+\mathbf{b})\cdot(-\mathbf{a}+\mathbf{b})\\
&= -\mathbf{a}^2+\mathbf{b}^2\\
&= -\mathbf{c}^2+\mathbf{c}^2\\
&= 0
\end{aligned}$$

$\therefore$ GF is perpendicular to AB. □

18. Let P be the point on the line closest to A.

$$\begin{aligned}
\overrightarrow{AP} &= \overrightarrow{OP}-\overrightarrow{OA}\\
&= -3\mathbf{i}+2\mathbf{j}+7\mathbf{k}+\mu\,(\mathbf{i}-\mathbf{j}+2\mathbf{k})-(-5\mathbf{i}+10\mathbf{j}-3\mathbf{k})\\
&= 2\mathbf{i}-8\mathbf{j}+10\mathbf{k}+\mu\,(\mathbf{i}-\mathbf{j}+2\mathbf{k})
\end{aligned}$$

AP is perpendicular to line L:

$$\begin{aligned}
\overrightarrow{AP}\cdot(\mathbf{i}-\mathbf{j}+2\mathbf{k}) &= 0\\
(2\mathbf{i}-8\mathbf{j}+10\mathbf{k}+\mu\,(\mathbf{i}-\mathbf{j}+2\mathbf{k}))\cdot(\mathbf{i}-\mathbf{j}+2\mathbf{k}) &= 0\\
2+8+20+\mu(1+1+4) &= 0\\
30+6\mu &= 0\\
\mu &= -5
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{AP} &= 2\mathbf{i}-8\mathbf{j}+10\mathbf{k}+(-5)\,(\mathbf{i}-\mathbf{j}+2\mathbf{k})\\
&= -3\mathbf{i}-3\mathbf{j}\\
|\overrightarrow{AP}| &= \sqrt{(-3)^2+(-3)^2}\\
&= 3\sqrt{2}
\end{aligned}$$

Chapter 7

Exercise 7A

1. I will use the "intelligent guess" method for this question, but my preference is for the "rearranging" method, so I will use that for most of the questions where one of these approaches is suitable.
   Guess $y = \frac{(2x+5)^5}{5}$
   $\frac{\mathrm{d}}{\mathrm{d}x}\frac{(2x+5)^5}{5} = (2x+5)^4(2)$
   Adjust by a factor of $\frac{1}{2}$:
   $$\frac{(2x+5)^5}{10} + c$$

2. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (3x+1)^3\\ &= \frac{1}{3}(3(3x+1)^3)\\ y &= \frac{1}{3}\frac{(3x+1)^4}{4} + c\\ &= \frac{(3x+1)^4}{12} + c\end{aligned}$$

3. This can not be rearranged to the form $\mathrm{f}'(x)(\mathrm{f}(x))^n$ so we expand:
   $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= x(3x+4)\\ &= 3x^2 + 4x\\ y &= \frac{3x^3}{3} + \frac{4x^2}{2} + c\\ &= x^3 + 2x^2 + c\end{aligned}$$

4. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 50(1+5x)^4\\ &= 10(5(1+5x)^4)\\ y &= 10\left(\frac{(1+5x)^5}{5}\right) + c\\ &= 2(1+5x)^5 + c\end{aligned}$$

5. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 24x(2-x^2)^3\\ &= -12(-2x(2-x^2)^3)\\ y &= -12\left(\frac{(2-x^2)^4}{4}\right) + c\\ &= -3(2-x^2)^4 + c\end{aligned}$$

6. This can not be rearranged to the form $\mathrm{f}'(x)(\mathrm{f}(x))^n$ so we expand:
   $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= x(1+4x)^2\\ &= x(1+8x+16x^2)\\ &= x + 8x^2 + 16x^3\\ y &= \frac{x^2}{2} + \frac{8x^3}{3} + \frac{16x^4}{4} + c\\ &= \frac{x^2}{2} + \frac{8x^3}{3} + 4x^4 + c\end{aligned}$$

7. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 30x(x^2-3)^2\\ &= 15(2x(x^2-3)^2)\\ y &= 15\left(\frac{(x^2-3)^3}{3}\right) + c\\ &= 5(x^2-3)^3 + c\end{aligned}$$

8. This can not be rearranged to the form $\mathrm{f}'(x)(\mathrm{f}(x))^n$ so we expand:
   $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= x(2x-3)^2\\ &= x(4x^2-12x+9)\\ &= 4x^3 - 12x^2 + 9x\\ y &= \frac{4x^4}{4} - \frac{12x^3}{3} + \frac{9x^2}{2} + c\\ &= x^4 - 4x^3 + \frac{9x^2}{2} + c\end{aligned}$$

9. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 12(2x+1)^3\\ &= 6(2(2x+1)^3)\\ y &= 6\left(\frac{(2x+1)^4}{4}\right) + c\\ &= \frac{3(2x+1)^4}{2} + c\end{aligned}$$

10. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 2(3x+1)^4\\ &= \frac{2}{3}(3(3x+1)^4)\\ y &= \frac{2}{3}\left(\frac{(3x+1)^5}{5}\right) + c\\ &= \frac{2(3x+1)^5}{15} + c\end{aligned}$$

11. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (2x-3)^4\\ &= \frac{1}{2}(2(2x-3)^4)\\ y &= \frac{1}{2}\left(\frac{(2x-3)^5}{5}\right) + c\\ &= \frac{(2x-3)^5}{10} + c\end{aligned}$$

12. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 5x^2(3-x^3)^4\\ &= \frac{5}{-3}(-3x^2(3-x^3)^4)\\ y &= -\frac{5}{3}\left(\frac{(3-x^3)^5}{5}\right) + c\\ &= -\frac{(3-x^3)^5}{3} + c\\ &= \frac{(x^3-3)^5}{3} + c\end{aligned}$$
    (Note that the last step is valid because $(-1)^5 =$

−1. Such a simplification would not be possible if we were raising to an even power.)

13. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (1+x)^4\\ y &= \frac{(1+x)^5}{5} + c\end{aligned}$$

14. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 4x(2+x^2)^4\\ &= 2(2x(2+x^2)^4)\\ y &= 2\frac{(2+x^2)^5}{5} + c\\ &= \frac{2(2+x^2)^5}{5} + c\end{aligned}$$

15. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (1+2x)^4\\ &= \frac{1}{2}(2(1+2x)^4)\\ y &= \frac{1}{2}\left(\frac{(1+2x)^5}{5}\right) + c\\ &= \frac{(1+2x)^5}{10} + c\end{aligned}$$

16. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 4x(1+x^2)\\ &= 2(2x(1+x^2))\\ y &= 2\left(\frac{(1+x^2)^2}{2}\right) + c\\ &= (1+x^2)^2 + c\end{aligned}$$

This question is probably just as easy to do by first expanding:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 4x(1+x^2)\\ &= 4x + 4x^3\\ y &= 4\frac{x^2}{2} + 4\frac{x^4}{3} + c\\ &= 2x^2 + x^4 + c\end{aligned}$$

Although these two solutions may not look quite the same, you should satisfy yourself that they are, in fact, both correct. (Remember, c is an *arbitrary* constant.)

17. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 60(2x-3)^5\\ &= 30(2(2x-3)^5\\ y &= 30\left(\frac{(2x-3)^6}{6}\right) + c\\ &= 5(2x-3)^6 + c\end{aligned}$$

18. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 60(3-2x)^5\\ &= -30(-2(3-2x)^5\\ y &= -30\left(\frac{(3-2x)^6}{6}\right) + c\\ &= -5(3-2x)^6 + c\end{aligned}$$

19. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{(x+2)^4}\\ &= (x+2)^{-4}\\ y &= \frac{(x+2)^{-3}}{-3} + c\\ &= -\frac{1}{3(x+2)^3} + c\end{aligned}$$

20. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{(2x+1)^4}\\ &= (2x+1)^{-4}\\ &= \frac{1}{2}(2(2x+1)^{-4}\\ y &= \frac{1}{2}\frac{(2x+1)^{-3}}{-3} + c\\ &= -\frac{1}{6(2x+1)^3} + c\end{aligned}$$

21. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{25x}{(x^2+1)^5}\\ &= -25x(x^2+1)^{-5}\\ &= \frac{-25}{2}(2x(x^2+1)^{-5})\\ y &= \frac{-25}{2}\frac{(x^2+1)^{-4}}{-4} + c\\ &= \frac{25}{8(x^2+1)^4} + c\end{aligned}$$

22. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{6}{(3x+5)^3}\\ &= 6(3x+5)^{-3}\\ &= 2(3(3x+5)^{-3})\\ y &= 2\left(\frac{(3x+5)^{-2}}{-2}\right) + c\\ &= -\frac{1}{(3x+5)^2} + c\end{aligned}$$

23. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{18x}{(3x^2+5)^3}\\ &= 3(6x)(3x^2+5)^{-3}\\ y &= 3\left(\frac{(3x^2+5)^{-2}}{-2}\right) + c\\ &= -\frac{3}{2(3x^2+5)^2} + c\end{aligned}$$

24. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 12\sqrt[3]{3x-2}\\ &= 4\left(3(3x-2)^{\frac{1}{3}}\right)\\ y &= 4\left(\frac{(3x-2)^{\frac{4}{3}}}{\frac{4}{3}}\right) + c\\ &= 4(3x-2)^{\frac{4}{3}}\left(\frac{3}{4}\right) + c\\ &= 3(3x-2)^{\frac{4}{3}} + c\end{aligned}$$

25. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 12\sqrt{3x+5}\\ &= 4\left(3(3x+5)^{\frac{1}{2}}\right)\\ y &= 4\left(\frac{(3x+5)^{\frac{3}{2}}}{\frac{3}{2}}\right)+c\\ &= 4(3x+5)^{\frac{3}{2}}\left(\frac{2}{3}\right)+c\\ &= \frac{8(3x+5)^{\frac{3}{2}}}{3}+c\end{aligned}$$

26. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{12}{\sqrt{3x+5}}\\ &= 4\left(3(3x+5)^{-\frac{1}{2}}\right)\\ y &= 4\left(\frac{(3x+5)^{\frac{1}{2}}}{\frac{1}{2}}\right)+c\\ &= 8(3x+5)^{\frac{1}{2}}+c\\ &= 8\sqrt{3x+5}+c\end{aligned}$$

27. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 3-12(-3x^2(1-x^3)^2)\\ y &= 3x-12\left(\frac{(1-x^3)^3}{3}\right)+c\\ &= 3x-4(1-x^3)^3+c\end{aligned}$$

28. $$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}t} &= 12(3(2+3t)^3)\\ P &= 12\left(\frac{(2+3t)^4}{4}\right)+c\\ &= 3(2+3t)^4+c\\ 50 &= 3(2+3(0))^4+c\\ 50 &= 3(2^4)+c\\ c &= 50-3\times 16\\ &= 2\\ \therefore\quad P &= 3(2+3t)^4+2\end{aligned}$$

29. $$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}t} &= 12(2t(t^2-5)^3\\ P &= 12\left(\frac{(t^2-5)^4}{4}\right)+c\\ &= 3(t^2-5)^4+c\\ 10 &= 3(2^2-5)^4+c\\ 10 &= 3(-1)^4+c\\ 10 &= 3+c\\ c &= 7\\ \therefore\quad P &= 3(t^2-5)^4+7\end{aligned}$$

Exercise 7B

1. $$\int 5\cos x\,\mathrm{d}x = 5\sin x + c$$

2. $$\int 2\sin x\,\mathrm{d}x = -2\cos x + c$$

3. $$\int -10\sin x\,\mathrm{d}x = 10\cos x + c$$

4. $$\int -2\cos x\,\mathrm{d}x = -2\sin x + c$$

5. $$\begin{aligned}\int 6\cos 2x\,\mathrm{d}x &= 3\int 2\cos 2x\,\mathrm{d}x\\ &= 3\sin 2x + c\end{aligned}$$

6. $$\begin{aligned}\int 2\cos 6x\,\mathrm{d}x &= \frac{1}{3}\int 6\cos 6x\,\mathrm{d}x\\ &= \frac{\sin 6x}{3}+c\end{aligned}$$

7. $$\begin{aligned}\int 12\sin 4x\,\mathrm{d}x &= -3\int -4\sin 4x\,\mathrm{d}x\\ &= -3\cos 4x + c\end{aligned}$$

8. $$\begin{aligned}\int -\sin 3x\,\mathrm{d}x &= \frac{1}{3}\int -3\sin 3x\,\mathrm{d}x\\ &= \frac{\cos 3x}{3}+c\end{aligned}$$

9. $$\begin{aligned}\int -8\cos 10x\,\mathrm{d}x &= -\frac{4}{5}\int 10\cos 10x\,\mathrm{d}x\\ &= -\frac{4\sin 10x}{5}+c\end{aligned}$$

10. $$\begin{aligned}\int \sin\frac{x}{2}\,\mathrm{d}x &= -2\int -\frac{1}{2}\sin\frac{x}{2}\,\mathrm{d}x\\ &= -2\cos\frac{x}{2}+c\end{aligned}$$

11. $$\begin{aligned}\int \cos\frac{3x}{2}\,\mathrm{d}x &= \frac{2}{3}\int \frac{3}{2}\cos\frac{3x}{2}\,\mathrm{d}x\\ &= \frac{2}{3}\sin\frac{3x}{2}+c\end{aligned}$$

12. $$\begin{aligned}\int -6\sin\frac{2x}{3}\,\mathrm{d}x &= \left(6\times\frac{3}{2}\right)\int -\frac{2}{3}\sin\frac{2x}{3}\,\mathrm{d}x\\ &= 9\cos\frac{2x}{3}+c\end{aligned}$$

13. For this question and the next, you should note that $\sin\left(x+\frac{\pi}{2}\right)=\cos x$ and $\cos\left(x-\frac{\pi}{2}\right)=\sin x$. You may make the substitution at the beginning and then antidifferentiate, or first antidifferentiate and then substitute.

14. See previous question.

15. $$\begin{aligned}\int \cos\left(2x+\frac{2\pi}{3}\right)\mathrm{d}x &= \frac{1}{2}\int 2\cos\left(2x+\frac{2\pi}{3}\right)\mathrm{d}x\\ &= \sin\left(2x+\frac{2\pi}{3}\right)+c\end{aligned}$$

16. $$\begin{aligned}\int \sin(-x)\,\mathrm{d}x &= \int -\sin x\,\mathrm{d}x\\ &= \cos x + c\end{aligned}$$

17. $$\int \frac{4}{\cos^2 x}\mathrm{d}x = 4\tan x + c$$

18. $$\begin{aligned}\int \frac{1}{\cos^2 4x}\mathrm{d}x &= \frac{1}{4}\int\frac{4}{\cos^2 4x}\mathrm{d}x\\ &= \frac{1}{4}\tan(4x)+c\end{aligned}$$

19. $$\int \frac{1}{\cos^2(x-1)}\mathrm{d}x = \tan(x-1)+c$$

20. $$\begin{aligned}&\int 6\cos 2x + 6\sin 3x\,\mathrm{d}x\\ &= 3\int 2\cos 2x\,\mathrm{d}x - 2\int -3\sin 3x\,\mathrm{d}x\\ &= 3\sin 2x - 2\cos 3x + c\end{aligned}$$

21. $$\begin{aligned}&\int \cos 8x - 4\sin 2x\,\mathrm{d}x\\ &= \frac{1}{8}\int 8\cos 8x\,\mathrm{d}x + 2\int -2\sin 2x\,\mathrm{d}x\\ &= \frac{1}{8}\sin 8x + 2\cos 2x + c\end{aligned}$$

22. $$\begin{aligned}&\int 2x + 4\cos x + 6\cos 2x\,\mathrm{d}x\\ &= \int 2x\,\mathrm{d}x + 4\int\cos x\,\mathrm{d}x + 3\int 2\cos 2x\,\mathrm{d}x\\ &= x^2 + 4\sin x + 3\sin 2x + c\end{aligned}$$

23. Although this looks long, you should by now be able to antidifferentiate it in a single step, simply working term by term.

$$\int 3+4x-6x^2\,\mathrm{d}x = 3x+2x^2-2x^3+c_1$$

$$\begin{aligned}\int 10\cos 5x\,\mathrm{d}x &= 2\int 5\cos 5x\,\mathrm{d}x\\ &= 2\sin 5x + c_2\end{aligned}$$

$$\begin{aligned}\int -2\sin 4x\,\mathrm{d}x &= \frac{1}{2}\int -4\sin 4x\,\mathrm{d}x\\ &= \frac{\cos 4x}{2}+c_3\end{aligned}$$

so the overall antiderivative is

$$3x+2x^2-2x^3+2\sin 5x+\frac{\cos 4x}{2}+c$$

24. $$\int \cos^3 x\sin x\,\mathrm{d}x = -\frac{\cos^4 x}{4}+c$$

25. $$\begin{aligned}\int 30\cos^5 x\sin x\,\mathrm{d}x &= -\frac{30\cos^6 x}{6}+c\\ &= -5\cos^6 x + c\end{aligned}$$

26. $$\int \sin^4 x\cos x\,\mathrm{d}x = \frac{\sin^5 x}{5}+c$$

27. $$\begin{aligned}\int 6\sin^3 x\cos x\,\mathrm{d}x &= \frac{6\sin^4 x}{4}+c\\ &= \frac{3\sin^4 x}{2}+c\end{aligned}$$

28. $$\int -2\cos^4 x\sin x\,\mathrm{d}x = \frac{2\cos^5 x}{5}+c$$

29. $$\begin{aligned}\int -2\sin^7 x\cos x\,\mathrm{d}x &= -\frac{2\sin^8 x}{8}+c\\ &= -\frac{\sin^8 x}{4}+c\end{aligned}$$

30. $$\begin{aligned}\int 32\sin^3 4x\cos 4x\,\mathrm{d}x &= 8\int \sin^3 4x(4\cos 4x)\,\mathrm{d}x\\ &= \frac{8\sin^4 4x}{4}+c\\ &= 2\sin^4 4x + c\end{aligned}$$

31. $$\begin{aligned}&\int -24\sin^3 2x\cos 2x\,\mathrm{d}x\\ &= -12\int \sin^3 2x(2\cos 2x)\,\mathrm{d}x\\ &= \frac{-12\sin^4 2x}{4}+c\\ &= -3\sin^4 2x + c\end{aligned}$$

32. $$\begin{aligned}&\int 20\sin^4 2x\cos 2x\,\mathrm{d}x\\ &= 10\int \sin^4 2x(2\cos 2x)\,\mathrm{d}x\\ &= \frac{10\sin^5 2x}{5}+c\\ &= 2\sin^5 2x + c\end{aligned}$$

33. $$\begin{aligned}&\int -6\cos^2 4x\sin 4x\,\mathrm{d}x\\ &= \frac{3}{2}\int \cos^2 4x(-4\sin 4x)\,\mathrm{d}x\\ &= \frac{3\cos^3 4x}{2\times 3}+c\\ &= \frac{\cos^3 4x}{2}+c\end{aligned}$$

34. $$\int \sin^3 x \, dx = \int \sin x(\sin^2 x) \, dx$$
$$= \int \sin x(1 - \cos^2 x) \, dx$$
$$= \int \sin x - \sin x \cos^2 x \, dx$$
$$= -\cos x + \frac{\cos^3 x}{3} + c$$

35. $$\int \cos^3 x \, dx = \int \cos x(\cos^2 x) \, dx$$
$$= \int \cos x(1 - \sin^2 x) \, dx$$
$$= \int \cos x - \cos x \sin^2 x \, dx$$
$$= \sin x - \frac{\sin^3 x}{3} + c$$

36. $$\int \cos^5 x \, dx$$
$$= \int \cos x(\cos^4 x) \, dx$$
$$= \int \cos x(\cos^2 x)^2 \, dx$$
$$= \int \cos x(1 - \sin^2 x)^2 \, dx$$
$$= \int \cos x(1 - 2\sin^2 x + \sin^4 x) \, dx$$
$$= \int \cos x - 2\cos x \sin^2 x + \cos x \sin^4 x \, dx$$
$$= \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + c$$

37. $$\int \cos^2 x \, dx = \frac{1}{2}\int 2\cos^2 x \, dx$$
$$= \frac{1}{2}\int 2\cos^2 x - 1 + 1 \, dx$$
$$= \frac{1}{2}\int \cos 2x + 1 \, dx$$
$$= \frac{1}{2}\left(\frac{\sin 2x}{2} + x\right) + c$$
$$= \frac{\sin 2x}{4} + \frac{x}{2} + c$$

38. $$\int \sin^2 x \, dx = -\frac{1}{2}\int -2\sin^2 x \, dx$$
$$= -\frac{1}{2}\int 1 - 2\sin^2 x - 1 \, dx$$
$$= -\frac{1}{2}\int \cos 2x - 1 \, dx$$
$$= -\frac{1}{2}\left(\frac{\sin 2x}{2} - x\right) + c$$
$$= -\frac{\sin 2x}{4} + \frac{x}{2} + c$$

39. $$\int 8\sin^4 x \, dx = 8\int (\sin^2 x)^2 \, dx$$
$$= 8\int \left(\frac{-2\sin^2 x}{-2}\right)^2 dx$$
$$= 8\int \frac{(-2\sin^2 x)^2}{4} \, dx$$
$$= 2\int (-2\sin^2 x)^2 \, dx$$
$$= 2\int (1 - 2\sin^2 x - 1)^2 \, dx$$
$$= 2\int (\cos 2x - 1)^2 \, dx$$
$$= 2\int \cos^2 2x - 2\cos 2x + 1 \, dx$$
$$= \int 2\cos^2 2x - 4\cos 2x + 2 \, dx$$
$$= \int 2\cos^2 2x - 1 - 4\cos 2x + 3 \, dx$$
$$= \int \cos 4x - 4\cos 2x + 3 \, dx$$
$$= \frac{\sin 4x}{4} - \frac{4\sin 2x}{2} + 3x + c$$
$$= \frac{\sin 4x}{4} - 2\sin 2x + 3x + c$$

40. $$\int \cos^2 x + \sin^2 x \, dx = \int 1 \, dx$$
$$= x + c$$
(Compare this with your answers for questions 37 and 38.)

41. $$\int \cos^2 x - \sin^2 x \, dx = \int \cos 2x \, dx$$
$$= \frac{\sin 2x}{2} + c$$
(Compare this with your answers for questions 37 and 38.)

42. $$\int \sin 5x \cos 2x + \cos 5x \sin 2x \, dx$$
$$= \int \sin(5x + 2x) \, dx$$
$$= \int \sin 7x \, dx$$
$$= -\frac{\cos 7x}{7} + c$$
(You need to know the angle sum trig identities well enough to recognise them.)

43. $$\int \sin 3x \cos x - \cos 3x \sin x \, dx$$
$$= \int \sin(3x - x) \, dx$$
$$= \int \sin 2x \, dx$$
$$= -\frac{\cos 2x}{2} + c$$

44. $\int \cos 5x \cos 2x - \sin 5x \sin 2x \,\mathrm{d}x$

$$= \int \cos(5x + 2x)\,\mathrm{d}x$$
$$= \int \cos 7x \,\mathrm{d}x$$
$$= \frac{\sin 7x}{7} + c$$

45. $\int \cos 5x \cos x + \sin 5x \sin x \,\mathrm{d}x$

$$= \int \cos(5x - x)\,\mathrm{d}x$$
$$= \int \cos 4x \,\mathrm{d}x$$
$$= \frac{\sin 4x}{4} + c$$

46. Refer to your answers to questions 34 and 37.

47. There are (at least) three different ways of approaching this question. You can treat it as
   - $\int 2f(x)f'(x)\,\mathrm{d}x = [f(x)]^2 + c$ where $f(x) = \sin x$ (i.e. $sin^2 x + c$);
   - $\int -2f(x)f'(x)\,\mathrm{d}x = -[f(x)]^2 + c$ where $f(x) = \cos x$ (i.e. $-cos^2 x + c$); or
   - Recognise the double angle formula for sine: $\int \sin 2x \,\mathrm{d}x = -\frac{1}{2}\cos 2x + c$.

   You should satisfy yourself that these answers are equivalent (differing only in the value of the constant of integration).

48. $\int \sin^3 x \cos^2 x \,\mathrm{d}x$

$$= \int \sin x \sin^2 x \cos^2 x \,\mathrm{d}x$$
$$= \int \sin x(1 - \cos^2 x)\cos^2 x \,\mathrm{d}x$$
$$= \int \sin x \cos^2 x - \sin x \cos^4 x \,\mathrm{d}x$$
$$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + c$$

49. $\int \cos^3 x \sin^2 x \,\mathrm{d}x$

$$= \int \cos x \cos^2 x \sin^2 x \,\mathrm{d}x$$
$$= \int \cos x(1 - \sin^2 x)\sin^2 x \,\mathrm{d}x$$
$$= \int \cos x \sin^2 x - \cos x \sin^4 x \,\mathrm{d}x$$
$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c$$

50.
$$\begin{aligned} C &= \int 4 \sin 2p \, dp \\ &= -2\cos 2p + k \\ 3 &= -2\cos 0 + k \\ k &= 5 \\ \therefore \quad C &= 5 - 2\cos 2p \end{aligned}$$

Note that I use k for the constant of integration here rather than the more commonly used c so as to avoid confusion with C.

51.
$$\begin{aligned} P &= \int 12 \cos 3x \,\mathrm{d}x \\ &= 4 \sin 3x + c \\ 10 &= 4 \sin \frac{3\pi}{2} + c \\ 10 &= -4 + c \\ c &= 14 \\ \therefore \quad P &= 4 \sin 3x + 14 \end{aligned}$$

52. $\int 2 \sec x \tan x \,\mathrm{d}x = 2 \sec x + c$

53. By the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sec 2x) = 2 \sec 2x \tan 2x$$

so

$$\int \sec 2x \tan 2x \,\mathrm{d}x = \frac{\sec 2x}{2} + c$$

54. By the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cot 2x) = -2 \operatorname{cosec}^2 2x$$

so

$$\int -4 \operatorname{cosec}^2 2x \,\mathrm{d}x = 2 \cot 2x + c$$

55.
$$\begin{aligned} \int \frac{1}{\sin^2 x}\,\mathrm{d}x &= \int \operatorname{cosec}^2 x \,\mathrm{d}x \\ &= -\cot x + c \end{aligned}$$

56.
$$\begin{aligned} \int \frac{\sin x}{\cos^2 x}\,\mathrm{d}x &= \int \frac{1}{\cos x}\frac{\sin x}{cos x}\,\mathrm{d}x \\ &= \int \sec x \tan x \,\mathrm{d}x \\ &= \sec x + c \end{aligned}$$

57.
$$\begin{aligned} \int \frac{\cos x}{\sin^2 x}\,\mathrm{d}x &= \int \operatorname{cosec} x \cot x \,\mathrm{d}x \\ &= -\operatorname{cosec} x + c \end{aligned}$$

58. This is of the form $\int f(x)^n f'(x)\,\mathrm{d}x$ where $f(x) = \operatorname{cosec} x$:

$$\begin{aligned} \int 20(\operatorname{cosec} x \cot x)(\operatorname{cosec}^3 x)\,\mathrm{d}x &= \frac{-20 \operatorname{cosec}^4 x}{4} + c \\ &= -5 \operatorname{cosec}^4 x + c \end{aligned}$$

59. $\int 20 \operatorname{cosec}^4 x \cot x \,\mathrm{d}x$

$$= \int 20(\operatorname{cosec} x \cot x)(\operatorname{cosec}^3 x)\,\mathrm{d}x$$
$$= -5 \operatorname{cosec}^4 x + c$$

Exercise 7C

In the following solutions I have used notation like $du = 2x\,\mathrm{d}x$. This is convenient in helping us make the variable substitutions, but it is important to remember that du and dx are meaningless outside the context of integration. We must not fall into the trap of treating them like real quantities.

1. $u = x^2 - 3 \quad du = 2x\,\mathrm{d}x$

$$\begin{aligned}\int 60x(x^2-3)^5\,\mathrm{d}x &= 30\int u^5\,2x\,\mathrm{d}x\\ &= 30\int u^5\,du\\ &= 5u^6 + c\\ &= 5(x^2-3)^6 + c\end{aligned}$$

2. $u = 1 - 2x \quad du = -2\,\mathrm{d}x$

   $x = \dfrac{1-u}{2} \quad dx = -\dfrac{1}{2}\,du$

$$\begin{aligned}&\int 80x(1-2x)^3\,\mathrm{d}x\\ &= \int 80\left(\frac{1-u}{2}\right)u^3\left(-\frac{1}{2}\right)du\\ &= \int -20(1-u)u^3\,du\\ &= 20\int u^4 - u^3\,du\\ &= 20\left(\frac{u^5}{5} - \frac{u^4}{4}\right) + c\\ &= 4u^5 - 5u^4 + c\\ &= u^4(4u-5) + c\\ &= (1-2x)^4\,(4(1-2x)-5) + c\\ &= (1-2x)^4(4-8x-5) + c\\ &= -(1-2x)^4(8x+1) + c\end{aligned}$$

3. $u = 3x + 1 \quad du = 3\,\mathrm{d}x$

   $x = \dfrac{u-1}{3} \quad dx = \dfrac{1}{3}\,du$

$$\begin{aligned}&\int 12x(3x+1)^5\,\mathrm{d}x\\ &= \int 12\left(\frac{u-1}{3}\right)u^5\left(\frac{1}{3}\right)du\\ &= \int \frac{4}{3}(u-1)u^5\,du\\ &= \frac{4}{3}\int u^6 - u^5\,du\\ &= \frac{4}{3}\left(\frac{u^7}{7} - \frac{u^6}{6}\right) + c\\ &= \frac{4u^7}{21} - \frac{2u^6}{9} + c\\ &= \frac{2u^6}{63}(6u-7) + c\\ &= \frac{2(3x+1)^6}{63}\,(6(3x+1)-7) + c\\ &= \frac{2(3x+1)^6}{63}(18x+6-7) + c\\ &= \frac{2}{63}(3x+1)^6(18x-1) + c\end{aligned}$$

4. $u = 2x^2 - 1 \quad du = 4x\,\mathrm{d}x$

$$\begin{aligned}\int 6x(2x^2-1)^5\,\mathrm{d}x &= \int \frac{3}{2}u^5 4x\,\mathrm{d}x\\ &= \frac{3}{2}\int u^5\,du\\ &= \frac{3}{2}\frac{u^6}{6} + c\\ &= \frac{1}{4}u^6 + c\\ &= \frac{1}{4}(2x^2-1)^6 + c\end{aligned}$$

5. $u = 3x^2 + 1 \quad du = 6x\,\mathrm{d}x$

$$\begin{aligned}\int 12x(3x^2+1)^5\,\mathrm{d}x &= \int 2u^5 6x\,\mathrm{d}x\\ &= 2\int u^5\,du\\ &= 2\frac{u^6}{6} + c\\ &= \frac{1}{3}u^6 + c\\ &= \frac{1}{3}(3x^2+1)^6 + c\end{aligned}$$

6. $u = x - 2 \quad du = dx$

   $x = u + 2 \quad dx = du$

$$\begin{aligned}\int 3x(x-2)^5\,\mathrm{d}x &= \int 3(u+2)u^5\,du\\ &= \int 3u^6+6u^5\,du\\ &= 3\frac{u^7}{7}+6\frac{u^6}{6}+c\\ &= 3\frac{u^7}{7}+u^6+c\\ &= \frac{1}{7}u^6(3u+7)+c\\ &= \frac{1}{7}(x-2)^6\left(3(x-2)+7\right)+c\\ &= \frac{1}{7}(x-2)^6(3x-6+7)+c\\ &= \frac{1}{7}(x-2)^6(3x+1)+c\end{aligned}$$

7. $u=3-x \quad du=-dx$
   $x=3-u \quad dx=-du$

$$\begin{aligned}\int 20x(3-x)^3\,\mathrm{d}x &= \int -20(3-u)u^3\,du\\ &= 20\int u^4-3u^3\,du\\ &= 20\left(\frac{u^5}{5}-\frac{3u^4}{4}\right)+c\\ &= 4u^5-15u^4+c\\ &= u^4(4u-15)+c\\ &= (3-x)^4\left(4(3-x)-15\right)+c\\ &= (3-x)^4(12-4x-15)+c\\ &= (3-x)^4(-4x-3)+c\\ &= -(3-x)^4(4x+3)+c\end{aligned}$$

8. $u=5-2x \quad du=-2\,\mathrm{d}x$
   $x=\frac{5-u}{2} \quad dx=-\frac{1}{2}\,du$

$$\begin{aligned}&\int 4x(5-2x)^5\,\mathrm{d}x\\ &= \int 4\frac{5-u}{2}u^5\left(-\frac{1}{2}\right)\,du\\ &= \int (u-5)u^5\,du\\ &= \frac{u^7}{7}-\frac{5u^6}{6}+c\\ &= \frac{1}{42}u^6(6u-35)+c\\ &= \frac{1}{42}(5-2x)^6\left(6(5-2x)-35\right)+c\\ &= \frac{1}{42}(5-2x)^6(-12x-5)+c\\ &= -\frac{1}{42}(5-2x)^6(12x+5)+c\end{aligned}$$

9. $u=2x+3 \quad du=2\,\mathrm{d}x$
   $x=\frac{u-3}{2} \quad dx=\frac{1}{2}\,du$

$$\begin{aligned}&\int 20x(2x+3)^3\,\mathrm{d}x\\ &= \int 5(u-3)u^3\,du\\ &= u^5-\frac{15u^4}{4}+c\\ &= \frac{1}{4}u^4(4u-15)+c\\ &= \frac{1}{4}(2x+3)^4\left(4(2x+3)-15\right)+c\\ &= \frac{1}{4}(2x+3)^4(8x-3)+c\end{aligned}$$

10. $u=3x+1 \quad du=3\,\mathrm{d}x$
    $x=\frac{u-1}{3} \quad dx=\frac{1}{3}\,du$

$$\begin{aligned}&\int 18x\sqrt{3x+1}\,\mathrm{d}x\\ &= \int 2(u-1)u^{\frac{1}{2}}\,du\\ &= 2\int u^{\frac{3}{2}}-u^{\frac{1}{2}}\,du\\ &= 2\left(\frac{2u^{\frac{5}{2}}}{5}-\frac{2u^{\frac{3}{2}}}{3}\right)+c\\ &= \frac{4}{15}\sqrt{u^3}(3u-5)+c\\ &= \frac{4}{15}\sqrt{(3x+1)^3}\left(3(3x+1)-5\right)+c\\ &= \frac{4}{15}\sqrt{(3x+1)^3}(9x-2)+c\end{aligned}$$

11. $u=3x^2+5 \quad du=6x\,\mathrm{d}x$

$$\begin{aligned}\int\frac{6x\,\mathrm{d}x}{\sqrt{3x^2+5}} &= \int\frac{du}{\sqrt{u}}\\ &= 2\sqrt{u}+c\\ &= 2\sqrt{3x^2+5}+c\end{aligned}$$

12. $u=1-2x \quad du=-2\,\mathrm{d}x$
    $x=\frac{1-u}{2} \quad dx=-\frac{1}{2}\,du$

$$\begin{aligned}\int \frac{3x\,\mathrm{d}x}{\sqrt{1-2x}} &= \int \frac{3(1-u)}{2\sqrt{u}}\left(-\frac{1}{2}\right)\,du\\ &= -\frac{3}{4}\int \frac{1-u}{\sqrt{u}}\,du\\ &= -\frac{3}{4}\int \frac{1}{\sqrt{u}} - \sqrt{u}\,du\\ &= -\frac{3}{4}\left(2\sqrt{u} - \frac{2}{3}u^{\frac{3}{2}}\right) + c\\ &= -\frac{1}{2}\sqrt{u}(3-u) + c\\ &= \frac{1}{2}\sqrt{u}(u-3) + c\\ &= \frac{1}{2}\sqrt{1-2x}(1-2x-3) + c\\ &= -\frac{1}{2}\sqrt{1-2x}(2x+2) + c\\ &= -\sqrt{1-2x}(x+1) + c\end{aligned}$$

13. $u = \sin 2x \quad du = 2\cos 2x\,\mathrm{d}x$

$$\begin{aligned}\int 8\sin^5 2x\cos 2x\,\mathrm{d}x &= \int 4u^5\,du\\ &= \frac{4u^6}{6} + c\\ &= \frac{2}{3}\sin^6 2x + c\end{aligned}$$

14. $u = \cos 3x \quad du = -3\sin 3x\,\mathrm{d}x$

$$\begin{aligned}\int 27\cos^7 3x\sin 3x\,\mathrm{d}x &= \int -9u^7\,du\\ &= -\frac{9}{8}u^8 + c\\ &= -\frac{9}{8}\cos^8 3x + c\end{aligned}$$

15. $u = x^2 + 4 \quad du = 2x\,\mathrm{d}x$

$$\begin{aligned}\int 6x\sin(x^2+4)\,\mathrm{d}x &= \int 3\sin u\,du\\ &= -3\cos u + c\\ &= -3\cos(x^2+4) + c\end{aligned}$$

16. $u = 2x + 1 \quad du = 2\,\mathrm{d}x$

$2x = u - 1 \quad dx = \frac{1}{2}\,du$

$$\begin{aligned}&\int (4x+3)(2x+1)^5\,\mathrm{d}x\\ &= \int \left(2(u-1)+3\right)u^5\frac{1}{2}\,du\\ &= \frac{1}{2}\int (2u+1)u^5\,du\\ &= \frac{1}{2}\left(\frac{2u^7}{7} + \frac{u^6}{6}\right) + c\\ &= \frac{1}{84}u^6(12u+7) + c\\ &= \frac{1}{84}(2x+1)^6\left(12(2x+1)+7\right) + c\\ &= \frac{1}{84}(2x+1)^6(24x+19) + c\end{aligned}$$

Exercise 7D

1. $\displaystyle\int (x + \sin 3x)dx = \frac{x^2}{2} - \frac{\cos 3x}{3} + c$

2. $\displaystyle\int 2\,\mathrm{d}x = 2x + c$

3. $\displaystyle\int \sin 8x\,\mathrm{d}x = -\frac{\cos 8x}{8} + c$

4. $$\begin{aligned}&\int (\cos x + \sin x)(\cos x - \sin x)dx\\ &= \int \cos^2 x - \sin^2 x\,\mathrm{d}x\\ &= \int \cos 2x\,\mathrm{d}x\\ &= \frac{1}{2}\sin 2x + c\end{aligned}$$

5. $$\begin{aligned}\int \frac{x^2+x}{\sqrt{x}}dx &= \int x^{\frac{3}{2}} + x^{\frac{1}{2}}\,\mathrm{d}x\\ &= \frac{2}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}} + c\end{aligned}$$

6. $$\int \sin^3 2x\,\mathrm{d}x = \int \sin 2x(1-\cos^2 2x)\,\mathrm{d}x$$
$$= -\frac{\cos 2x}{2} + \frac{1}{2}\frac{\cos^3 2x}{3} + c$$
$$= \frac{\cos 2x}{6}(\cos^2 2x - 3) + c$$

7. $$\int (3+\cos^2 x)\,\mathrm{d}x = \frac{1}{2}\int 6 + 2\cos^2 x\,\mathrm{d}x$$
$$= \frac{1}{2}\int 7 + 2\cos^2 x - 1\,\mathrm{d}x$$
$$= \frac{1}{2}\int 7 + \cos 2x\,\mathrm{d}x$$
$$= \frac{7}{2}x + \frac{1}{4}\sin 2x + c$$

8. $$\int 4x\sin x^2\,\mathrm{d}x = 2\int 2x\sin x^2\,\mathrm{d}x$$
$$= 2(-\cos x^2) + c$$
$$= -2\cos x^2 + c$$

9. $u = x^2 - 3 \quad du = 2x\,\mathrm{d}x$
$$\int 8x\sin(x^2-3)\,\mathrm{d}x = 4\int \sin u\,du$$
$$= -4\cos u + c$$
$$= -4\cos(x^2-3) + c$$

10. $u = 1+3x \quad du = 3\,\mathrm{d}x$
$$\int 24\sqrt{1+3x}\,\mathrm{d}x = 8\int \sqrt{u}\,du$$
$$= 8\left(\frac{2}{3}\right)u^{\frac{3}{2}} + c$$
$$= \frac{16(1+3x)^{\frac{3}{2}}}{3} + c$$

11. $u = 1+3x \quad du = 3\,\mathrm{d}x$

$3x = u-1 \qquad dx = \frac{1}{3}\,du$
$$\int 15x\sqrt{1+3x}\,\mathrm{d}x$$
$$= \int 5(u-1)\sqrt{u}\frac{1}{3}\,du$$
$$= \frac{5}{3}\int u^{\frac{3}{2}} - \sqrt{u}\,du$$
$$= \frac{5}{3}(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}) + c$$
$$= \frac{2}{3}u^{\frac{5}{2}} - \frac{10}{9}u^{\frac{3}{2}} + c$$
$$= \frac{2}{9}u^{\frac{3}{2}}(3u-5) + c$$
$$= \frac{2}{9}(1+3x)^{\frac{3}{2}}\,(3(1+3x)-5) + c$$
$$= \frac{2}{9}(1+3x)^{\frac{3}{2}}(3+9x-5) + c$$
$$= \frac{2}{9}(1+3x)^{\frac{3}{2}}(9x-2) + c$$

12. $$\int \sin^4 2x\cos 2x\,\mathrm{d}x = \frac{1}{2}\int \sin^4 2x(2\cos 2x)\,\mathrm{d}x$$
$$= \frac{1}{10}\sin^5 2x$$

13. $u = 2x+7 \quad du = 2\,\mathrm{d}x$

$2x = u-7 \qquad dx = \frac{1}{2}\,du$
$$\int 6x(2x{+}7)^5\,\mathrm{d}x$$
$$= \int 3(u-7)u^5\frac{1}{2}\,du$$
$$= \frac{3}{2}\int u^6 - 7u^5\,du$$
$$= \frac{3}{2}(\frac{1}{7}u^7 - \frac{7}{6}u^6) + c$$
$$= \frac{3}{84}u^6(6u-49) + c$$
$$= \frac{3}{84}(2x+7)^6\,(6(2x+7)-49) + c$$
$$= \frac{3}{84}(2x+7)^6(12x-7) + c$$

14. $$\int 6(2x+7)^5\,\mathrm{d}x = 3\int 2(2x+7)^5\,\mathrm{d}x$$
$$= \frac{1}{2}(2x+7)^6 + c$$

15. $$\int (3x^2-2)dx = x^3 - 2x + c$$

16. $$\int 4x(3x^2-3)^7\,\mathrm{d}x = \frac{2}{3}\int 6x(3x^2-3)^7\,\mathrm{d}x$$
$$= \frac{1}{12}(3x^2-2)^8 + c$$

17. $$\int (\cos x + \sin 2x)dx = \sin x - \frac{1}{2}\cos 2x + c$$

18. $u = 3x-2 \quad du = 3\,\mathrm{d}x$

$3x = u+2 \qquad dx = \frac{1}{3}\,du$
$$\int 6x(3x{-}2)^7\,\mathrm{d}x$$
$$= \int 2(u+2)u^7\frac{1}{3}\,du$$
$$= \frac{2}{3}\int u^8 + 2u^7\,du$$
$$= \frac{2}{3}(\frac{1}{9}u^9 + \frac{1}{4}u^8) + c$$
$$= \frac{1}{54}u^8(4u+9) + c$$
$$= \frac{1}{54}(3x-2)^8\,(4(3x-2)+9) + c$$
$$= \frac{1}{54}(3x-2)^8(12x+1) + c$$

19. $$\int x\,\mathrm{d}x = \frac{x^2}{2} + c$$

20. $u = 1 + 2x \quad du = 2\,\mathrm{d}x$

$$\begin{aligned}\int \frac{6}{\sqrt{1+2x}}\,\mathrm{d}x &= \int \frac{3}{\sqrt{u}}\,du\\ &= 6\sqrt{u} + c\\ &= 6\sqrt{1+2x} + c\end{aligned}$$

21. $u = 1 + 2x \quad du = 2\,\mathrm{d}x$

$2x = u - 1 \quad \mathrm{d}x = \frac{1}{2}\,du$

$$\begin{aligned}\int \frac{6}{x}\sqrt{1+2x}\,\mathrm{d}x &= \int \frac{3(u-1)}{2\sqrt{u}}\,du\\ &= \frac{3}{2}\int \frac{u-1}{\sqrt{u}}\,du\\ &= \frac{3}{2}\int \sqrt{u} - \frac{1}{\sqrt{u}}\,du\\ &= \frac{3}{2}(\frac{2}{3}u^{\frac{3}{2}} - 2\sqrt{u}) + c\\ &= u^{\frac{3}{2}} - 3\sqrt{u} + c\\ &= \sqrt{u}(u-3) + c\\ &= \sqrt{1+2x}(1+2x-3) + c\\ &= \sqrt{1+2x}(2x-2) + c\end{aligned}$$

22. $$\begin{aligned}\int 6\sin 2x\cos x\,\mathrm{d}x &= \int 6(2\sin x\cos x)\cos x\,\mathrm{d}x\\ &= 12\int \sin x\cos^2 x\,\mathrm{d}x\\ &= -4\cos^3 x + c\end{aligned}$$

23. $$\begin{aligned}\int 6\cos 2x\sin x\,\mathrm{d}x &= \int 6(2\cos^2 x - 1)\sin x\,\mathrm{d}x\\ &= 6\int 2\cos^2 x\sin x - \sin x\,\mathrm{d}x\\ &= 6(-\frac{2}{3}\cos^3 x + \cos x) + c\\ &= -4\cos^3 x + 6\cos x + c\end{aligned}$$

24. $$\int (x^2+x+1)^8(2x+1)\,\mathrm{d}x = \frac{1}{9}(x^2+x+1)^9 + c$$

25. $u = x^2 + 3 \quad du = 2x\,\mathrm{d}x$

$$\begin{aligned}\int 24x\sin(x^2+3)\,\mathrm{d}x &= 12\int 2x\sin u\,\mathrm{d}x\\ &= 12\int \sin u\,du\\ &= -12\cos u + c\\ &= -12\cos(x^2+3) + c\end{aligned}$$

26. $u = x - 5 \quad du = \mathrm{d}x$

$x = u + 5 \quad \mathrm{d}x = du$

$$\begin{aligned}&\int (2x+1)\sqrt[3]{x-5}\,\mathrm{d}x\\ &= \int (2u+10+1)u^{\frac{1}{3}}\,\mathrm{d}x\\ &= \int (2u+11)u^{\frac{1}{3}}\,\mathrm{d}x\\ &= \int 2u^{\frac{4}{3}} + 11u^{\frac{1}{3}}\,\mathrm{d}x\\ &= \frac{6u^{\frac{7}{3}}}{7} + \frac{33u^{\frac{4}{3}}}{4} + c\\ &= \frac{3u^{\frac{4}{3}}}{28}(8u+77) + c\\ &= \frac{3}{28}(x-5)^{\frac{4}{3}}(8(x-5)+77) + c\\ &= \frac{3}{28}(x-5)^{\frac{4}{3}}(8x+37) + c\end{aligned}$$

27. $u = \sqrt{x} + 5 \quad du = \frac{1}{2\sqrt{x}}\,\mathrm{d}x$

$$\begin{aligned}\int \frac{(\sqrt{x}+5)^5}{\sqrt{x}}\,\mathrm{d}x &= 2\int \frac{(\sqrt{x}+5)^5}{2\sqrt{x}}\,\mathrm{d}x\\ &= 2\int u^5\,du\\ &= \frac{2u^6}{6} + c\\ &= \frac{(\sqrt{x}+5)^6}{3} + c\end{aligned}$$

28. $$\begin{aligned}\int 4(2x-1)^5\,\mathrm{d}x &= 2\int 2(2x-1)^5\,\mathrm{d}x\\ &= \frac{2(2x-1)^6}{6} + c\\ &= \frac{(2x-1)^6}{3} + c\end{aligned}$$

29. $u = 2x - 1 \quad du = 2\,\mathrm{d}x$

$2x = u + 1$

$$\begin{aligned}&\int 4x(2x-1)^5\,\mathrm{d}x\\ &= \int 2x\,u^5\,du\\ &= \int (u+1)u^5\,du\\ &= \int u^6 + u^5\,du\\ &= \frac{u^7}{7} + \frac{u^6}{6} + c\\ &= \frac{(2x-1)^7}{7} + \frac{(2x-1)^6}{6} + c\\ &= \frac{1}{42}(2x-1)^6(6(2x-1)+7) + c\\ &= \frac{1}{42}(2x-1)^6(12x+1) + c\end{aligned}$$

30. $\displaystyle\int \cos^3 6x \sin 6x \, dx = -\frac{1}{6}\int -6\cos^3 6x \sin 6x \, dx$
$\displaystyle = -\frac{\cos^4 6x}{24} + c$

31. $\displaystyle\int \frac{6x}{\sqrt{x^2-3}} \, dx = 3\int \frac{2x}{\sqrt{x^2-3}} \, dx$
$= 6\sqrt{x^2-3} + c$

32. $\displaystyle\int \sin 2x \cos 2x \, dx = \int \sin 4x \, dx$
$\displaystyle = -\frac{\cos 4x}{4} + c$

33. $u = 2x - 1 \quad du = 2\, dx$
$2x = u + 1$

$$\begin{aligned}
&\int 8x^2(2x-1)^5 \, dx \\
&= \int (2x)^2 \, u^5 \, du \\
&= \int (u+1)^2 u^5 \, du \\
&= \int (u^2 + 2u + 1)u^5 \, du \\
&= \int u^7 + 2u^6 + u^5 \, du \\
&= \frac{u^8}{8} + \frac{2u^7}{7} + \frac{u^6}{6} + c \\
&= \frac{u^6}{168}(21u^2 + 48u + 28) + c \\
&= \frac{(2x-1)^6}{168}(21(2x-1)^2 + 48(2x-1) + 28) + c \\
&= \frac{(2x-1)^6}{168}(21(4x^2 - 4x + 1) + 96x - 48 + 28) + c \\
&= \frac{(2x-1)^6}{168}(84x^2 - 84x + 21 + 96x - 20) + c \\
&= \frac{(2x-1)^6}{168}(84x^2 + 12x + 1) + c
\end{aligned}$$

Exercise 7E

1. $\displaystyle\int \frac{dy}{dx} \, dx = \int 6x - 5 \, dx$
$y = 3x^2 - 5x + c$

2. $\displaystyle\int \frac{dy}{dx} \, dx = \int 6\sqrt{x} \, dx$
$y = 4x^{\frac{3}{2}} + c$

3. $\displaystyle\int 8y \, dy = \int 4x - 1 \, dx$
$4y^2 = 2x^2 - x + c$

4. $\displaystyle\int 3y \, dy = \int \frac{5}{x^2} \, dx$
$\displaystyle\frac{3}{2}y^2 = -\frac{5}{x} + c$

5. $\displaystyle 6y\frac{dy}{dx} = \frac{1}{x^2}$
$\displaystyle\int 6y \, dy = \int \frac{dx}{x^2}$
$\displaystyle 3y^2 = -\frac{1}{x} + c$

6. $\displaystyle\sin 2y \frac{dy}{dx} = \frac{5}{4x^2}$
$\displaystyle\int \sin 2y \, dy = \int \frac{5}{4x^2} \, dx$
$\displaystyle -\frac{\cos 2y}{2} = -\frac{5}{4x} + c$
$\displaystyle 2\cos 2y = \frac{5}{x} + c$

7. $\displaystyle(2y-3)\frac{dy}{dx} = 8x + 1$
$\displaystyle\int 2y - 3 \, dy = \int 8x + 1 \, dx$
$y^2 - 3y = 4x^2 + x + c$

8. $\displaystyle(4y-5)\frac{dy}{dx} = x(2-3x)$
$\displaystyle\int 4y - 5 \, dy = \int 2x - 3x^2 \, dx$
$2y^2 - 5y = x^2 - x^3 + c$

9. $\displaystyle\cos y \frac{dy}{dx} = \frac{1}{x^2}$
$\displaystyle\int \cos y \, dy = \int \frac{1}{x^2} \, dx$
$\displaystyle\sin y = -\frac{1}{x} + c$

10. $$2y(y^2+1)^5\frac{\mathrm{d}y}{\mathrm{d}x} = x$$
$$\int 2y(y^2+1)^5\,dy = \int x\,\mathrm{d}x$$
$$\frac{(y^2+1)^6}{6} = \frac{x^2}{2} + c$$
$$(y^2+1)^6 = 3x^2 + c$$

11. $$\int dy = \int 6x\,\mathrm{d}x$$
$$y = 3x^2 + c$$
$$4 = 3(-1)^2 + c$$
$$c = 1$$
$$\therefore \quad y = 3x^2 + 1$$

12. $$6y\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5}{x^2}$$
$$\int 6y\,dy = \int \frac{5}{x^2}\,\mathrm{d}x$$
$$3y^2 = -\frac{5}{x} + c$$
$$3(1)^2 = -\frac{5}{0.5} + c$$
$$3 = -10 + c$$
$$x = 13$$
$$\therefore \quad 3y^2 = 13 - \frac{5}{x}$$

13. $$\int 2 + \cos y\,dy = \int 2x + 3\,\mathrm{d}x$$
$$2y + \sin y = x^2 + 3x + c$$
$$\pi + 1 = 1 + 3 + c$$
$$c = \pi - 3$$
$$\therefore \quad 2y + \sin y = x^2 + 3x + \pi - 3$$

14. $$2y + 3\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 + 8x$$
$$\int 2y + 3\,dy = \int 4x^3 + 8x\,\mathrm{d}x$$
$$y^2 + 3y = x^4 + 4x^2 + c$$
$$4 + 6 = 1 + 4 + c$$
$$c = 5$$
$$\therefore \quad y^2 + 3y = x^4 + 4x^2 + 5$$

15. $$\int v\,dv = \int 6s^2\,ds$$
$$\frac{v^2}{2} = 2s^3 + c_1$$
$$v^2 = 4s^3 + c$$
$$6^2 = 4(2)^3 + c$$
$$36 = 32 + c$$
$$c = 4$$
$$\therefore \quad v^2 = 4s^3 + 4$$
$$v^2 = 4(3)^3 + 4$$
$$= 112$$
$$v = \pm\sqrt{112}$$
$$= 4\sqrt{7}$$

(Ignore the negative solution, because v is a speed so can not be negative.)

16. $$y\frac{\mathrm{d}y}{\mathrm{d}x} = -\sin x$$
$$\int y\,dy = \int -\sin x\,\mathrm{d}x$$
$$y^2 = 2\cos x + c$$
$$2^2 = 2\cos(\pi/3) + c$$
$$4 = 1 + c$$
$$c = 3$$
$$\therefore \quad y^2 = 2\cos x + 3$$

(a) $$a^2 = 2\cos\pi + 3$$
$$a = \sqrt{-2+3}$$
$$= 1$$
$\therefore$ Point A is $(\pi, 1)$

(b) $$b^2 = 2\cos(\pi/6) + 3$$
$$b = \sqrt{\sqrt{3}+3}$$
$\therefore$ Point B is $(\pi/6, \sqrt{\sqrt{3}+3})$

At B, $$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\sin(\pi/6)}{\sqrt{\sqrt{3}+3}}$$
$$= -\frac{1/2}{\sqrt{\sqrt{3}+3}}$$
$$= -\frac{1}{2\sqrt{\sqrt{3}+3}}$$

17. $$2V\frac{\mathrm{d}V}{\mathrm{d}t} = 25$$
$$\int 2V\,dV = \int 25\,dt$$
$$V^2 = 25t + c$$
$$20^2 = 25(0) + c$$
$$c = 400$$
$$\therefore \quad V^2 = 25t + 400$$

(a) $$V^2 = 25(20) + 400$$
$$= 900$$
$$V = 300\text{cm}^3$$

(b) $$40^2 = 25t + 400$$
$$1600 = 25t + 400$$
$$1200 = 25t$$
$$t = 48$$

Exercise 7F

1. $$\int_0^2 10x^4\,\mathrm{d}x = \left[2x^5\right]_0^2 = 2(2^5) - 2(0^5) = 64$$

2. $$\int_2^4 2\,\mathrm{d}x = [2x]_2^4 = 2\times 4 - 2\times 2 = 4$$

3. $$\int_1^3 x^2\,\mathrm{d}x = \left[\frac{1}{3}x^3\right]_1^3 = \frac{1}{3}(3^3 - 1^3) = \frac{26}{3}$$

4. $$\int_{-1}^1 (2x-3)\,\mathrm{d}x = \left[x^2 - 3x\right]_{-1}^1 = ((1)^2 - 3(1)) - ((-1)^2 - 3(-1)) = -2 - 4 = -6$$

5. It is not necessary to do any work for this. Any integral with equal upper and lower bounds is equal to zero.

6. $$\int_2^3 (2+6x)\,\mathrm{d}x = \left[2x + 3x^2\right]_2^3 = (2(3) + 3(3^2)) - (2(2) + 3(2^2)) = 33 - 16 = 17$$

7. $$\int_{-1}^2 (x-1)^2\,\mathrm{d}x = \left[\frac{1}{3}(x-1)^3\right]_{-1}^2 = \frac{1}{3}((2-1)^3 - (-1-1)^3) = \frac{1}{3}(1-(-8)) = 3$$

8. $$\int_0^1 (2x-1)^4\,\mathrm{d}x = \frac{1}{2}\int_0^1 2(2x-1)^4\,\mathrm{d}x = \frac{1}{2}\left[\frac{1}{5}(2x-1)^5\right]_0^1 = \frac{1}{10}((2(1)-1)^5 - (2(0)-1)^5) = \frac{1}{10}(1-(-1)) = 0.2$$

9. $u = x+1 \quad du = dx$
   $x = u - 1 \quad dx = du$

$$\int_3^8 (\frac{1}{\sqrt{x+1}}\,\mathrm{d}x = \int_{u=4}^9 \frac{1}{\sqrt{u}}\,du = \left[2\sqrt{u}\right]_4^9 = 6 - 4 = 2$$

10. $$\int_0^{\frac{\pi}{2}} \sin x\,\mathrm{d}x = [-\cos x]_0^{\frac{\pi}{2}} = -(0-1) = 1$$

11. $$\int_0^{\frac{\pi}{2}} \cos x\,\mathrm{d}x = [\sin x]_0^{\frac{\pi}{2}} = (1-0) = 1$$

12. $$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2 x\,\mathrm{d}x = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} -0.5(1 - 2\sin^2 x - 1)\,\mathrm{d}x = -0.5(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos 2x - 1\,\mathrm{d}x = -0.5\left[\frac{\sin 2x}{2} - x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$
$$= -0.5\left(\frac{\sin\frac{2\pi}{3}}{2} - \frac{\pi}{3} - \frac{\sin\frac{\pi}{3}}{2} + \frac{\pi}{6}\right) = -0.5\left(\frac{\sqrt{3}}{4} - \frac{\pi}{3} - \frac{\sqrt{3}}{4} + \frac{\pi}{6}\right) = -0.5\left(-\frac{\pi}{6}\right) = \frac{\pi}{12}$$

13. $$\int_{\frac{\pi}{2}}^{\pi} \cos\frac{x}{2}\,\mathrm{d}x = \left[2\sin\frac{x}{2}\right]_{\frac{\pi}{2}}^{\pi} = 2\left(\sin\frac{\pi}{2} - \sin\frac{\pi}{4}\right) = 2\left(1 - \frac{\sqrt{2}}{2}\right) = 2 - \sqrt{2}$$

14. $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 4\cos^2 x\sin x\,\mathrm{d}x = \left[\frac{4}{3}\cos^3 x\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{4}{3}\left(\cos^3\frac{\pi}{4} - \cos^3 -\frac{\pi}{4}\right) = 0$$

15. $\int 3x^2\,\mathrm{d}x = x^3 + c$ so

   (a) $\int_0^1 3x^2\,\mathrm{d}x = \left[x^3\right]_0^1 = 1^3 - 0^3 = 1$

   (b) $\int_1^3 3x^2\,\mathrm{d}x = \left[x^3\right]_1^3 = 3^3 - 1^3 = 26$

   (c) $\int_0^3 3x^2\,\mathrm{d}x = \left[x^3\right]_0^3 = 3^3 - 0^3 = 27$

16. $\int 2x + x^2\,\mathrm{d}x = x^2 + \frac{x^3}{3} + c$ so

(a)
$$\begin{aligned}\int_0^3 2x + x^2\,\mathrm{d}x &= \left[x^2 + \frac{x^3}{3}\right]_0^3\\ &= 3^2 + \frac{3^3}{3} - 0^2 - \frac{0^3}{3}\\ &= 9 + 9 - 0 - 0\\ &= 18\end{aligned}$$

(b)
$$\begin{aligned}\int_3^4 2x + x^2\,\mathrm{d}x &= \left[x^2 + \frac{x^3}{3}\right]_3^4\\ &= 4^2 + \frac{4^3}{3} - 3^2 - \frac{3^3}{3}\\ &= 16 + \frac{64}{3} - 9 - 9\\ &= \frac{58}{3}\end{aligned}$$

(c)
$$\begin{aligned}\int_0^4 2x + x^2\,\mathrm{d}x &= \left[x^2 + \frac{x^3}{3}\right]_0^4\\ &= 4^2 + \frac{4^3}{3} - 0^2 - \frac{0^3}{3}\\ &= 16 + \frac{64}{3} - 0 - 0\\ &= \frac{112}{3}\end{aligned}$$

17. $\int \sin x\,\mathrm{d}x = -\cos x + c$ so

(a)
$$\begin{aligned}\int_0^{\frac{\pi}{4}} \sin x\,\mathrm{d}x &= [-\cos x]_0^{\frac{\pi}{4}}\\ &= -\cos\frac{\pi}{4} + \cos 0\\ &= 1 - \frac{\sqrt{2}}{2}\end{aligned}$$

(b)
$$\begin{aligned}\int_{-\frac{\pi}{4}}^0 \sin x\,\mathrm{d}x &= [\cos x]_{-\frac{\pi}{4}}^0\\ &= -\cos 0 + \cos -\frac{\pi}{4}\\ &= \frac{\sqrt{2}}{2} - 1\end{aligned}$$

18. (a) $\int_0^4 x\,\mathrm{d}x = \left[\frac{x^2}{2}\right]_0^4 = \frac{4^2}{2} - \frac{0^2}{2} = 8$

(b) $\int_0^4 3x\,\mathrm{d}x = 3\int_0^4 x\,\mathrm{d}x = 3 \times 8 = 24$

(c) $\int_0^4 5x\,\mathrm{d}x = 5\int_0^4 x\,\mathrm{d}x = 5 \times 8 = 40$

19. $u = 2x + 1 \quad du = 2dx$

$x = \dfrac{u-1}{2} \quad dx = \dfrac{du}{2}$

$$\begin{aligned}\int_0^1 16(2x+1)^3\,\mathrm{d}x &= 16\int_{u=1}^3 u^3\,\frac{du}{2}\\ &= 16\left[\frac{u^4}{8}\right]_1^3\\ &= 16\left(\frac{81-1}{8}\right)\\ &= 160\end{aligned}$$

20. Using the same substitutions as the previous question,

$$\begin{aligned}\int_0^1 16x(2x+1)^3\,\mathrm{d}x &= 16\int_{u=1}^3 \left(\frac{u-1}{2}\right) u^3\,\frac{du}{2}\\ &= 4\int_{u=1}^3 u^4 - u^3\,du\\ &= 4\left[\frac{u^5}{5} - \frac{u^4}{4}\right]_1^3\\ &= 4\left(\frac{243}{5} - \frac{81}{4} - \frac{1}{5} + \frac{1}{4}\right)\\ &= \frac{968}{5} - \frac{400}{5}\\ &= \frac{568}{5}\\ &= 113.6\end{aligned}$$

21. $u = x + 5 \quad du = dx$

$x = u - 5 \quad dx = du$

$$\begin{aligned}\int_0^1 \frac{6x}{25}(x+5)^4\,\mathrm{d}x &= \int_{u=5}^6 \frac{6(u-5)}{25}(u)^4\,du\\ &= \int_5^6 \frac{6u^5}{25} - \frac{6u^4}{5}\,du\\ &= \left[\frac{u^6}{25} - \frac{6u^5}{25}\right]_5^6\\ &= \frac{1}{25}\left[u^5(u-6)\right]_5^6\\ &= \frac{1}{25}\left(6^5(6-6) - 5^5(5-6)\right)\\ &= \frac{1}{25} \times 5^5\\ &= 125\end{aligned}$$

22. $u = \sin x \quad du = \cos x\,\mathrm{d}x$

$$\begin{aligned}\int_0^{\frac{\pi}{2}} 12\sin^5 x\cos x\,\mathrm{d}x &= \int_{u=0}^1 12u^5\,du\\ &= \left[2u^6\right]_0^1\\ &= 2(1-0)\\ &= 2\end{aligned}$$

23. $u = 5x + 6 \quad du = 5dx$

$$x = \frac{u-6}{5} \quad dx = \frac{du}{5}$$

$$\begin{aligned}\int_2^6 \frac{3x}{\sqrt{5x+6}}\,\mathrm{d}x &= \int_{u=16}^{36} \frac{\frac{3(u-6)}{5}}{\sqrt{u}}\frac{du}{5} \\ &= \int_{16}^{36} \frac{3u-18}{25\sqrt{u}}\,du \\ &= \int_{16}^{36} \frac{3\sqrt{u}}{25} - \frac{18}{25\sqrt{u}}\,du \\ &= \left[\frac{3u^{\frac{3}{2}}}{25}\times\frac{2}{3} - \frac{18u^{\frac{1}{2}}}{25}\times\frac{2}{1}\right]_{16}^{36} \\ &= \left[\frac{2u^{\frac{3}{2}}}{25} - \frac{36u^{\frac{1}{2}}}{25}\right]_{16}^{36} \\ &= \frac{2}{25}\left[\sqrt{u}(u-18)\right]_{16}^{36} \\ &= \frac{2}{25}\left(6(36-18) - 4(16-18)\right) \\ &= \frac{2}{25}(108+8) \\ &= \frac{232}{25} \\ &= 9.28\end{aligned}$$

24. $u = x - 1 \quad du = dx$

$$x = u + 1 \quad dx = du$$

$$\begin{aligned}\int_2^5 \frac{x+3}{\sqrt{x-1}}\,\mathrm{d}x &= \int_{u=1}^{4} \frac{u+4}{\sqrt{u}}\,du \\ &= \int_1^4 \sqrt{u} + \frac{4}{\sqrt{u}}\,du \\ &= \left[u^{\frac{3}{2}}\times\frac{2}{3} + 4u^{\frac{1}{2}}\times\frac{2}{1}\right]_1^4 \\ &= \left[\frac{2}{3}u^{\frac{3}{2}} + 8u^{\frac{1}{2}}\right]_1^4 \\ &= \frac{2}{3}\left[u^{\frac{3}{2}} + 12u^{\frac{1}{2}}\right]_1^4 \\ &= \frac{2}{3}\left[\sqrt{u}(u+12)\right]_1^4 \\ &= \frac{2}{3}\left(2(4+12) - 1(1+12)\right) \\ &= \frac{2}{3}(32-13) \\ &= \frac{38}{3} \\ &= 12\frac{2}{3}\end{aligned}$$

Exercise 7G

1. Area is always positive, but

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x$$

is signed, so they will only be equal where

$$\mathrm{f}(x) \geq 0 \quad \forall x \in [a,b]$$

i.e. (a), (b), (e), and (f)

2. Since $3x^2 > 0$ for $x \in [1,3]$, the area is given by

$$A = \int_1^3 3x^2\,\mathrm{d}x = \left[x^3\right]_1^3 = 27 - 1 = 26 \text{ units}^2$$

3. $y = 4 - x^2$ crosses the $x-$axis at $x = 2$, so

(a)
$$\begin{aligned} A &= \left|\int_0^2 4 - x^2\,\mathrm{d}x\right| \\ &= \left|\left[4x - \frac{x^3}{3}\right]_0^2\right| \\ &= \left|0 - 8 + \frac{8}{3}\right| \\ &= \frac{16}{3} \quad = 5\frac{1}{3} \text{ units}^2\end{aligned}$$

(b)
$$\begin{aligned} A &= \left|\int_0^2 4 - x^2\,\mathrm{d}x\right| + \left|\int_2^3 4 - x^2\,\mathrm{d}x\right| \\ &= \frac{16}{3} + \left|\left[4x - \frac{x^3}{3}\right]_2^3\right| \\ &= \frac{16}{3} + \left|8 - \frac{8}{3} - 12 + 9\right| \\ &= \frac{16}{3} + \frac{7}{3} \\ &= \frac{23}{3} \quad = 7\frac{2}{3} \text{ units}^2\end{aligned}$$

4. (a) Area$=\left|[x^4]_0^2\right| = 16$ units^2

(b) Area$=\left|[x^4]_{-2}^0\right| + \left|[x^4]_0^2\right| = 32$ units^2

5. $y = \sin x$ is positive between $x = 0$ and $x = \pi$ so

(a) Area$=[-\cos x]_0^{\frac{\pi}{2}} = 0 + 1 = 1$ units^2

(b) Area$=[-\cos x]_0^{\pi} = 1 + 1 = 2$ units^2

6. $x-$intercepts are at $x = 0$ and $x = 1$ so we must find the area in two pieces: $x \in [-1, 0]$ and

$x \in [0, 1]$.

$$\begin{aligned}\int 4x(1-x^2)^3\,\mathrm{d}x &= -2\int -2x(1-x^2)^3\,\mathrm{d}x\\ &= -2\frac{(1-x^2)^4}{4}+c\\ &= -0.5(1-x^2)^4+c\end{aligned}$$

$$\begin{aligned}A &= \left|\int_{-1}^{0} y\,\mathrm{d}x\right| + \left|\int_{0}^{1} y\,\mathrm{d}x\right|\\ &= \left|\left[-0.5(1-x^2)^4\right]_{-1}^{0}\right| + \left|\left[-0.5(1-x^2)^4\right]_{0}^{1}\right|\\ &= \left|-0.5(1)^4+0.5(0)^4\right| + \left|-0.5(0)^4+0.5(1)^4\right|\\ &= |-0.5|+|0.5|\\ &= 1 \text{ unit}^2\end{aligned}$$

7. $x-$intercepts are at $x = \mp\sqrt{5}$.

$$\begin{aligned}A &= \int_{-\sqrt{5}}^{\sqrt{5}} 5-x^2\,\mathrm{d}x\\ &= \left[5x-\frac{x^3}{3}\right]_{-\sqrt{5}}^{\sqrt{5}}\\ &= \left(5\sqrt{5}-\frac{(\sqrt{5})^3}{3}+5\sqrt{5}+\frac{(-\sqrt{5})^3}{3}\right)\\ &= \left(10\sqrt{5}-\frac{5\sqrt{5}}{3}-\frac{5\sqrt{5}}{3}\right)\\ &= \sqrt{5}\left(10-\frac{10}{3}\right)\\ &= \frac{20\sqrt{5}}{3} \text{ units}^2\end{aligned}$$

8. (a)
$$\begin{aligned}\int_0^2 \left((2x-1)^3-8\right)dx &= \left[\frac{(2x-1)^4}{8}-8x\right]_0^2\\ &= \frac{3^4}{8}-16-\frac{(-1)^4}{8}\\ &= \frac{81}{8}-16-\frac{1}{8}\\ &= -6\end{aligned}$$

(b)
$$\begin{aligned}\left|\int_0^2 \left((2x-1)^3-8\right)dx\right| &= |-6|\\ &= 6\end{aligned}$$

(c) Roots are given by

$$\begin{aligned}(2x-1)^3-8 &= 0\\ (2x-1)^3 &= 8\\ 2x-1 &= 2\\ 2x &= 3\\ x &= 1.5\end{aligned}$$

Thus the area is to be found in two parts:

$$\begin{aligned}A_1 &= \left|\int_0^{1.5} \left((2x-1)^3-8\right)dx\right|\\ &= \left|\left[\frac{(2x-1)^4}{8}-8x\right]_0^{1.5}\right|\\ &= \left|\frac{2^4}{8}-12-\frac{1}{8}\right|\\ &= \left|2-12-\frac{1}{8}\right|\\ &= 10\frac{1}{8}\\ A_2 &= \left|\int_{1.5}^{2} \left((2x-1)^3-8\right)dx\right|\\ &= \left|\left[\frac{(2x-1)^4}{8}-8x\right]_{1.5}^{2}\right|\\ &= \left|\frac{3^4}{8}-16-\frac{2^4}{8}+12\right|\\ &= \left|\frac{81}{8}-16-2+12\right|\\ &= 4\frac{1}{8}\\ A &= A_1+A_2\\ &= 14.25 \text{ units}^2\end{aligned}$$

9. We know that $y = \sin^2 x$ is non-negative for all x. Referring to earlier work (e.g. 7B question 38) to integrate sin^2x we get

$$\begin{aligned}A &= \int_0^{2\pi} \sin^2 x\,\mathrm{d}x\\ &= \left[\frac{x}{2}-\frac{\sin 2x}{4}\right]_0^{2\pi}\\ &= (\pi-0-0+0)\\ &= \pi \text{ units}^2\end{aligned}$$

10. We want $\int_a^c |\mathrm{f}(x)-\mathrm{g}(x)|\,\mathrm{d}x$, so this immediately includes both (e) and (g). Between a and b $\mathrm{f}(x) \geq \mathrm{g}(x)$ and between b and c $\mathrm{g}(x) \geq \mathrm{f}(x)$ so both parts of (b) are positive so we include that. Both (d) and (f) will give us the difference between the area from a to b and that from b to c and (c) gives us the absolute value of this difference, so these are wrong. Part (a) deals with areas between one or other curve and the $x-$ axis so this is entirely wrong.

11. $$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} |2\sin x - \sin x| \, dx \\ &= \int_0^{\frac{\pi}{2}} |\sin x| \, dx \\ &= \int_0^{\frac{\pi}{2}} \sin x \, dx \\ &= [-\cos x]_0^{\frac{\pi}{2}} \\ &= 0 + 1 \\ &= 1 \text{ unit}^2 \end{aligned}$$

12. First find where the curves intersect:

$$\begin{aligned} x^2 + 3 &= x + 5 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \\ x &= -1 \\ \text{or } x &= 2 \end{aligned}$$

Decide which curve is greater between $x = -1$ and $x = 2$ by choosing any convenient point. At $x = 0$, $x + 5 > x^2 + 3$. Thus the area is given by

$$\begin{aligned} A &= \int_{-1}^{2} (x+5) - (x^2+3) \, dx \\ &= \left[\frac{x^2}{2} - \frac{x^3}{3} + 2x \right]_{-1}^{2} \\ &= (2 - \frac{8}{3} + 4) - (\frac{1}{2} + \frac{1}{3} - 2) \\ &= 6 - 2\frac{2}{3} - \frac{1}{2} - \frac{1}{3} + 2 \\ &= 4.5 \text{ units}^2 \end{aligned}$$

13. First determine the points of intersection at $x \in \{-1, 2, 5\}$.

$$\begin{aligned} A_1 &= \left| \int_{-1}^{2} \left(x^3 - 5x^2 + 6x - (x^2 + 3x - 10) \right) dx \right| \\ &= \left| \int_{-1}^{2} \left(x^3 - 6x^2 + 3x + 10 \right) dx \right| \\ &= \left| \left[\frac{x^4}{4} - 2x^3 + \frac{3x^2}{2} + 10x \right]_{-1}^{2} \right| \\ &= \left| 4 - 16 + 6 + 20 - (\frac{1}{4} + 2 + \frac{3}{2} - 10) \right| \\ &= |14 - (-6.25)| \\ &= 20.25 \\ A_2 &= \left| \int_{2}^{5} \left(x^3 - 5x^2 + 6x - (x^2 + 3x - 10) \right) dx \right| \\ &= \left| \left[\frac{x^4}{4} - 2x^3 + \frac{3x^2}{2} + 10x \right]_{2}^{5} \right| \\ &= |-6.25 - 14| \\ &= 20.25 \\ A &= A_1 + A_2 \\ &= 40.5 \text{ units}^2 \end{aligned}$$

14. First determine the points of intersection: $x^2 = 8$, i.e. $x = \mp 2\sqrt{2}$

$$\begin{aligned} A &= \left| \int_{-2\sqrt{2}}^{2\sqrt{2}} \left(x^2 - 8 \right) dx \right| \\ &= \left| \left[\frac{x^3}{3} - 8x \right]_{-2\sqrt{2}}^{2\sqrt{2}} \right| \\ &= \left| \frac{16\sqrt{2}}{3} - 16\sqrt{2} - \left(-frac16\sqrt{2}3 + 16\sqrt{2} \right) \right| \\ &= \left| \frac{32\sqrt{2}}{3} - 32\sqrt{2} \right| \\ &= \frac{64\sqrt{2}}{3} \text{ units}^2 \end{aligned}$$

15. (a) $$\begin{aligned} A &= \int_0^{\frac{5\pi}{6}} \left(\sin x - \frac{3x}{5\pi} \right) dx \\ &= \left[-\cos x - \frac{3x^2}{10\pi} \right]_0^{\frac{5\pi}{6}} \\ &= \left(-\cos\frac{5\pi}{6} - \frac{3\left(\frac{5\pi}{6}\right)^2}{10\pi} \right) \\ &\quad - \left(-\cos 0 - \frac{3\,(0)^2}{10\pi} \right) \\ &= \left(\frac{\sqrt{3}}{2} - \frac{75\pi^2}{360\pi} \right) + 1 \\ &= \frac{\sqrt{3}}{2} - \frac{5\pi}{24} + 1 \text{ units}^2 \end{aligned}$$

(b) It should be clear from your graph that the line and curve enclose only two regions. From the symmetry of the curve and line it should also be clear that these two regions have equal area. Thus the total area enclosed is

$$\begin{aligned} A &= 2\left(\frac{\sqrt{3}}{2} - \frac{5\pi}{24} + 1 \right) \\ &= \sqrt{3} - \frac{5\pi}{12} + 2 \text{ units}^2 \end{aligned}$$

16. (a) By the null factor theorem points A, B and C must be where either $\cos x = 0$ or $\sin x = 0$, i.e. A= $(\frac{\pi}{2}, 0)$, B= $(\pi, 0)$ and C= $(\frac{3\pi}{2}, 0)$.

(b) $$\int 6\cos x \sin^2 x \, dx = 2\sin^3 x + c$$

$$\begin{aligned}\therefore \quad A_1 &= \int_0^{\frac{\pi}{2}} 6\cos x \sin^2 x \,\mathrm{d}x\\ &= \left[2\sin^3 x\right]_0^{\frac{\pi}{2}}\\ &= 2-0\\ &= 2\end{aligned}$$

$$\begin{aligned}\text{and} \quad A_2 &= -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 6\cos x \sin^2 x \,\mathrm{d}x\\ &= -\left[2\sin^3 x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\\ &= -(-2-2)\\ &= 4\end{aligned}$$

$$\therefore \quad \text{total } A = A_1 + A_2 = 6 \text{ units}^2$$

17.
$$\begin{aligned}a &= 0^3 - 3(0)^2 + 3(0) + 0.8\\ &= 0.8\text{m}\\ b &= 2.2^3 - 3(2.2)^2 + 3(2.2) + 0.8\\ &= 3.528\text{m}\\ A &= \int_0^{2.2} x^3 - 3x^2 + 3x + 0.8 \,\mathrm{d}x\\ &= \left[0.25x^4 - x^3 + 1.5x^2 + 0.8x\right]_0^{2.2}\\ &= (0.25(2.2)^4 - (2.2)^3 + 1.5(2.2)^2 + 0.8(2.2))\\ &\quad - (0.25(0)^4 - (0)^3 + 1.5(0)^2 + 0.8(0))\\ &= 4.2284\text{m}^2\end{aligned}$$

18. (a) First consider the point $(5, 4)$ from the persepctive of the curved edge:

$$\begin{aligned}y &= ax^2\\ 4 &= a(5)^2\\ a &= \frac{4}{25}\\ &= 0.16\end{aligned}$$

The gradient at $(5, 4)$ is

$$\begin{aligned}m &= \frac{\mathrm{d}y}{\mathrm{d}x}\\ &= 2ax\\ &= 2 \times 0.16 \times 5\\ &= 1.6\end{aligned}$$

The equation of the line is

$$\begin{aligned}y &= 1.6x + c\\ 4 &= 1.6(5) + c\\ 4 &= 8 + c\\ c &= -4\end{aligned}$$

Now d:

$$\begin{aligned}d &= 1.6(10) - 4\\ &= 12\text{m}\end{aligned}$$

(b) The area can be found by considering the curved section and straight section separately:

$$\begin{aligned}A_1 &= \int_0^5 0.16x^2 \,\mathrm{d}x\\ &= \left[\frac{0.16x^3}{3}\right]_0^5\\ &= \frac{20}{3}\\ A_2 &= \int_5^{10} 1.6x - 4 \,\mathrm{d}x\\ &= \left[0.8x^2 - 4x\right]_5^{10}\\ &= 80 - 40 - (20 - 20)\\ &= 40\\ A &= A_1 + A_2\\ &= 46\frac{2}{3}\text{m}^2\end{aligned}$$

19. First find the points of intersection between the curve and the line.

$$\begin{aligned}\frac{60x - x^2}{25} &= 20\\ 60x - x^2 &= 500\\ x^2 - 60x + 500 &= 0\\ (x-10)(x-50) &= 0\\ x &= 10\\ \text{or } x &= 50\end{aligned}$$

Now find the $x-$values where the curve meets the water level.

$$\begin{aligned}\frac{60x - x^2}{25} &= 0\\ x(60 - x) &= 0\\ x &= 0\\ \text{or } x &= 60\end{aligned}$$

(which values could have been deduced from the symmetry of the problem).

Now find the area in three parts:

$$\begin{aligned}
A_1 &= \int_0^{10} 20 - \frac{60x - x^2}{25}\,\mathrm{d}x \\
&= \left[20x - \frac{60x^2}{50} + \frac{x^3}{75}\right]_0^{10} \\
&= 200 - 120 + 13.33 \\
&= 93.33 \\
A_2 &= \int_{10}^{50} \frac{60x - x^2}{25} - 20\,\mathrm{d}x \\
&= \left[\frac{6x^2}{5} - \frac{x^3}{75} - 20x\right]_{10}^{50} \\
&= 3000 - 1666.67 - 1000 - (-93.33) \\
&= 426.67 \\
A_3 &= A_1 \text{ by symmetry} \\
&= 93.33 \\
A_{\text{total}} &= A_1 + A_2 + A_3 \\
&= 613\text{m}^2 \text{ to the nearest square metre}
\end{aligned}$$

20. The coordinates of point B in the cross-section as shown are $(30, 40)$. This gives the coordinate of the curve as

$$\begin{aligned}
y &= ax^2 \\
40 &= a(30)^2 \\
a &= \frac{2}{45} \\
\therefore \quad y &= \frac{2x^2}{45}
\end{aligned}$$

From this the area of region ABC is

$$\begin{aligned}
A &= \int_0^{30} 40 - \frac{2x^2}{45}\,\mathrm{d}x \\
&= \left[40x\frac{2x^3}{135}\right]_0^{30} \\
&= 800\text{cm}^2
\end{aligned}$$

Hence the total area of the cross section is 1600cm^2 and the capacity is

$$\begin{aligned}
V &= 1600 \times 200 \\
&= 320\,000\text{cm}^3 \\
&= 320\text{L or } 0.32\text{m}^3
\end{aligned}$$

21.
$$\begin{aligned}
A &= \int_0^{60} \frac{600 + 60x - x^2}{30} - \frac{4500 + 60x - x^2}{225} dx \\
&= \int_0^{60} 20 + 2x - \frac{x^2}{30} - \left(20 + \frac{4x}{15} - \frac{x^2}{225}\right) dx \\
&= \int_0^{60} \frac{26x}{15} - \frac{13x^2}{450}\,\mathrm{d}x \\
&= \left[\frac{13x^2}{15} - \frac{13x^3}{1350}\right]_0^{60} \\
&= 1040\text{m}^2
\end{aligned}$$

22. The first two $x-$intercepts can be easily determined to be where $x = 0$ and $x = 1$. For the third,

$$\begin{aligned}
\sin\left(\frac{4\pi}{3}(x-1)\right) &= 0 \\
\frac{4\pi}{3}(x-1) &= \pi \\
x - 1 &= \frac{3}{4} \\
x &= 1.75
\end{aligned}$$

Thus the area is

$$\begin{aligned}
A_1 &= \left|\int_0^1 -\sin(\pi x)\,\mathrm{d}x\right| \\
&= \left|\left[\frac{\cos(\pi x)}{\pi}\right]_0^1\right| \\
&= \left|\frac{\cos\pi - \cos 0}{\pi}\right| \\
&= \frac{2}{\pi} \\
A_2 &= \left|\int_1^{1.75} -\frac{1}{2}\sin\left(\frac{4\pi}{3}(x-1)\right)\,\mathrm{d}x\right| \\
&= \left|\left[\frac{3\cos\left(\frac{4\pi}{3}(x-1)\right)}{8\pi}\right]_1^{1.75}\right| \\
&= \frac{3}{8\pi}\left|\frac{\cos\pi - \cos 0}{\pi}\right| \\
&= \frac{3}{4\pi} \\
A_{\text{total}} &= A_1 + A_2 \\
&= \frac{11}{4\pi} \text{ units}^2
\end{aligned}$$

Miscellaneous Exercise 7

1. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 3(2x+1)^2(2)\\ &= 6(2x+1)^2\end{aligned}$$

2. $$\frac{\mathrm{d}y}{\mathrm{d}x} = -12\sin 3x + 12\cos 4x$$

3. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(4\sin^3 x\cos x)x - \sin^4 x}{x^2}\\ &= \frac{\sin^3 x(4x\cos x - \sin x}{x^2}\end{aligned}$$

4. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2\cos x(1+\cos x) - (1+2\sin x)(-\sin x)}{(1+\cos x)^2}\\ &= \frac{2\cos x + 2\cos^2 x + \sin x + 2\sin^2 x}{(1+\cos x)^2}\\ &= \frac{2\cos x + \sin x + 2(\cos^2 x + \sin^2 x)}{(1+\cos x)^2}\\ &= \frac{2\cos x + \sin x + 2}{(1+\cos x)^2}\end{aligned}$$

5. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2\cos 2x(1+\sin 2x) - \sin 2x(2\cos 2x)}{(1+\sin 2x)^2}\\ &= \frac{2\cos 2x + 2\sin 2x\cos 2x - 2\sin 2x\cos 2x}{(1+\sin 2x)^2}\\ &= \frac{2\cos 2x}{(1+\sin 2x)^2}\end{aligned}$$

6. $$\begin{aligned}5y + 5x\frac{\mathrm{d}y}{\mathrm{d}x} + 6y^2\frac{\mathrm{d}y}{\mathrm{d}x} &= 6x\\ 5x\frac{\mathrm{d}y}{\mathrm{d}x} + 6y^2\frac{\mathrm{d}y}{\mathrm{d}x} &= 6x - 5y\\ (5x+6y^2)\frac{\mathrm{d}y}{\mathrm{d}x} &= 6x - 5y\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{6x-5y}{5x+6y^2}\end{aligned}$$

7. $$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= 6t - 5\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -12t^2\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}\\ &= \frac{-12t^2}{6t-5}\end{aligned}$$

8. $$\begin{aligned}\cos y + x(-\sin y\frac{\mathrm{d}y}{\mathrm{d}x}) &= \frac{\mathrm{d}y}{\mathrm{d}x}\sin x + y\cos x\\ \cos y - y\cos x &= \frac{\mathrm{d}y}{\mathrm{d}x}(\sin x + x\sin y)\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\cos y - ycosx}{\sin x + x\sin y}\end{aligned}$$

9. Given $\sin A = \frac{3}{5}$, we can first deduce $\cos A = \pm\frac{4}{5}$ (depending on whether A is in the first or second quadrant).

   (a) $$\begin{aligned}\sin 2A &= 2\sin A\cos A\\ &= 2\left(\frac{3}{5}\right)\left(\pm\frac{4}{5}\right)\\ &= \pm\frac{24}{25}\end{aligned}$$

   (b) $$\begin{aligned}\cos 2A &= 1 - 2\sin^2 A\\ &= 1 - 2\left(\frac{3}{5}\right)^2\\ &= \pm\frac{7}{25}\end{aligned}$$

   (c) $$\begin{aligned}\tan 2A &= \frac{\sin 2A}{\cos 2A}\\ &= \pm\frac{24}{7}\end{aligned}$$

10. (a) $$\begin{aligned}2x + y + x\frac{\mathrm{d}y}{\mathrm{d}x} &= 2y\frac{\mathrm{d}y}{\mathrm{d}x}\\ 2x + y &= (2y - x)\frac{\mathrm{d}y}{\mathrm{d}x}\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x+y}{2y-x}\\ &= \frac{4+3}{6-2}\\ &= \frac{7}{4}\\ (y - y_1) &= m(x - x_1)\\ y - 3 &= \frac{7}{4}(x-2)\\ 4y - 12 &= 7x - 14\\ 7x - 4y &= 2\end{aligned}$$

    (b) $$\begin{aligned}3x^2 + 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x^2}{y^2}\\ &= -\frac{4}{9}\\ (y - y_1) &= m(x - x_1)\\ y - 3 &= -\frac{4}{9}(x-2)\\ 9(y-3) &= -4(x-2)\\ 9y - 27 &= -4x + 8\\ 4x + 9y &= 35\end{aligned}$$

11. (a) $$\frac{4\sin 8x}{8} + c = 0.5\sin 8x + c$$

    (b) $$\frac{(3+x^2)^6}{6} + c$$

    (c) Let $u = x + 3 \quad du = dx$
    $x = u - 3 \quad dx = du$

$$\int (2-3x)\sqrt[3]{x+3}\,\mathrm{d}x$$
$$= \int (2-3(u-3))\sqrt[3]{u}\,du$$
$$= \int (2-3u+9)u^{\frac{1}{3}}\,du$$
$$= \int 11u^{\frac{1}{3}} - 3u^{\frac{4}{3}}\,du$$
$$= 11u^{\frac{4}{3}} \times \frac{3}{4} - 3u^{\frac{7}{3}} \times \frac{3}{7} + c$$
$$= \frac{33}{4}u^{\frac{4}{3}} - \frac{9}{7}u^{\frac{7}{3}} + c$$
$$= \frac{3u^{\frac{4}{3}}}{28}(77-12u) + c$$
$$= \frac{3(x+3)^{\frac{4}{3}}}{28}(77-12(x+3)) + c$$
$$= \frac{3(x+3)^{\frac{4}{3}}}{28}(41-12x) + c$$

(d) $\dfrac{\sin^6 2x}{6} \times \dfrac{1}{2} + c = \dfrac{\sin^6 2x}{1}2 + c$

(e) $$\int \sin^2 \frac{x}{2}\,\mathrm{d}x = -0.5\int -2\sin^2\frac{x}{2}\,\mathrm{d}x$$
$$= -0.5\int \left(1-2\sin^2\frac{x}{2}\right) - 1\,\mathrm{d}x$$
$$= -0.5\int \cos x - 1\,\mathrm{d}x$$
$$= -0.5(\sin x - x) + c$$
$$= \frac{x}{2} - \frac{\sin x}{2} + c$$

(f) $$\int \cos^3\frac{x}{2}\,\mathrm{d}x = \int \cos\frac{x}{2}\cos^2\frac{x}{2}\,\mathrm{d}x$$
$$= \int \cos\frac{x}{2}\left(1-\sin^2\frac{x}{2}\right)\mathrm{d}x$$
$$= \int \cos\frac{x}{2} - \cos\frac{x}{2}\sin^2\frac{x}{2}\,\mathrm{d}x$$
$$= 2\sin\frac{x}{2} - 2\left(\frac{1}{3}\sin^3\frac{x}{2}\right) + c$$
$$= 2\sin\frac{x}{2} - \frac{2}{3}\sin^3\frac{x}{2} + c$$

12. (a) Change the sign of the argument:
$$\bar{z} = 8\operatorname{cis} -\frac{\pi}{6}$$

(b) A complex number plus its conjugate gives double the real component:
$$2 \times 8\cos\frac{\pi}{6} = 8\sqrt{3} = 8\sqrt{3}\operatorname{cis} 0$$

(c) A complex number minus its conjugate gives double the imaginary part:
$$2 \times 8\mathrm{i}\sin\frac{\pi}{6} = 8\mathrm{i} = 8\operatorname{cis}\frac{\pi}{2}$$

(d) When multiplying complex numbers in polar form, add the arguments and multiply the moduli:
$$8^2\operatorname{cis}\left(\frac{\pi}{6} + -\frac{\pi}{6}\right) = 64\operatorname{cis} 0$$

(e) When dividing complex numbers in polar form, subtract the arguments and divide the moduli:
$$\frac{8}{8}\operatorname{cis}\left(\frac{\pi}{6} - -\frac{\pi}{6}\right) = 1\operatorname{cis}\frac{\pi}{3}$$

13. L.H.S.:
$$\frac{\sin\theta}{1-\cos\theta} + \frac{1-\cos\theta}{\sin\theta}$$
$$= \frac{\sin^2\theta + (1-\cos\theta)^2}{\sin\theta(1-\cos\theta)}$$
$$= \frac{\sin^2\theta + 1 - 2\cos\theta + \cos^2\theta}{\sin\theta(1-\cos\theta)}$$
$$= \frac{2-2\cos\theta}{\sin\theta(1-\cos\theta)}$$
$$= \frac{2(1-\cos\theta)}{\sin\theta(1-\cos\theta)}$$
$$= \frac{2}{\sin\theta}$$
= R.H.S.

□

14. $$\text{L.H.S.} = \frac{\mathrm{d}}{\mathrm{d}x}(3x^2 - x + 1)$$
$$= \lim_{h\to 0}\frac{\left((3(x+h)^2 - (x+h) + 1\right) - (3x^2 - x + 1)}{h}$$
$$= \lim_{h\to 0}\frac{3x^2 + 6xh + 3h^2 - x - h + 1 - 3x^2 + x - 1}{h}$$
$$= \lim_{h\to 0}\frac{6xh + 3h^2 - h}{h}$$
$$= \lim_{h\to 0}(6x + 3h - 1)$$
$$= 6x - 1$$
= R.H.S.

□

15. To prove: for z a non-zero complex number,
$$\frac{1}{z} = \frac{\bar{z}}{|z|}$$

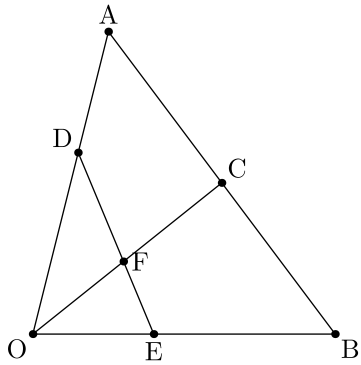
Proof:
Let $z = a + b\mathrm{i}$ for $a, b \in \Re$. Then $\bar{z} = a - b\mathrm{i}$ and $|z| = a^2 + b^2$.

$$\text{L.H.S.} = \frac{1}{z}$$
$$= \frac{1}{a+b\mathrm{i}}$$
$$= \frac{1}{a+b\mathrm{i}} \times \frac{a-b\mathrm{i}}{a-b\mathrm{i}}$$
$$= \frac{a-b\mathrm{i}}{a^2 - b^2\mathrm{i}^2}$$
$$= \frac{a-b\mathrm{i}}{a^2+b^2}$$
$$= \frac{\bar{z}}{|z|}$$
= R.H.S.

□

16. $y = \cos^3 x$ is non-negative between $x = 0$ and $x = \frac{\pi}{2}$ so the area is the simple integral

$$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} \cos^3 x \, dx \\ &= \int_0^{\frac{\pi}{2}} \cos x (cos^2 x) \, dx \\ &= \int_0^{\frac{\pi}{2}} \cos x (1 - \sin^2 x) \, dx \\ &= \int_0^{\frac{\pi}{2}} \cos x - \cos x \sin^2 x \, dx \\ &= \left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{\frac{\pi}{2}} \\ &= \left(1 - \frac{1}{3}\right) - (0 - 0) \\ &= \frac{2}{3} \text{ units}^2 \end{aligned}$$

17. 

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= \mathbf{b} - \mathbf{a} \\ \overrightarrow{AC} &= 0.5\overrightarrow{AB} \\ &= 0.5(\mathbf{b} - \mathbf{a}) \\ \overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AC} \\ &= \mathbf{a} + 0.5(\mathbf{b} - \mathbf{a}) \\ &= 0.5(\mathbf{a} + \mathbf{b}) \end{aligned}$$

(a) $$\begin{aligned} \overrightarrow{DF} &= \overrightarrow{DO} + \overrightarrow{OF} \\ &= -h\mathbf{a} + m\overrightarrow{OC} \\ &= -h\mathbf{a} + 0.5m(\mathbf{a} + \mathbf{b}) \\ &= (0.5m - h)\mathbf{a} + 0.5m\mathbf{b} \end{aligned}$$

(b) $$\begin{aligned} \overrightarrow{FE} &= \overrightarrow{FO} + \overrightarrow{OE} \\ &= -m\overrightarrow{OC} + k\mathbf{b} \\ &= -0.5m(\mathbf{a} + \mathbf{b}) + k\mathbf{b} \\ &= -0.5m\mathbf{a} + (k - 0.5m)\mathbf{b} \end{aligned}$$

(c) $$\begin{aligned} \overrightarrow{DF} &= \overrightarrow{FE} \\ (0.5m - h)\mathbf{a} + 0.5m\mathbf{b} &= -0.5m\mathbf{a} + (k - 0.5m)\mathbf{b} \\ (m - h)\mathbf{a} &= (k - m)\mathbf{b} \\ \therefore \quad m - h &= 0 \\ m &= h \\ \text{and} \quad k - m &= 0 \\ k &= m \\ \therefore \quad h = k &= m \end{aligned}$$

□

(d) $$\begin{aligned} \overrightarrow{DE} &= \overrightarrow{DO} + \overrightarrow{OE} \\ &= -h\mathbf{a} + k\mathbf{b} \\ &= h(\mathbf{b} - \mathbf{a}) \\ &= h\overrightarrow{AB} \\ \therefore \quad \overrightarrow{DE} &\parallel \overrightarrow{AB} \end{aligned}$$

□

18. (a) $$\begin{aligned} \frac{dy}{dt} &= 2t \\ \frac{dx}{dt} &= 2 - \frac{1}{t^2} \\ \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx} \\ &= \frac{2t}{2 - \frac{1}{t^2}} \\ &= \frac{2t^3}{2t^2 - 1} \end{aligned}$$

(b) $$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{6t^2(2t^2 - 1) - 2t^3(4t)}{(2t^2 - 1)^2} \times \frac{1}{2 - \frac{1}{t^2}} \\ &= \frac{12t^4 - 6t^2 - 8t^4}{(2t^2 - 1)^2} \times \frac{t^2}{2t^2 - 1} \\ &= \frac{(4t^4 - 6t^2)(t^2)}{(2t^2 - 1)^3} \\ &= \frac{2t^4(2t^2 - 3)}{(2t^2 - 1)^3} \end{aligned}$$

19. (a) For the first statement:

$$\begin{aligned} \text{L.H.S.} &= \cos(A - B) + \cos(A + B) \\ &= (\cos A \cos B + \sin A \sin B) \\ &\quad + (\cos A \cos B - \sin A \sin B) \\ &= 2\cos A \cos B \\ &= \text{R.H.S.} \end{aligned}$$

□

Similarly the second statement:

$$\begin{aligned} \text{L.H.S.} &= \cos(A - B) - \cos(A + B) \\ &= (\cos A \cos B + \sin A \sin B) \\ &\quad - (\cos A \cos B - \sin A \sin B) \\ &= 2\sin A \sin B \\ &= \text{R.H.S.} \end{aligned}$$

□

(b) From the preceding we can draw the following simplifications:

$$\begin{aligned} 2\sin 4x \sin x &= \cos(4x - x) - \cos(4x + x) \\ &= \cos 3x - \cos 5x \end{aligned}$$

and

$$\begin{aligned} 6\cos 7x \cos 2x &= 3\left(\cos(7x - 2x)\right. \\ &\quad \left. + \cos(7x + 2x)\right) \\ &= 3(\cos 5x + \cos 9x) \end{aligned}$$

From this,

i. $\int (2\sin 4x \sin x)dx$

$$= \int (\cos 3x - \cos 5x)dx$$
$$= \frac{1}{3}\sin 3x - \frac{1}{5}\sin 5x + c$$

ii. $\int (6\cos 7x \cos 2x)dx$

$$= 3\int (\cos 5x + \cos 9x)dx$$
$$= \frac{3}{5}\sin 5x + \frac{1}{3}\sin 9x + c$$

20. (a) $\int_0^{\frac{5\pi}{6}} (0.5 - \sin x)\,dx$

$$= [0.5x + \cos x]_0^{\frac{5\pi}{6}}$$
$$= \left(\frac{5\pi}{12} + \cos\frac{5\pi}{6}\right) - (0 + \cos 0)$$
$$= \frac{5\pi}{12} - \frac{\sqrt{3}}{2} - 1$$
$$= \frac{5\pi - 6\sqrt{3} - 12}{12}$$

(b) $\int_0^{\frac{5\pi}{6}} (0.5 - \sin x)\,dx$

$$= \frac{|5\pi - 6\sqrt{3} - 12|}{12}$$
$$= \frac{6\sqrt{3} + 12 - 5\pi}{12}$$

(If you had to do this without a calculator, you should be able to estimate the value of 5π (about 15 or 16) and $6\sqrt{3}$ (more than 6 and less than 12) sufficiently to be able to be confident that $5\pi - 6\sqrt{3} - 12 < 0$.)

(c) $0.5 - \sin x$ crosses the x−axis where $\sin x = 0.5$, i.e. at $x = \frac{\pi}{6}$. We need to find the area in two parts:

$$A_1 = \int_0^{\frac{\pi}{6}} (0.5 - \sin x)\,dx$$
$$= [0.5x + \cos x]_0^{\frac{\pi}{6}}$$
$$= \left(\frac{\pi}{12} + \cos\frac{\pi}{6}\right) - (0 + \cos 0)$$
$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$
$$= \frac{\pi + 6\sqrt{3} - 12}{12}$$

$$A_2 = -\int_{\frac{\pi}{6}}^{\frac{\pi}{6}} (0.5 - \sin x)\,dx$$
$$= -[0.5x + \cos x]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$
$$= -\left(\frac{5\pi}{12} + \cos\frac{5\pi}{6}\right) - \left(\frac{\pi}{12} + \frac{\sqrt{3}}{2}\right)$$
$$= -\left(\frac{5\pi}{12} - \frac{\sqrt{3}}{2} - \frac{\pi}{12} - \frac{\sqrt{3}}{2}\right)$$
$$= \frac{12\sqrt{3} - 4\pi}{12}$$

$$A_{\text{total}} = A_1 + A_2$$
$$= \frac{\pi + 6\sqrt{3} - 12}{12} + \frac{12\sqrt{3} - 4\pi}{12}$$
$$= \frac{18\sqrt{3} - 3\pi - 12}{12}$$
$$= \frac{6\sqrt{3} - \pi - 4}{4} \text{ units}^2$$

21. John's conjecture is easily disproven by finding a single counter-example (e.g. $x = 6$ or $x = 10$). However even these give a result that is a multiple of six.

Conjecture: $x^3 - x$ is a multiple of 6 for $x \geq 2$.

Proof: Consider the factorisation

$$x^3 - x = x(x-1)(x+1)$$

At least one of these factors must be even. (If x is not even, then both $x - 1$ and $x + 1$ are even.)

Similarly, and one factor must be a multiple of 3. (If x has a remainder of 1 when divided by 3, then $x - 1$ is a multiple of 3. If x has a remainder of 2 when divided by 3, then $x + 1$ is a multiple of 3. If x has no remainder when divided by 3, it is itself a multiple of 3. There are no other possibilities.)

Since $x^3 - x$ is has both a factor that is even and a factor that is a multiple of 3, it must be a multiple of 6. □

22. Let A be the given position of the shuttle, O be the position of the satellite and P be the position of the shuttle at closest approach t seconds later.

$$\overrightarrow{OP} \cdot \overrightarrow{AP} = 0$$
$$\left(\overrightarrow{OA} + \overrightarrow{AP}\right) \cdot \overrightarrow{AP} = 0$$
$$(\mathbf{r} + t\mathbf{v}) \cdot \mathbf{v} = 0$$
$$\begin{pmatrix} 30\,000 \\ -39\,000 \\ 12\,750 \end{pmatrix} \cdot \begin{pmatrix} -150 \\ 180 \\ -60 \end{pmatrix} + t\begin{pmatrix} -150 \\ 180 \\ -60 \end{pmatrix} \cdot \begin{pmatrix} -150 \\ 180 \\ -60 \end{pmatrix} = 0$$
$$-12\,285\,000 + 58\,500t = 0$$
$$t = 210\text{s}$$

$$|\overrightarrow{OP}| = |\mathbf{r} + 210\mathbf{v}|$$
$$= 1930\text{m (to the nearest 10m)}$$

Let B be the position of the shuttle at $t = 100$. Let $\mathbf{w}$ be the new velocity.

$$\overrightarrow{OB} = \mathbf{r} + 100\mathbf{v}$$
$$= \begin{pmatrix} 15\,000 \\ -21000 \\ 6\,750 \end{pmatrix} \text{m}$$
$$\mathbf{w} = -\frac{1}{12.5 \times 60}\begin{pmatrix} 15\,000 \\ -21000 \\ 6\,750 \end{pmatrix}$$
$$= \begin{pmatrix} -20 \\ 28 \\ -9 \end{pmatrix} \text{ms}^{-1}$$

23. Starting with the first expression stripped of the $+c$:

$$\frac{3}{8}x + \frac{\sin 2x}{4} + \frac{\sin 4x}{32}$$
$$= \frac{3x}{8} + \frac{2\sin x\cos x}{4} + \frac{2\sin 2x\cos 2x}{32}$$
$$= \frac{3x}{8} + \frac{\sin x\cos x}{2} + \frac{\sin 2x\cos 2x}{16}$$
$$= \frac{3x}{8} + \frac{\sin x\cos x}{2} + \frac{(2\sin x\cos x)(2\cos^2 x - 1)}{16}$$
$$= \frac{3x}{8} + \frac{\sin x\cos x}{2} + \frac{4\sin x\cos^3 x - 2\sin x\cos x}{16}$$
$$= \frac{3x}{8} + \frac{\sin x\cos x}{2} + \frac{\sin x\cos^3 x}{4} - \frac{\sin x\cos x}{8}$$
$$= \frac{\sin x\cos^3 x}{4} + \frac{3\sin x\cos x}{8} + \frac{3x}{8}$$

Chapter 8

Exercise 8A

1-9 No working required. You should be able to differentiate these by observation.

10. $$\begin{aligned}\frac{d}{dx}e^x\sqrt{x} &= e^x\sqrt{x} + e^x\left(-\frac{1}{2\sqrt{x}}\right)\\ &= \frac{e^x(2x-1)}{2\sqrt{x}}\end{aligned}$$

11. $$\begin{aligned}\frac{d}{dx}e^x\sin x &= e^x\sin x + e^x\cos x\\ &= e^x(\sin x + \cos x)\\ &= \sqrt{2}e^x\sin\left(x+\frac{\pi}{4}\right)\end{aligned}$$
The last step is not really necessary here. You should, however, understand how it is obtained. (But see question 16.)

12. $$\begin{aligned}\frac{d}{dx}e^x\cos 2x &= e^x\cos 2x - 2e^x\sin 2x\\ &= e^x(\cos 2x - 2\sin 2x)\end{aligned}$$

13. $$\begin{aligned}\frac{d}{dx}e^x\sin^2 x &= e^x\sin^2 x + e^x 2\sin x\cos x\\ &= e^x\sin x(\sin x + 2\cos x)\end{aligned}$$

14. Simply apply the chain rule and you should be able to do this in a single step. No working required.

15. Again, one step using the chain rule.

16. $\frac{dy}{dx} = \sqrt{2}e^x\sin\left(x+\frac{\pi}{4}\right)$ (see question 11). Where the gradient is zero,
$$\begin{aligned}\frac{dy}{dx} &= 0\\ \sqrt{2}e^x\sin\left(x+\frac{\pi}{4}\right) &= 0\\ \sin\left(x+\frac{\pi}{4}\right) &= 0\\ x+\frac{\pi}{4} &= 0 + k\pi \qquad (k\in\mathbb{I})\\ x &= -\frac{\pi}{4}+k\pi\\ x &\in \left\{-\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}\right\}\end{aligned}$$

17. $$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{dx}\frac{dx}{dt}\\ &= (e^x(\sin x+\cos x))(2)\\ &= 2e^x(\sin x + \cos x)\end{aligned}$$
When $x = \pi$,
$$\begin{aligned}\frac{dy}{dt} &= 2e^{\pi}(\sin\pi + \cos\pi)\\ &= -2e^{\pi}\end{aligned}$$

18. (a) $$\begin{aligned}e^x\sin y &= x\\ e^x\sin y + e^x\cos y\frac{dy}{dx} &= 1\\ e^x\cos y\frac{dy}{dx} &= 1 - e^x\sin y\\ \frac{dy}{dx} &= \frac{1-e^x\sin y}{e^x\cos y}\end{aligned}$$

(b) $$\begin{aligned}y &= x + e^{x+y}\\ \frac{dy}{dx} &= 1 + e^{x+y}\left(1+\frac{dy}{dx}\right)\\ &= 1 + e^{x+y} + e^{x+y}\frac{dy}{dx}\\ \frac{dy}{dx} - e^{x+y}\frac{dy}{dx} &= 1 + e^{x+y}\\ \frac{dy}{dx}\left(1-e^{x+y}\right) &= 1 + e^{x+y}\\ \frac{dy}{dx} &= \frac{1+e^{x+y}}{1-e^{x+y}}\end{aligned}$$

(c) $$\begin{aligned}x^2 + xy &= 2 + y^2e^x\\ 2x + y + x\frac{dy}{dx} &= 2y\frac{dy}{dx}e^x + y^2e^x\\ x\frac{dy}{dx} - 2ye^x\frac{dy}{dx} &= y^2e^x - 2x - y\\ \frac{dy}{dx}(x - 2ye^x) &= y^2e^x - 2x - y\\ \frac{dy}{dx} &= \frac{y^2e^x - 2x - y}{x - 2ye^x}\end{aligned}$$

Exercise 8B

1. No working needed: integrate by observation.

2. $\frac{d}{dx}e^{6x} = 6e^{6x}$ so $\int e^{6x}\,dx = \frac{e^{6x}}{6} + c$

3. $\frac{d}{dx}e^{2x} = 2e^{2x}$ so $\int 5e^{2x}\,dx = \frac{5e^{2x}}{2} + c$

4. $\frac{1}{e^x} = e^{-x}$ so
$$\begin{aligned}\int\frac{6}{e^x}\,dx &= \int 6e^{-x}\,dx\\ &= -6e^{-x} + c\\ &= -\frac{6}{e^x} + c\end{aligned}$$

5. $\frac{d}{dx}e^{0.5x} = 0.5e^{0.5x}$ so $\int 8e^{0.5x}\,dx = 16e^{0.5x} + c$

6. $\frac{d}{dx}\sqrt{e^x} = \frac{d}{dx}e^{0.5x} = 0.5e^{0.5x}$ so

$$\int 2\sqrt{e^x}\,dx = 4\sqrt{e^x} + c$$

7. $$\int (6e^{3x} + 2x)dx = \int 6e^{3x}\,dx + \int 2x\,dx = 2e^{3x} + x^2 + c$$

8. $$\int (2e^{3x} + 3e^{2x})dx = \int 2e^{3x}\,dx + \int 3e^{2x}\,dx = \frac{2e^{3x}}{3} + \frac{3e^{2x}}{2} + c$$

9. $\frac{d}{dx}e^{-2x} = -2e^{-2x}$ so $\int 4e^{2x}\,dx = -2e^{-2x} + c$

10. $$\begin{aligned}\int \left(\frac{3}{e^{2x}} + \frac{e^{2x}}{3}\right) dx &= \int \frac{3}{e^{2x}}\,dx + \int \frac{e^{2x}}{3}\,dx \\ &= \int 3e^{-2x}\,dx + \frac{e^{2x}}{6} + c \\ &= -\frac{3}{2}e^{-2x} + \frac{e^{2x}}{6} + c \\ &= \frac{e^{2x}}{6} - \frac{3}{2e^2x} + c\end{aligned}$$

11. $\frac{d}{dx}e^{3x^2} = 6xe^{3x^2}$ so

$$\int 12xe^{3x^2}\,dx = 2e^{3x^2} + c$$

12. $\frac{d}{dx}e^{3x-2} = 3e^{3x-2}$ so

$$\int 6xe^{3x-2}\,dx = 2e^{3x-2} + c$$

With a little practice you should be able to do problems like this by observation. Questions 11–15 are all in the form

$$\int af'(x)e^{f(x)}\,dx$$

so all that needs to be done is to determine what $f(x)$ and a are then

$$\int af'(x)e^{f(x)}\,dx = ae^{f(x)}$$

(essentially the chain rule in reverse). For questions 13–15 these solutions will simply state $f(x)$ since the rest should be obvious.

13. $f(x) = x^2 + 1$

14. $f(x) = \sin x$

15. $f(x) = 2\sin x$

To answer questions 16–20 you should use

$$\int af'(x)\,(f(x))^n = a\frac{(f(x))^{n+1}}{n+1} + c$$

For these questions these solutions will simply state $f(x)$ and $f'(x)$ since the rest should be obvious.

16. $f(x) = 1 + e^x \qquad f'(x) = e^x$

17. $f(x) = 1 + 2e^x \qquad f'(x) = 2e^x$

18. $f(x) = 1 + e^x \qquad f'(x) = e^x \qquad (n = \frac{1}{2})$

19. $f(x) = 1 - 2e^x \qquad f'(x) = -2e^x$

20. $f(x) = 2 + e^{\sin x} \qquad f'(x) = \cos x\, e^{\sin x}$

21–23 No working needed.

24. $$\begin{aligned}\int_0^3 3e^x\,dx &= [3e^x]_0^3 \\ &= 3e^3 - 3e^0 \\ &= 3(e^3 - 1)\end{aligned}$$

25. $$\begin{aligned}\int_0^2 \left(2e^x + 3e^{2x}\right) dx &= \left[2e^x + \frac{3e^{2x}}{2}\right]_0^2 \\ &= \left(2e^2 + \frac{3e^4}{2}\right) - \left(2e^0 + \frac{3e^0}{2}\right) \\ &= \left(2e^2 + \frac{3e^4}{2}\right) - \left(2 + \frac{3}{2}\right) \\ &= \left(2e^2 + \frac{3e^4}{2}\right) - \frac{7}{2} \\ &= \frac{1}{2}(4e^2 + 3e^4 - 7) \\ &= \frac{1}{2}(3e^2 + 7)(e^2 - 1)\end{aligned}$$

The factorization in the last step is not strictly necessary, but since the result factors nicely and since factor form is frequently more useful than expanded form it is reasonable to leave the result thus.

26. $$\begin{aligned}\int_0^{\frac{\pi}{3}} 8\sin x\, e^{4\cos x}\,dx &= \left[-2e^{4\cos x}\right]_0^{\frac{\pi}{3}} \\ &= (-2e^{4\cos\frac{\pi}{3}}) - (-2e^{4\cos 0}) \\ &= -2e^2 + 2e^4 \\ &= 2e^2(e^2 - 1)\end{aligned}$$

27. (a) $$\begin{aligned}f(x) &= \int 4x - 6e^{3x}\,dx \\ &= 2x^2 - 2e^{3x} + c \\ f(0) &= 3 \\ -2e^0 + c &= 3 \\ c &= 5 \\ \therefore\quad f(x) &= 2x^2 - 2e^{3x} + 5\end{aligned}$$

(b) $$\begin{aligned}f(1) &= 2 \times 1^2 - 2e^{3\times 1} + 5 \\ &= 2 - 2e^3 + 7 \\ &= 7 - 2e^3\end{aligned}$$

Exercise 8C

1-29 No working required. You should be able to do all these in a single step.

30. $\log_2 21 = \dfrac{\ln 21}{\ln 2} = \dfrac{\ln(3\times 7)}{\ln 2} = \dfrac{\ln 3 + \ln 7}{\ln 2}$

31. $\log_3 200 = \dfrac{\ln 200}{\ln 3} = \dfrac{\ln(2^3\times 5^2)}{\ln 3} = \dfrac{3\ln 2 + 2\ln 5}{\ln 3}$

32. $\log_5 50 = \log_5(5^2\times 2) = \log_5(5^2) + \log_5 2 = 2 + \dfrac{\ln 2}{\ln 5}$

33. $\log_6 9 = \dfrac{\ln 3^2}{\ln(3\times 2)} = \dfrac{2\ln 3}{\ln 3 + \ln 2}$

34. $\log_9 6 = \dfrac{\ln(3\times 2)}{\ln 3^2} = \dfrac{\ln 3 + \ln 2}{2\ln 3} = \dfrac{1}{2} + \dfrac{\ln 2}{2\ln 3}$

35. $\log_4 300 = \dfrac{\ln(2^2\times 3\times 5^2)}{\ln 2^2} = \dfrac{2\ln 2 + \ln 3 + 2\ln 5}{2\ln 2} = 1 + \dfrac{\ln 3 + 2\ln 5}{2\ln 2}$

36. $\log_8 220 = \dfrac{\ln(2^2\times 5\times 11)}{\ln 2^3} = \dfrac{2\ln 2 + \ln 5 + \ln 11}{3\ln 2} = \dfrac{2}{3} + \dfrac{\ln 5 + \ln 11}{3\ln 2}$

37. $e^{x+1} = 12$
$x + 1 = \ln 12$
$x = \ln 12 - 1$

38. $e^{x+2} = 25$
$x + 2 = \ln 25$
$x = \ln 25 - 2$

39. $e^{x-1} = 150$
$x - 1 = \ln 150$
$x = \ln 150 + 1$

40. $e^{2x+1} = 34$
$2x + 1 = \ln 34$
$x = \dfrac{\ln 34 - 1}{2}$

41. $5e^{x+1} + 3e^{x+1} = 200$
$8e^{x+1} = 200$
$e^{x+1} = 25$
$x + 1 = \ln 25$
$x = \ln 25 - 1$

42. $e^{2x} - 12e^x = -35$
$(e^x)^2 - 12e^x + 35 = 0$
$(e^x - 5)(e^x - 7) = 0$
$e^x = 5$ or $e^x = 7$
$x = \ln 5$ $\quad x = \ln 7$

43. $3\log x + \log y = \log x^3 + \log y = \log(x^3 y)$

44. $2\log x - 3\log y = \log x^2 - \log y^3 = \log\dfrac{x^2}{y^3}$

45. $2\log a + \log b - 3\log c = \log a^2 + \log b - \log c^3 = \log\dfrac{a^2 b}{c^3}$

46. $3 + \log x = \log 10^3 + \log x = \log(1000x)$

47. $2 + \ln x = \ln e^2 + \ln x = \ln(e^2 x)$

48. $3 - \ln x + 2\ln y = \ln e^3 - \ln x + \ln y^2 = \ln\dfrac{e^3 y^2}{x}$

Exercise 8D

1. $$\begin{aligned} y &= \ln 5x \\ &= \ln 5 + \ln x \\ \frac{dy}{dx} &= \frac{1}{x} \end{aligned}$$

2. $$\begin{aligned} y &= 3x + \ln 3x \\ &= 3x + \ln 3 + \ln x \\ \frac{dy}{dx} &= 3 + \frac{1}{x} \end{aligned}$$

3. $$\frac{dy}{dx} = \frac{2}{x}$$

4. $$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2x+3}(2) \\ &= \frac{2}{2x+3} \end{aligned}$$

5. $$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2x-3}(2) \\ &= \frac{2}{2x-3} \end{aligned}$$

6. $$\begin{aligned} y &= 2\ln(x^3) \\ &= 6\ln x \\ \frac{dy}{dx} &= \frac{6}{x} \end{aligned}$$

7. $$\begin{aligned} \frac{dy}{dx} &= \frac{\cos x}{(} - \sin x) \\ &= -\tan x \end{aligned}$$

8. $$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sin 2x}(\cos 2x)(2) \\ &= \frac{2}{\tan 2x} \end{aligned}$$

9. $$\begin{aligned} y &= \ln(2\sqrt{x}) \\ &= \ln 2 + \frac{1}{2}\ln x \\ \frac{dy}{dx} &= \frac{1}{2x} \end{aligned}$$

10. $$\begin{aligned} \frac{dy}{dx} &= \ln x + x\left(\frac{1}{x}\right) \\ &= \ln x + 1 \end{aligned}$$

11. $$\begin{aligned} \frac{dy}{dx} &= 2\log_e x\left(\frac{1}{x}\right) \\ &= \frac{2\log_e x}{x} \end{aligned}$$

12. $$\begin{aligned} \frac{dy}{dx} &= 2x\ln x + x^2\left(\frac{1}{x}\right) \\ &= x(2\ln x + 1) \end{aligned}$$

13. $$\begin{aligned} \frac{dy}{dx} &= 2(3+\ln x)\left(\frac{1}{x}\right) \\ &= \frac{6+2\ln x}{x} \end{aligned}$$

14. $$\begin{aligned} \frac{dy}{dx} &= -\frac{2}{x^2} + \frac{2}{x} \\ &= \frac{2}{x} - \frac{2}{x^2} \end{aligned}$$

15. $$\begin{aligned} y &= \ln\left(\frac{2}{x}\right) \\ &= \ln 2 - \ln x \\ \frac{dy}{dx} &= -\frac{1}{x} \end{aligned}$$

16. $$\begin{aligned} \frac{dy}{dx} &= -\frac{1}{(\ln x)^2}\left(\frac{1}{x}\right) \\ &= -\frac{1}{x(\ln x)^2} \end{aligned}$$

17. $$\begin{aligned} \frac{dy}{dx} &= \frac{\ln x - x\left(\frac{1}{x}\right)}{(\ln x)^2} \\ &= \frac{\ln x - 1}{(\ln x)^2} \\ &= \frac{1}{\ln x} - \frac{1}{(\ln x)^2} \end{aligned}$$

18. $$\begin{aligned} y &= \log_e\left[(x^2+1)^3\right] \\ &= 3\log_e(x^2+1) \\ \frac{dy}{dx} &= \frac{3}{x^2+1}(2x) \\ &= \frac{6x}{x^2+1} \end{aligned}$$

19. $$\begin{aligned} y &= \ln\left[\frac{(x-1)^3}{x+1}\right] \\ &= 3\ln(x-1) - \ln(x+1) \\ \frac{dy}{dx} &= \frac{3}{x-1} - \frac{1}{x+1} \end{aligned}$$

20. $$\begin{aligned} y &= \log_5 x \\ &= \frac{\ln x}{\ln 5} \\ \frac{dy}{dx} &= \frac{1}{x\ln 5} \end{aligned}$$

21. $$\begin{aligned} y &= \log_7 x \\ &= \frac{\ln x}{\ln 7} \\ \frac{dy}{dx} &= \frac{1}{x\ln 7} \end{aligned}$$

22. $$\frac{dy}{dx} = \frac{3}{x}$$
at $(e, 3)$: $$\frac{dy}{dx} = \frac{3}{e}$$

23. $$\frac{dy}{dx} = \ln x + 1 \quad \text{(see q.10)}$$
at (e, e): $$\begin{aligned} \frac{dy}{dx} &= \ln e + 1 \\ &= 2 \end{aligned}$$

24. (a)
$$\begin{aligned}\frac{dy}{dx} &= 1 + \frac{1}{x}\\ 1.5 &= 1 + \frac{1}{x}\\ \frac{1}{x} &= 0.5\\ x &= 2\\ y &= 2 + \ln(2\times 2)\end{aligned}$$
coordinates are $(2, 2 + \ln 4)$

(b)
$$\begin{aligned}y &= \ln x + \ln(x+3)\\ \frac{dy}{dx} &= \frac{1}{x} + \frac{1}{x+3}\\ &= \frac{2x+3}{x^2+3x}\\ 0.5 &= \frac{2x+3}{x^2+3x}\\ 1 &= \frac{4x+6}{x^2+3x}\\ x^2+3x &= 4x+6\\ x^2 - x - 6 &= 0\\ (x-3)(x+2) &= 0\\ x &= 3\\ \text{or } x &= -2\\ y &= \ln 18\\ \text{or } y &= \ln -2\end{aligned}$$
coordinates are $(3, \ln 18)$
(rejecting the second solution because $ln - 2$ is not a real number.)

25.
$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{x}\\ &= \frac{1}{e^2}\\ y - y_1 &= m(x - x_1)\\ y - 2 &= \frac{1}{e^2}(x - e^2)\\ y - 2 &= \frac{x}{e^2} - 1\\ y &= \frac{x}{e^2} + 1\end{aligned}$$

26.
$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sin x}\cos x\\ &= \frac{1}{\tan x}\\ y - y_1 &= m(x - x_1)\\ y - 0 &= \frac{1}{\tan\frac{\pi}{6}}\left(x - \frac{\pi}{6}\right)\\ y &= \sqrt{3}\left(x - \frac{\pi}{6}\right)\end{aligned}$$

27. (a)
$$\begin{aligned}2x + 6y\frac{dy}{dx} &= \frac{dy}{dx}\ln x + \frac{y}{x}\\ \frac{dy}{dx}(6y - \ln x) &= \frac{y}{x} - 2x\\ \frac{dy}{dx} &= \frac{\frac{y}{x} - 2x}{6y - \ln x}\\ &= \frac{y - 2x^2}{x(6y - \ln x)}\end{aligned}$$

(b)
$$\begin{aligned}5 + \frac{3}{2y+1}\left(2\frac{dy}{dx}\right) &= 3y + 3x\frac{dy}{dx}\\ \frac{dy}{dx}\left(\frac{6}{2y+1} - 3x\right) &= 3y - 5\\ \frac{dy}{dx}\left(\frac{6 - (6xy + 3x)}{2y+1}\right) &= 3y - 5\\ \frac{dy}{dx} &= \left(\frac{2y+1}{6 - 6xy - 3x}\right)(3y-5)\\ &= \frac{(2y+1)(3y-5)}{6 - 6xy - 3x}\\ &= \frac{6y^2 - 7y - 5}{6 - 6xy - 3x}\end{aligned}$$

28. (a)
$$\begin{aligned}\ln y &= x\ln 2\\ \frac{1}{y}\frac{dy}{dx} &= \ln 2\\ \frac{dy}{dx} &= y\ln 2\\ &= 2^x\ln 2\end{aligned}$$

(b)
$$\begin{aligned}\ln y &= x\ln 4\\ \frac{1}{y}\frac{dy}{dx} &= \ln 4\\ \frac{dy}{dx} &= y\ln 4\\ &= 4^x\ln 4\end{aligned}$$

Exercise 8E

1–4 No working required: do these in a single step.

5.
$$\text{Try } y = \ln|x^2+1|$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{x^2+1}$$
$$\therefore \quad \int \frac{2x}{x^2+1}\,\mathrm{d}x = \ln|x^2+1|$$

6–8 No working required: integrate in a single step.

9.
$$\text{Try } y = \ln|x^2-3|$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{x^2-3}$$
$$\therefore \quad \int \frac{8x}{x^2-3}\,\mathrm{d}x = 4\ln|x^2-3| + c$$

10.
$$\text{Try } y = \ln|5x-3|$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5}{5x-3}$$
$$\therefore \quad \int \frac{5}{5x-3}\,\mathrm{d}x = \ln|5x-3| + c$$

11.
$$\text{Try } y = \ln|2x+1|$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{2x+1}$$
$$\therefore \quad \int \frac{10}{2x+1}\,\mathrm{d}x = 5\ln|2x+1| + c$$

12.
$$\text{Try } y = \ln|x^2+1|$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{x^2+1}$$
$$\therefore \quad \int \frac{6x}{x^2+1}\,\mathrm{d}x = 3\ln|x^2+1| + c$$

13.
$$\text{Try } y = \ln|\cos x|$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\sin x}{\cos x}$$
$$\therefore \quad \int \frac{\sin x}{\cos x}\,\mathrm{d}x = -\ln|\cos x| + c$$

14.
$$\text{Try } y = \ln|\sin x|$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x}{\sin x}$$
$$\therefore \quad \int \frac{\cos x}{\sin x}\,\mathrm{d}x = \ln|\sin x| + c$$

15.
$$\text{Try } y = \ln|\cos 2x|$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2\sin 2x}{\cos 2x}$$
$$\therefore \quad \int \frac{\sin 2x}{\cos 2x}\,\mathrm{d}x = -\frac{1}{2}\ln|\cos 2x| + c$$

16.
$$\int \tan x\,\mathrm{d}x = \int \frac{\sin x}{\cos x}\,\mathrm{d}x$$
$$= -\ln|\cos x| + c$$

17.
$$\text{Try } y = \ln|\cos 5x|$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-5\sin 5x}{\cos 5x}$$
$$= -5\tan 5x$$
$$\therefore \quad \int \tan 5x\,\mathrm{d}x = -\frac{1}{5}\ln|\cos 5x| + c$$

18.
$$\int 6\tan 2x\,\mathrm{d}x = 6\int \tan 2x\,\mathrm{d}x$$
$$= 6\left(-\frac{1}{2}\ln|\cos 2x|\right) + c$$
$$= -3\ln|\cos 2x| + c$$

19.
$$\text{Try } y = \ln|\sin x + \cos x|$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x - \sin x}{\sin x + \cos x}$$
$$\therefore \quad \int \frac{\sin x - \cos x}{\sin x + \cos x}\,\mathrm{d}x = -\ln|\sin x + \cos x| + c$$

20.
$$\text{Try } y = \ln|4x + \sin 2x|$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4 + 2\cos 2x}{4x + \sin 2x}$$
$$\therefore \quad \int \frac{2 + \cos 2x}{4x + \sin 2x}\,\mathrm{d}x = \frac{1}{2}\ln|4x + \sin 2x| + c$$

21.
$$\text{Try } y = \ln|\mathrm{e}^x + x|$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^x + 1}{\mathrm{e}^x + x}$$
$$\therefore \quad \int \frac{\mathrm{e}^x + 1}{\mathrm{e}^x + x}\,\mathrm{d}x = \ln|\mathrm{e}^x + x| + c$$

22.
$$\int_1^3 \frac{1}{x}\,\mathrm{d}x = \big[\ln|x|\big]_1^3$$
$$= \ln 3 - \ln 1$$
$$= \ln 3$$

23.
$$\int_{-3}^{-2} \frac{3}{x}\,\mathrm{d}x = \big[3\ln|x|\big]_{-3}^{-2}$$
$$= 3\ln 2 - 3\ln 3)$$
$$= 3\ln\frac{2}{3}$$

(Make sure you understand how this is equivalent to the answer Sadler gives.)

24.
$$\int_1^2 \left(\mathrm{e}^x + \frac{1}{x}\right)\mathrm{d}x = \big[\mathrm{e}^x + \ln|x|\big]_1^2$$
$$= (\mathrm{e}^2 + \ln 2) - (\mathrm{e} + \ln 1)$$
$$= \mathrm{e}^2 - \mathrm{e} + \ln 2$$

25.
$$\begin{aligned}\text{Try } y &= 4^x\\ \text{then } \ln y &= x\ln 4\\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \ln 4\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= y\ln 4\\ &= 4^x\ln 4\\ \therefore\quad \int 4^x\,\mathrm{d}x &= \frac{4^x}{\ln 4}+c\end{aligned}$$

26.
$$\begin{aligned}\text{Area} &= \int_1^3 |\frac{2x+1}{x}|\,\mathrm{d}x\\ &= \int_1^3 |2+\frac{1}{x}|\,\mathrm{d}x\\ &= [2x+\ln x]_1^3\\ &= (6+\ln 3)-(2+\ln 1)\\ &= 4+\ln 3 \text{ square units}\end{aligned}$$

(Remember to use your CAS tool to answer questions like this in calculator-assumed assessments.)

27. One bound for the definite integral will be the y−axis, i.e. $x = 0$. To find the other,
$$\begin{aligned}\frac{1}{x+2}-1 &= 0\\ x &= -1\\ \text{Area} &= \int_{-1}^0 |\frac{1}{x+2}-1|\,\mathrm{d}x\\ &= \int_{-1}^0 1-\frac{1}{x+2}\,\mathrm{d}x\\ &= [x-\ln|x+2|]_{-1}^0\\ &= (0-\ln 2)-(-1-\ln 1)\\ &= 1-\ln 2\end{aligned}$$

28.
$$\begin{aligned}\text{Area} &= \int_0^{\frac{\pi}{6}} |\tan x|\,\mathrm{d}x\\ &= [-\ln|\cos x|]_0^{\frac{\pi}{6}}\\ &= (-\ln\cos\frac{\pi}{6})-(-\ln\cos 0)\\ &= -\ln\frac{\sqrt{3}}{2}+\ln 1\\ &= -\left(\frac{\ln 3}{2}-\ln 2\right)\\ &= \ln 2-\frac{\ln 3}{2}\end{aligned}$$

29.
$$\begin{aligned}\frac{a}{x+4}+\frac{b}{x+2} &= \frac{a(x+2)+b(x+4)}{(x+4)(x+2)}\\ &= \frac{(a+b)x+2a+4b}{(x+4)(x+2)}\\ \therefore\quad (a+b)x+2(a+2b) &= 2(4x+13)\\ a+b &= 8\\ \text{and } a+2b &= 13\\ \therefore\quad b &= 5\\ \text{and } a &= 3\\ \therefore\quad \int \frac{2(4x+13)}{(x+4)(x+2)}\,\mathrm{d}x &= \int\left(\frac{3}{x+4}+\frac{5}{x+2}\right)\mathrm{d}x\\ &= 3\ln|x+4|+5\ln|x+2|+c\end{aligned}$$

30. (a)
$$\begin{aligned}\int_1^k \frac{2}{x}\,\mathrm{d}x &= 1\\ [2\ln|x|]_1^k &= 1\\ 2\ln k-2\ln 1 &= 1\\ 2\ln k &= 1\\ \ln k &= \frac{1}{2}\\ k &= \mathrm{e}^{\frac{1}{2}}\end{aligned}$$

(b)
$$\begin{aligned}\int_1^b \frac{2}{x}\,\mathrm{d}x &= 0.5\\ [2\ln|x|]_1^b &= 0.5\\ 2\ln b-2\ln 1 &= 0.5\\ 2\ln b &= 0.5\\ \ln b &= 0.25\\ b &= \mathrm{e}^{0.25}\end{aligned}$$

(c)
$$\begin{aligned}c &= \frac{1+k}{2}\\ &= \frac{1+e^{0.5}}{2}\end{aligned}$$
$$\begin{aligned}\int_1^c \frac{2}{x}\,\mathrm{d}x &= [2\ln|x|]_1^c\\ &= 2\ln c-2\ln 0\\ &= 2\ln c\\ &= 2\ln\frac{1+e^{0.5}}{2}\\ &= 2\ln(1+e^{0.5})-2\ln 2\\ &= 2\ln(1+\mathrm{e}^{0.5})-\ln 4\end{aligned}$$

Miscellaneous Exercise 8

1. (a)
$$\begin{aligned} r &= 5\sqrt{\sqrt{3})^2+1^2} \\ &= 3 \\ \tan\theta &= \frac{1}{\sqrt{3}} \\ \theta &= \frac{\pi}{6}+n\pi \quad (3^{\text{rd}} \text{ quadrant.}) \\ \theta &= -\frac{5\pi}{6} \\ -5(\sqrt{3}+\mathrm{i}) &= 3\,\mathrm{cis}-\frac{5\pi}{6} \end{aligned}$$

   (b)
$$\begin{aligned} a &= 6\cos\frac{3\pi}{4} \\ &= 6\left(-\frac{\sqrt{2}}{2}\right) \\ &= -3\sqrt{2} \\ b &= 6\sin\frac{3\pi}{4} \\ &= 6\left(\frac{\sqrt{2}}{2}\right) \\ &= 3\sqrt{2} \\ 6\,\mathrm{cis}\,\frac{3\pi}{4} &= -3\sqrt{2}+3\sqrt{2}\mathrm{i} \end{aligned}$$

2. (a) Do in a single step. No working needed.

   (b) Do in a single step. No working needed.

   (c) Do in a single step. No working needed.

   (d)
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2(5-3x)-(2x+3)(-3)}{(5-3x)^2} \\ &= \frac{10-6x+6x+9}{(5-3x)^2} \\ &= \frac{19}{(5-3x)^2} \end{aligned}$$

   (e)
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 3(2x+3)^2(2) \\ &= 6(2x+3)^2 \end{aligned}$$

   (f)
$$\begin{aligned} 2y+2x\frac{\mathrm{d}y}{\mathrm{d}x}+2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 3\cos x \\ \frac{\mathrm{d}y}{\mathrm{d}x}(2x+2y) &= 3\cos x-2y \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3\cos x-2y}{2(x+y)} \end{aligned}$$

   (g)
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} \\ &= \frac{6\cos 3t}{10\sin 2t} \\ &= \frac{3\cos 3t}{5\sin 2t} \end{aligned}$$

3. (a)
$$\begin{aligned} \int \sin^3 x\,\mathrm{d}x &= \int \sin x\sin^2 x\,\mathrm{d}x \\ &= \int \sin x(1-\cos^2 x)\,\mathrm{d}x \\ &= \int \sin x\,\mathrm{d}x - \int \sin x\cos^2 x\,\mathrm{d}x \\ &= -\cos x+\frac{\cos^3 x}{3}+c \end{aligned}$$

   (b)
$$\begin{aligned} \int 2x^7(1+x)\,\mathrm{d}x &= \int (2x^7+2x^8)\,\mathrm{d}x \\ &= \frac{2x^8}{8}+\frac{2x^9}{9}+c \\ &= \frac{x^8}{4}+\frac{2x^9}{9}+c \\ &= \frac{x^8}{3}6(9+8x)+c \end{aligned}$$

   (c) Let $u=1+x$, $x=u-1$, $\mathrm{d}u=\mathrm{d}x$
$$\begin{aligned} \int 2x(1+x)^7\,\mathrm{d}x &= \int 2u^7(u-1)\,\mathrm{d}u \\ &= \int 2u^8-2u^7\,\mathrm{d}u \\ &= \frac{2u^9}{9}-\frac{u^8}{4}+c \\ &= \frac{u^8}{3}6(8u-9)+c \\ &= \frac{(x+1)^8}{3}6(8(x+1)-9)+c \\ &= \frac{(x+1)^8}{3}6(8x-1)+c \end{aligned}$$

   (d) Observe that we have a multiple of $\mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)}$
   Guess $y=\mathrm{e}^{x^2+5}$,
   then $\frac{\mathrm{d}y}{\mathrm{d}x}=2x\mathrm{e}^{x^2+5}$,
   hence $\int 6x\mathrm{e}^{x^2+5}\,\mathrm{d}x = 3\mathrm{e}^{x^2+5}+c$

4. (a)
$$\begin{aligned} \int \frac{x^2+1}{x}\,\mathrm{d}x &= \int x+\frac{1}{x}\,\mathrm{d}x \\ &= 0.5x^2+\ln x+c \end{aligned}$$

   (b) Observe that we have a multiple of $\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}$
   Guess $y=\ln(x^2+1)$,
   then $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x}{x^2+1}$,
   hence $\int \frac{x}{x^2+1}\,\mathrm{d}x = 0.5\ln(x^2+1)+c$

   (c) Observe that we have a multiple of $\mathrm{f}'(x)(\mathrm{f}(x))^n$ for $n=-\frac{1}{2}$
   Guess $y=\sqrt{x^2+1}$,
   then $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x}{\sqrt{x^2+1}}$,
   hence $\int \frac{x}{\sqrt{x^2+1}}\,\mathrm{d}x = \sqrt{x^2+1}+c$

   (d)

5. You should recognise this limit as having the form of a first-principles differentiation.
$$\begin{aligned} \lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} \\ &= \frac{1}{2}x^{-\frac{1}{2}} \end{aligned}$$

6.
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \mathrm{e}^x\cos x - \mathrm{e}^x\sin x &= 0\\ \mathrm{e}^x(\cos x - \sin x) &= 0\\ \cos x &= \sin x\\ \tan x &= 1\\ x &= \frac{\pi}{4} + n\pi\\ x &\in \left\{-\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}\right\}\end{aligned}$$

7.
$$\begin{aligned}\overrightarrow{\mathrm{OP}} &= \overrightarrow{\mathrm{OA}} + \frac{4}{5}\overrightarrow{\mathrm{AB}}\\ &= \begin{pmatrix}1\\6\\-7\end{pmatrix} + \frac{4}{5}\left(\begin{pmatrix}-4\\1\\3\end{pmatrix} - \begin{pmatrix}1\\6\\-7\end{pmatrix}\right)\\ &= \begin{pmatrix}1\\6\\-7\end{pmatrix} + \frac{4}{5}\begin{pmatrix}-5\\-5\\10\end{pmatrix}\\ &= \begin{pmatrix}1\\6\\-7\end{pmatrix} + \begin{pmatrix}-4\\-4\\8\end{pmatrix}\\ &= \begin{pmatrix}-3\\2\\1\end{pmatrix}\end{aligned}$$

8. (a)
$$\begin{aligned}|\mathbf{a}| &= \sqrt{1^2+1^2+3^2}\\ &= \sqrt{11}\\ |\mathbf{b}| &= \sqrt{2^2+1^2+1^2}\\ &= \sqrt{6}\\ \cos\theta &= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\\ &= \frac{1\times 2 + -1\times -1 + 3\times 1}{\sqrt{11}\sqrt{6}}\\ &= \frac{6}{\sqrt{11}\sqrt{6}}\\ \theta &\approx 42^\circ\end{aligned}$$

(b) Since we know the magnitude of **a** the two unit vectors parallel to **a** can be obtained without any further working.

(c) Any of the following will yield a suitable equation:

- $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{a} - \mathbf{b})$
- $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$
- $\mathbf{r} = \mathbf{b} + \lambda(\mathbf{a} - \mathbf{b})$
- $\mathbf{r} = \mathbf{b} + \lambda(\mathbf{b} - \mathbf{a})$

Simply substitute the values for **a** and **b** into one of the above.

(d) We have a point on the plane and a perpendicular or *normal*, so we use the normal form for the equation of a plane:

$$\begin{aligned}\mathbf{r}\cdot\overrightarrow{\mathrm{AB}} &= \mathbf{a}\cdot\overrightarrow{\mathrm{AB}}\\ \mathbf{r}\cdot(\mathbf{b}-\mathbf{a}) &= \mathbf{a}\cdot(\mathbf{b}-\mathbf{a})\\ \mathbf{r}\cdot\begin{pmatrix}1\\0\\-2\end{pmatrix} &= \begin{pmatrix}1\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-2\end{pmatrix}\\ \mathbf{r}\cdot\begin{pmatrix}1\\0\\-2\end{pmatrix} &= -5\end{aligned}$$

9. L.H.S.:

$$\begin{aligned}&(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)\\ &= \cos^2\theta - \sin^2\theta\\ &= (1-\sin^2\theta) - \sin^2\theta\\ &= 1 - 2\sin^2\theta\\ &= \text{R.H.S.}\end{aligned}$$

□

10. L.H.S.:

$$\begin{aligned}\cos 3x &= \cos(x + 2x)\\ &= \cos x\cos 2x - \sin x\sin 2x\\ &= \cos x(2\cos^2 x - 1) - \sin x(2\sin x\cos x)\\ &= 2\cos^3 x - \cos x - 2\sin^2 x\cos x\\ &= 2\cos^3 x - \cos x - 2\cos x(1-\cos^2 x)\\ &= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x\\ &= 4\cos^3 x - 3\cos x\\ &= \text{R.H.S.}\end{aligned}$$

□

11.
$$\begin{aligned}xy &= \cos x\\ y + x\frac{\mathrm{d}y}{\mathrm{d}x} &= -\sin x\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-\sin x - y}{x}\\ &= \frac{-\sin\left(\frac{\pi}{3}\right) - \frac{3}{2\pi}}{\frac{\pi}{3}}\\ &= \left(-\sqrt{3}2 - \frac{3}{2\pi}\right)\times\frac{3}{\pi}\\ &= -3\sqrt{3}2\pi - \frac{9}{2\pi^2}\\ &= -\frac{3(\sqrt{3}\pi + 3)}{2\pi^2}\end{aligned}$$

12.
$$\begin{aligned}x^3 + 2x^2y + y^3 &= 10\\ 3x^2 + 4xy + 2x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ (2x^2+3y^2)\frac{\mathrm{d}y}{\mathrm{d}x} &= -3x^2 - 4xy\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x(3x+4y)}{2x^2+3y^2}\end{aligned}$$

Note that although the question asks for $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a function of x, the answer provided is a function of x and y. You are *not* expected to be able to eliminate y from this expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. I believe the question is in error and should not specify "as a function of x".

13. (a)
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{x^2\sin 2x}\left((2x)(\sin 2x) + (x^2)(2\cos 2x)\right)\\ &= \frac{2x\sin 2x + 2x^2\cos 2x}{x^2\sin 2x}\\ &= \frac{2\sin 2x + 2x\cos 2x}{x\sin 2x}\\ &= \frac{2}{x} + \frac{2}{\tan 2x}\end{aligned}$$

(b)
$$\begin{aligned}y &= \frac{\ln x}{\ln 2}\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{x\ln 2}\end{aligned}$$

(c)
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4\mathrm{e}^{4x}(3x\sin^4 x) - \mathrm{e}^{4x}(3\sin^4 x + 3x(4\sin^3 x\cos x))}{(3x\sin^4 x)^2}\\ &= \frac{12x\mathrm{e}^{4x}\sin^4 x - (3\mathrm{e}^{4x}\sin^4 x + 12x\mathrm{e}^{4x}\sin^3 x\cos x)}{(3x\sin^4 x)^2}\\ &= \frac{12x\mathrm{e}^{4x}\sin^4 x - 3\mathrm{e}^{4x}\sin^4 x - 12x\mathrm{e}^{4x}\sin^3 x\cos x}{(3x\sin^4 x)^2}\\ &= \frac{3\mathrm{e}^{4x}\sin^3 x(4x\sin x - \sin x - 4x\cos x)}{9x^2\sin^8 x}\\ &= \frac{\mathrm{e}^{4x}(\sin x(4x-1) - 4x\cos x)}{3x^2\sin^5 x}\end{aligned}$$

14.
$$\begin{aligned}2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x}{y}\\ &= -\frac{3}{4}\\ y - y_1 &= m(x - x_1)\\ y - 4 &= -\frac{3}{4}(x-3)\\ 4(y-4) &= -3(x-3)\\ 4y - 16 &= -3x + 9\\ 4y &= -3y + 25\\ 3x + 4y &= 25\end{aligned}$$

15. When $x = 1$,

$$\begin{aligned}y^2 + 5y + 1 &= 15\\ y^2 + 5y - 14 &= 0\\ (y+7)(y-2) &= 0\\ y &= -7\\ \text{or } y &= 2\end{aligned}$$

$$\begin{aligned}\text{Then } 2y\frac{\mathrm{d}y}{\mathrm{d}x} + 5y + 5x\frac{\mathrm{d}y}{\mathrm{d}x} + 2x &= 0\\ \frac{\mathrm{d}y}{\mathrm{d}x}(2y + 5x) &= -(2x + 5y)\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{2x+5y}{5x+2y}\end{aligned}$$

At $(1, -7)$:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{2-35}{5-14}\\ &= -\frac{33}{9}\\ &= -\frac{11}{3}\\ y + 7 &= -\frac{11}{3}(x-1)\\ 3(y+7) + 11(x-1) &= 0\\ 3y + 11x &= -10\end{aligned}$$

At $(1, 2)$:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{2+10}{5+4}\\ &= -\frac{12}{9}\\ &= -\frac{4}{3}\\ y - 2 &= -\frac{4}{3}(x-1)\\ 3(y-2) &= -4(x-1)\\ 4x + 3y &= 10\end{aligned}$$

16. First find find $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}\\ &= \frac{1 + \frac{1}{t^2}}{2t - 6}\end{aligned}$$

The curve cuts the $y-$axis when $x = 0$

$$\begin{aligned}t^2 - 6t + 5 &= 0\\ (t-1)(t-5) &= 0\\ t &= 1\\ \text{or } t &= 5\end{aligned}$$

At $t = 1$:

$$\begin{aligned}y &= 1 - \frac{1}{1}\\ &= 0\\ \text{and } \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1 + \frac{1}{1}}{2-6}\\ &= -\frac{2}{4}\\ &= -0.5\end{aligned}$$

giving the tangent line

$$\begin{aligned}y - 0 &= -0.5(x - 0)\\ y &= -0.5x\end{aligned}$$

At $t = 5$:

$$\begin{aligned} y &= 5 - \frac{1}{5} \\ &= 4.8 \\ \text{and } \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1 + \frac{1}{25}}{10 - 6} \\ &= \frac{1.04}{4} \\ &= 0.26 \end{aligned}$$

giving the tangent line

$$\begin{aligned} y - 4.8 &= 0.26(x - 0) \\ y &= 0.26x + 4.8 \end{aligned}$$

17. (a)
$$\begin{aligned} 1 + \frac{1}{y} &= x \\ -\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= y^2 \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 2y\frac{\mathrm{d}y}{\mathrm{d}x} \\ &= 2y(y^2) \\ &= 2y^3 \end{aligned}$$

(b)
$$\begin{aligned} y^2 - \frac{5}{y} &= x \\ 2y\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{5}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x}\left(2y + \frac{5}{y^2}\right) &= 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{2y^3 + 5}{y^2}\right) &= 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y^2}{2y^3 + 5} \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{2y\frac{\mathrm{d}y}{\mathrm{d}x}(2y^3 + 5) - y^2(6y^2\frac{\mathrm{d}y}{\mathrm{d}x})}{(2y^3 + 5)^2} \\ &= \frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{4y^4 + 10y - 6y^4}{(2y^3 + 5)^2}\right) \\ &= \frac{y^2}{2y^3 + 5} \times \frac{10y - 2y^4}{(2y^3 + 5)^2} \\ &= \frac{10y^3 - 2y^6}{(2y^3 + 5)^3} \\ &= \frac{2y^3(5 - y^3)}{(2y^3 + 5)^3} \end{aligned}$$

18. (a) For $0 < k < \frac{\pi}{2}$ (i.e. first quadrant) y is positive so the area between the curve and the $x-$axis is equal to the definite integral:

$$\begin{aligned} A &= \int_0^k 3\sin^2 x \cos x \,\mathrm{d}x \\ &= \left[\sin^3 x\right]_0^k \\ &= \sin^3 k - \sin^3 0 \\ &= (\sin^3 k) \text{ units}^2 \end{aligned}$$

(b) For $\frac{\pi}{2} < k < \pi$ (i.e. second quadrant) as x goes from 0 to k the value of y has a positive component and a negative component so so the area between the curve and the $x-$axis must be taken piecewise. (If you were doing this numerically with a calculator it would be simplest to take the definite integral of the absolute value of y.)

$$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} 3\sin^2 x \cos x \,\mathrm{d}x - \int_{\frac{\pi}{2}}^k 3\sin^2 x \cos x \,\mathrm{d}x \\ &= \left[\sin^3 x\right]_0^{\frac{\pi}{2}} - \left[\sin^3 x\right]_{\frac{\pi}{2}}^k \\ &= \left(\sin^3 \frac{\pi}{2} - \sin^3 0\right) - \left(\sin^3 k - \sin^3 \frac{\pi}{2}\right) \\ &= (1 - 0) - (\sin^3 k - 1) \\ &= (2 - \sin^3 k) \text{ units}^2 \end{aligned}$$

19. (a)
$$\begin{aligned} y &= 0 \\ (\log_{\mathrm{e}} x)^2 - 1 &= 0 \\ (\log_{\mathrm{e}} x)^2 &= 1 \\ \log_{\mathrm{e}} x &= \pm 1 \\ x &= \mathrm{e}^{\pm 1} \\ &\mathrm{A} : (\mathrm{e}^{-1}, 0) \\ &\mathrm{B} : (\mathrm{e}, 0) \end{aligned}$$

Having found all the roots will suffice to prove that the function does not cut the $x-$axis at any other point. For the $y-$axis: Suppose that a point exists where the function cuts the $y-$ axis. Then the function is defined for $x = 0$. But $\log_{\mathrm{e}} 0$ is undefined: a contradiction. Therefore there is no point where the function cuts the $y-$axis. □

(b) At stationary points the first derivative is zero.

$$\begin{aligned} \frac{2\log_{\mathrm{e}} x}{x} &= 0 \\ \log_{\mathrm{e}} x &= 0 \\ x &= 1 \end{aligned}$$

That this has only solution is sufficient proof that there are no other stationary points.

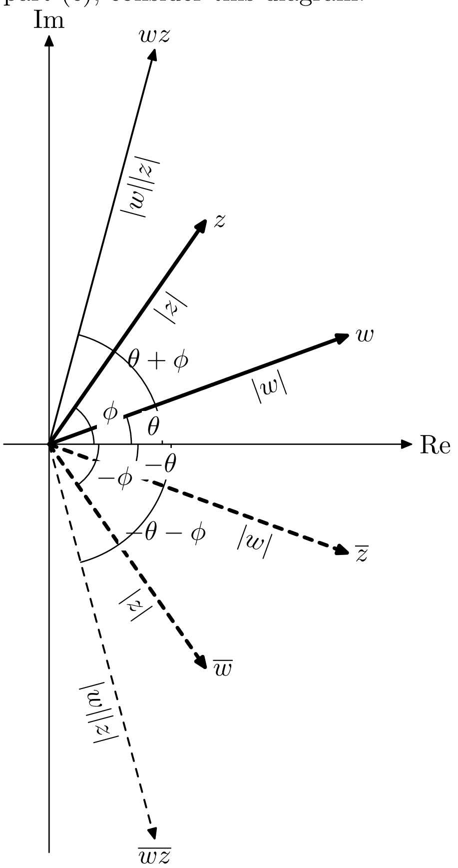
$$\begin{aligned} y &= (\log_{\mathrm{e}} 1) - 1 \\ &= -1 \\ &\mathrm{C} : (1, -1) \end{aligned}$$

(c) At points of inflection the second derivative is zero.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\frac{2\log_e x}{x} &= 0\\ \frac{2-2\log_e x}{x^2} &= 0\\ 1-\log_e x &= 0\\ \log_e x &= 1\\ x &= \mathrm{e}\end{aligned}$$

Therefore there is one point of inflection at point B:$(\mathrm{e}, 0)$.

20. Refer to Sadler's answers for (a) and (b). For part (c), consider this diagram:



Let $\theta = \arg(w)$

$\phi = \arg(z)$

A complex conjugate has the same modulus and opposite argument.

Then $\arg(\bar{w}) = -\theta$

$\arg(\bar{z}) = -\phi$

and $|\bar{w}| = |w|$

$|\bar{z}| = |z|$

We obtain the products by multiplying the moduli and adding the arguments:

$$\begin{aligned}|wz| &= |w||z|\\ |\overline{wz}| &= |wz|\\ &= |w||z|\\ &= |\bar{w}||\bar{z}|\\ &= |\bar{w}\bar{z}|\\ \arg(wz) &= \theta+\phi\\ \arg(\overline{wz}) &= -\arg(wz)\\ &= -(\theta+\phi)\\ &= (-\theta)+(-\phi)\\ &= \arg(\bar{w})+\arg(\bar{z})\\ &= \arg(\bar{w}\bar{z})\end{aligned}$$

Thus by comparing modulus and argument we can see that $\overline{wz} = \bar{w}\bar{z}$ as required.

21. $$\begin{aligned}\frac{ax}{x^2-1}+\frac{b}{x+1} &= \frac{ax}{(x-1)(x+1)}+\frac{b}{x+1}\\ &= \frac{ax+b(x-1)}{x^2-1}\\ &= \frac{ax+bx-b}{x^2-1}\\ &= \frac{(a+b)x-b}{x^2-1}\\ \therefore\quad b &= 5\\ a+b &= 7\\ a &= 2\end{aligned}$$

$$\begin{aligned}\int \frac{7x-5}{x^2-1}\,\mathrm{d}x &= \int \left(\frac{2x}{x^2-1}+\frac{5}{x+1}\right)\mathrm{d}x\\ &= \ln|x^2-1| + 5\ln|x+1| + c\end{aligned}$$

22. The non-zero point where graphs intersect is given by

$$\begin{aligned}\sin^2 x &= \sin x\cos x\\ \sin x &= \cos x\end{aligned}$$

(where $\sin x = 0$ gives the points where they intersect on the $x-$axis.)

$$x = \frac{\pi}{4}$$

(a) $$\begin{aligned}A &= \int_0^{\frac{\pi}{4}} (\sin x\cos x - \sin^2 x)\,\mathrm{d}x\\ &= \int_0^{\frac{\pi}{4}} (\frac{1}{2}\sin 2x + \frac{1}{2} - \sin^2 x - \frac{1}{2})\,\mathrm{d}x\\ &= \int_0^{\frac{\pi}{4}} (\frac{1}{2}\sin 2x + \frac{1}{2}(1-2\sin^2 x) - \frac{1}{2})\,\mathrm{d}x\\ &= \frac{1}{2}\int_0^{\frac{\pi}{4}} (\sin 2x + \cos 2x - 1)\,\mathrm{d}x\\ &= \frac{1}{4}\int_0^{\frac{\pi}{4}} (2\sin 2x + 2\cos 2x - 2)\,\mathrm{d}x\\ &= \frac{1}{4}\left[-\cos 2x + \sin 2x - 2x\right]_0^{\frac{\pi}{4}}\\ &= \frac{1}{4}\Big((-\cos\frac{\pi}{2} + \sin\frac{\pi}{2} - \frac{\pi}{2}) - (-\cos 0 + \sin 0 - 0)\Big)\\ &= \frac{1}{4}\Big((0 + 1 - \frac{\pi}{2}) - (-1 + 0)\Big)\\ &= \frac{4-\pi}{8}\ \text{units}^2\end{aligned}$$

(b)
$$\begin{aligned}A &= \int_0^{\pi} |\sin x \cos x - \sin^2 x|\,\mathrm{d}x\\ &= \frac{4-\pi}{8} + \int_{\frac{\pi}{4}}^{\pi} (\sin^2 x - \sin x \cos x)\,\mathrm{d}x\\ &= \frac{4-\pi}{8} + \frac{1}{4}\left[\cos 2x - \sin 2x + 2x\right]_{\frac{\pi}{4}}^{\pi}\\ &= \frac{4-\pi}{8} + \frac{1}{4}\left((\cos 2\pi - \sin 2\pi + 2\pi) - (\cos\frac{\pi}{2} - \sin\frac{\pi}{2} + \frac{\pi}{2})\right)\\ &= \frac{4-\pi}{8} + \frac{1}{4}\left((1 - 0 + 2\pi) - (0 - 1 + \frac{\pi}{2})\right)\\ &= \frac{4-\pi}{8} + \frac{1}{4}\left(2 + \frac{3\pi}{2}\right)\\ &= \frac{4-\pi}{8} + \frac{4+3\pi}{8}\\ &= \frac{8+2\pi}{8}\\ &= \frac{4+\pi}{4}\text{ units}^2\end{aligned}$$

23.
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \sqrt{3}\cos x - \sin x &= 0\\ \sqrt{3}\cos x &= \sin x\\ \tan x &= \sqrt{3}\\ x &\in \left\{-\frac{5\pi}{3}, -\frac{2\pi}{3}, \frac{\pi}{3}, \frac{4\pi}{3}\right\}\end{aligned}$$

and substituting to obtain values for y gives the coordinates

$$(-\frac{5\pi}{3}, 2); (-\frac{2\pi}{3}, -2); (\frac{\pi}{3}, 2); (\frac{4\pi}{3}, -2)$$

Now using the second derivative test

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -\sqrt{3}\sin x - \cos x\\ &= -y\end{aligned}$$

Hence $(-\frac{5\pi}{3}, 2)$ and $(\frac{\pi}{3}, 2)$ have negative second derivative are so local maxima and similarly $(-\frac{2\pi}{3}, -2)$ and $(\frac{4\pi}{3}, -2)$ are local minima.

24. (a) If the normal of plane A is parallel to plane B then planes A and B are perpendicular. If the normal of plane A is perpendicular to the normal of plane B then the normal of plane A is parallel to plane B and the planes are perpendicular. Thus we can test for perpendicularlity of planes by testing for perpendicularity of their respective normals.

$$(2\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (4\mathbf{i} - \mathbf{j} + 7\mathbf{k}) = 0$$

$\therefore$ the first and second planes are perpendicular.

$$(2\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (-\mathbf{i} + 3\mathbf{j} + \mathbf{k}) = 0$$

$\therefore$ the first and third planes are perpendicular.

$$(4\mathbf{i} - \mathbf{j} + 7\mathbf{k}) \cdot (-\mathbf{i} + 3\mathbf{j} + \mathbf{k}) = 0$$

$\therefore$ the second and third planes are perpendicular.

□

(b)
$$\begin{aligned}(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} - \mathbf{k}) &= 4\\ 2a + b - c &= 4\\ (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (4\mathbf{i} - \mathbf{j} + 7\mathbf{k}) &= 8\\ 4a - b + 7c &= 8\\ (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (-\mathbf{i} + 3\mathbf{j} + \mathbf{k}) &= 9\\ -a + 3b + c &= 9\end{aligned}$$

(c) You should be using your calculator for this, but if you wish to see how it would be done without, use the elimination method just as you would for solving two equations in two unknowns.

$$\begin{aligned}2a + b - c &= 4 && ①\\ 4a - b + 7c &= 8 && ②\\ -a + 3b + c &= 9 && ③\\ 7b + c &= 22 && ①+2\times③\rightarrow④\\ -3b + 9c &= 0 && -2\times①+②\rightarrow⑤\\ 22c &= 22 && (④+\tfrac{7}{3}⑤)\\ c &= 1\\ 7b + 1 &= 22\\ b &= 3\\ 2a + 3 - 1 &= 4\\ a &= 1\end{aligned}$$

Thus the point that lies on all three planes is $(\mathbf{i} + 3\mathbf{j} + \mathbf{k})$.

25. Refer to Sadler page 87.

26. (a) To prove: $\operatorname{cis} 0 = 1$
Proof:

$$\begin{aligned}\text{L.H.S.} &= \operatorname{cis} 0\\ &= \cos 0 + \mathrm{i}\sin 0\\ &= 1 + 0\mathrm{i}\\ &= 1\\ &= \text{R.H.S.}\end{aligned}$$

□

(b) To prove: $\operatorname{cis}\alpha \operatorname{cis}\beta = \operatorname{cis}(\alpha + \beta)$
Proof:

$$\begin{aligned}\text{L.H.S.} &= \operatorname{cis}\alpha \operatorname{cis}\beta\\ &= (\cos\alpha + \mathrm{i}\sin\alpha)(\cos\beta + \mathrm{i}\sin\beta)\\ &= \cos\alpha\cos\beta + \mathrm{i}\cos\alpha\sin\beta\\ &\quad + \mathrm{i}\sin\alpha\cos\beta - \sin\alpha\sin\beta\\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\ &\quad + \mathrm{i}(\cos\alpha\sin\beta + \sin\alpha\cos\beta)\\ &= \cos(\alpha + \beta) + \mathrm{i}\sin(\alpha + \beta)\\ &= \operatorname{cis}(\alpha + \beta)\\ &= \text{R.H.S.}\end{aligned}$$

□

(c) To prove: $\text{cis}(-\alpha) = (\text{cis}\,\alpha)^{-1}$
Proof:

$$\begin{aligned}\text{R.H.S.} &= (\text{cis}\,\alpha)^{-1}\\ &= \frac{1}{\cos\alpha + \mathrm{i}\sin\alpha}\\ &= \frac{\cos\alpha - \mathrm{i}\sin\alpha}{(\cos\alpha + \mathrm{i}\sin\alpha)(\cos\alpha - \mathrm{i}\sin\alpha)}\\ &= \frac{\cos(-\alpha) + \mathrm{i}\sin(-\alpha)}{\cos^2\alpha - \mathrm{i}^2\sin^2\alpha}\\ &= \frac{\text{cis}(-\alpha)}{\cos^2\alpha + \sin^2\alpha}\\ &= \text{cis}(-\alpha)\\ &= \text{L.H.S.}\end{aligned}$$

□

27. For this question it is essential that you understand that the given set of points describes the locus of a circle on the Argand plane centred at $12 + 5\mathrm{i}$ and having a radius of 4.

(a) The minimum $\Im(z)$ occurs directly below the centre, i.e. $5 - 4 = 1$.

(b) The maximum and minimum $\Re(z)$ occur right and left of the centre, i.e. at 12 ± 4, so the maximum of $|\Re(z)|$ is $12 + 4 = 16$.

(c) The maximum $|z|$ is the furthest point on the circle away from the origin. Hence it is the point 4 units from the centre that lies along the line between the centre and the origin. The centre lies 13 units from the origin, so the maximum of $|z|$ is $13 + 4 = 17$

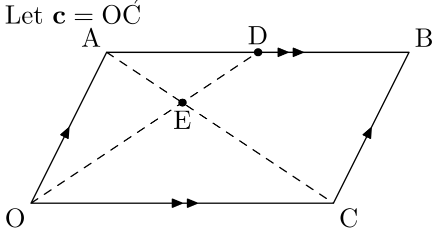
(d) The minimum possible $|z|$ is by similar reasoning $13 - 4 = 9$.

(e) Let A be the point on the circle with minimum $\arg(z)$.
Let C be the centre of the circle at $12 + 5\mathrm{i}$.
Let O be the origin at $0 + 0\mathrm{i}$.
It follows that OA is a tangent to the circle (since otherwise there would be a point on the circle below the line OA with a smaller argument).
Thus $\angle CAO$ is a right angle.

$$\begin{aligned}CA &= 4\\ OC &= 13\\ \angle COA &= \sin^{-1}\frac{4}{13}\\ &= 0.3128\\ \arg(C) &= \tan^{-1}\frac{5}{12}\\ &= 0.3948\\ \therefore\quad \arg(A) &= 0.3948 - 0.3128\\ &= 0.08 \quad \text{(2d.p.)}\end{aligned}$$

(f) By similar reasoning, the maximum $\arg z$ is $0.3948 + 0.3128 = 0.71$ (2d.p.).

28. Let D be the midpoint of AB.
Let E be the intersection between OD and AC.
Let $\mathbf{a} = \overrightarrow{\mathrm{OA}}$
Let $\mathbf{c} = \overrightarrow{\mathrm{OC}}$



To prove:

$$\overrightarrow{\mathrm{AE}} = \frac{1}{3}\overrightarrow{\mathrm{AC}}$$

Proof:

$$\begin{aligned}\overrightarrow{\mathrm{OD}} &= \mathbf{a} + 0.5\mathbf{c}\\ \overrightarrow{\mathrm{AE}} &= p\overrightarrow{\mathrm{AC}}\\ &= p(-\mathbf{a} + \mathbf{c})\\ \overrightarrow{\mathrm{AE}} &= -\mathbf{a} + q\overrightarrow{\mathrm{OD}}\\ &= -\mathbf{a} + q(\mathbf{a} + 0.5\mathbf{c})\\ &= (q-1)\mathbf{a} + 0.5q\mathbf{c}\\ \therefore\quad p(-\mathbf{a} + \mathbf{c}) &= (q-1)\mathbf{a} + 0.5q\mathbf{c}\end{aligned}$$

equating corresponding components

$$\begin{aligned}p &= 1 - q\\ p &= 0.5q\\ q &= 2p\\ p &= 1 - 2p\\ 3p &= 1\\ \therefore\quad \overrightarrow{\mathrm{AE}} &= \frac{1}{3}\overrightarrow{\mathrm{AC}}\end{aligned}$$

□

Given the content of the course, a vector proof would be the anticipated approach for this question. However, since the question does not specify that you must use vector methods, you could take a purely geometric approach:

$$\begin{aligned}\angle AED &\cong \angle OEC && \text{(vertically opposite)}\\ \angle EAB &\cong \angle ECO && \text{(alternate angles)}\\ \therefore\quad \triangle AED &\sim \triangle CEO && \text{(AAA)}\\ AD &= \frac{1}{2}AB && \text{(given)}\\ OC &= AB && \text{(opposite sides of a parallelogram)}\\ \therefore\quad AD &= \frac{1}{2}OC\\ \therefore\quad AE &= \frac{1}{2}EC && \text{(corresponding sides of similar }\triangle\text{s)}\\ \therefore\quad AE &= \frac{1}{3}AC\end{aligned}$$

□

29. (a)

$$\begin{aligned}
\frac{dy}{dx} &= 3(2x+3)^2(2) \\
&= 6(2x+3)^2 \\
\frac{d^2y}{dx^2} &= 12(2x+3)(2) \\
&= 24(2x+3) \\
\frac{dx}{dy} &= \frac{1}{6}y^{-\frac{2}{3}} \\
\frac{d^2x}{dy^2} &= -\frac{1}{9}y^{-\frac{5}{3}} \\
&= -\frac{1}{9}\left((2x+3)^3\right)^{-\frac{5}{3}} \\
&= -\frac{1}{9}(2x+3)^{-5} \\
-\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3 &= \frac{1}{9}(2x+3)^{-5}\left(6(2x+3)^2\right)^3 \\
&= \frac{6^3}{9}(2x+3)^{-5}(2x+3)^6 \\
&= 24(2x+3) \\
&= \frac{d^2y}{dx^2}
\end{aligned}$$

(b) Start with $y = f(x)$ and differentiate with respect to y:

$$\begin{aligned}
y &= f(x) \\
1 &= \frac{dy}{dx}\frac{dx}{dy} \qquad \text{(chain rule)} \\
0 &= \left(\frac{d^2y}{dx^2}\frac{dx}{dy}\right)\frac{dx}{dy} + \frac{dy}{dx}\frac{d^2x}{dy^2} \\
\frac{d^2y}{dx^2}\left(\frac{dx}{dy}\right)^2 &= -\frac{dy}{dx}\frac{d^2x}{dy^2} \\
\frac{d^2y}{dx^2} &= \frac{-\frac{dy}{dx}\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^2} \\
\frac{d^2y}{dx^2} &= \frac{-\frac{dy}{dx}\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^2} \times \frac{\left(\frac{dy}{dx}\right)^2}{\left(\frac{dy}{dx}\right)^2} \\
\frac{d^2y}{dx^2} &= \frac{-\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3}{\left(\frac{dx}{dy}\frac{dy}{dx}\right)^2} \\
\frac{d^2y}{dx^2} &= -\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3
\end{aligned}$$

□

30. (a) Let θ be the angle between vectors **a** and **b**.

$$\begin{aligned}
|\mathbf{a}\cdot\mathbf{b}| &= |(|\mathbf{a}||\mathbf{b}|\cos\theta)| \\
&= |\mathbf{a}||\mathbf{b}||\cos\theta| \\
|\cos\theta| &\le 1 \\
\therefore \quad |\mathbf{a}\cdot\mathbf{b}| &\le |\mathbf{a}||\mathbf{b}|
\end{aligned}$$

□

(b)

$$\begin{aligned}
(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) &= \mathbf{a}\cdot\mathbf{a} + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b} \\
&= |\mathbf{a}|^2 + 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2 \\
|\mathbf{a}+\mathbf{b}|^2 &= |\mathbf{a}|^2 + 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2 \\
(|\mathbf{a}|+|\mathbf{b}|)^2 &= |\mathbf{a}|^2 + 2|\mathbf{a}||\mathbf{b}| + |\mathbf{b}|^2 \\
\therefore \quad |\mathbf{a}+\mathbf{b}|^2 &\le (|\mathbf{a}|+|\mathbf{b}|)^2 \\
\therefore \quad |\mathbf{a}+\mathbf{b}| &\le |\mathbf{a}|+|\mathbf{b}|
\end{aligned}$$

□

31. The radius of each ball is the height of its centre above the table, i.e. the **k** component.

The minimum possible distance between centres is twice the radius of the balls, i.e. 5cm. The cue ball will strike the 12 ball if the minimum distance between the centre of the 12 ball and the line along which the centre of the cue ball is travelling is less than 5cm.

Let A be the initial position of the cue ball, B be the position of the 12 ball and P be the point of closest approach between B and the line along which the cue ball is travelling.

$$\begin{aligned}
\overrightarrow{BP} &= \overrightarrow{BA} + \overrightarrow{AP} \\
&= (40-120)\mathbf{i} + (69-20)\mathbf{j} + (2.5-2.5)\mathbf{k} \\
&\quad + t\,(30\mathbf{i} - 20\mathbf{j}) \\
&= (-80\mathbf{i} + 49\mathbf{j}) + t\,(30\mathbf{i} - 20\mathbf{j})
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{BP}\cdot\overrightarrow{AP} &= 0 \\
((-80\mathbf{i}+49\mathbf{j}) + t\,(30\mathbf{i}-20\mathbf{j}))\cdot(30\mathbf{i}-20\mathbf{j}) &= 0 \\
(-80\mathbf{i}+49\mathbf{j})\cdot(30\mathbf{i}-20\mathbf{j}) & \\
+t\,(30\mathbf{i}-20\mathbf{j})\cdot(30\mathbf{i}-20\mathbf{j}) &= 0 \\
-2400 - 980 + t(900+400) &= 0 \\
1300t &= 3380 \\
t &= 2.6
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{BP} &= (-80\mathbf{i}+49\mathbf{j}) + 2.6\,(30\mathbf{i}-20\mathbf{j}) \\
&= (-2\mathbf{i} - 3\mathbf{j}) \\
|\overrightarrow{BP}| &= \sqrt{13} \\
&\approx 3.6
\end{aligned}$$

Therefore, the cue ball does strike the 12 ball.

32. Starting with the first expression stripped of the constant of integration:

$$\begin{aligned}
&-\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x\\
&=\cos x\left(-1+\frac{2}{3}\cos^2 x-\frac{1}{5}\cos^4 x\right)\\
&=\cos x\left(-1+\frac{2(1-\sin^2 x)}{3}-\frac{(1-\sin^2 x)^2}{5}\right)\\
&=\cos x\left(-1+\frac{2}{3}-\frac{2\sin^2 x}{3}-\frac{1-2\sin^2 x+\sin^4 x}{5}\right)\\
&=\cos x\left(-1+\frac{2}{3}-\frac{1}{5}-\frac{2\sin^2 x}{3}+\frac{2\sin^2 x}{5}-\frac{\sin^4 x}{5}\right)\\
&=\cos x\left(\frac{-15+10-3}{15}+\frac{(-10+6)\sin^2 x}{15}-\frac{\sin^4 x}{5}\right)\\
&=\cos x\left(\frac{-8}{15}-\frac{4\sin^2 x}{15}-\frac{\sin^4 x}{5}\right)\\
&=\cos x\left(\frac{-\sin^4 x}{5}-\frac{4\sin^2 x}{15}-\frac{8}{15}\right)
\end{aligned}$$

which gives the first calculator display.

Now starting with some parts of the second calculator display:

$$\begin{aligned}
\cos 3x&=\cos(x+2x)\\
&=\cos x\cos 2x-\sin x\sin 2x\\
&=\cos x(2\cos^2 x-1)-\sin x(2\sin x\cos x)\\
&=2\cos^3 x-\cos x-2\sin^2 x\cos x\\
&=2\cos^3 x-\cos x-2(1-\cos^2 x)\cos x\\
&=2\cos^3 x-\cos x-2\cos x+2\cos^3 x\\
&=4\cos^3 x-3\cos x
\end{aligned}$$

$$\begin{aligned}
\cos 5x&=\cos(3x+2x)\\
&=\cos 3x\cos 2x-\sin 3x\sin 2x\\
&=(4\cos^3 x-3\cos x)(2\cos^2 x-1)-\sin(x+2x)\sin 2x\\
&=(8\cos^5 x-4\cos^3 x-6\cos^3 x+3\cos x)\\
&\quad-(\sin x\cos 2x+\cos x\sin 2x)(2\sin x\cos x)\\
&=(8\cos^5 x-10\cos^3 x+3\cos x)\\
&\quad-(\sin x(2\cos^2 x-1)+\cos x(2\sin x\cos x))(2\sin x\cos x)\\
&=8\cos^5 x-10\cos^3 x+3\cos x\\
&\quad-(2\sin x\cos^2 x-\sin x+2\sin x\cos^2 x)(2\sin x\cos x)\\
&=8\cos^5 x-10\cos^3 x+3\cos x\\
&\quad-(4\sin x\cos^2 x-\sin x)(2\sin x\cos x)\\
&=8\cos^5 x-10\cos^3 x+3\cos x\\
&\quad-(8\sin^2 x\cos^3 x-2\sin^2 x\cos x)\\
&=8\cos^5 x-10\cos^3 x+3\cos x\\
&\quad-\sin^2 x(8\cos^3 x-2\cos x)\\
&=8\cos^5 x-10\cos^3 x+3\cos x\\
&\quad-(1-\cos^2 x)(8\cos^3 x-2\cos x)\\
&=8\cos^5 x-10\cos^3 x+3\cos x\\
&\quad-(8\cos^3 x-2\cos x)+(8\cos^5 x-2\cos^3 x)\\
&=8\cos^5 x-10\cos^3 x+3\cos x\\
&\quad-8\cos^3 x+2\cos x+8\cos^5 x-2\cos^3 x\\
&=16\cos^5 x-20\cos^3 x+5\cos x
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{-(150\cos x+3\cos 5x-25\cos 3x)}{240}\\
&\quad=\frac{-150\cos x}{240}\\
&\qquad-\frac{48\cos^5 x-60\cos^3 x+15\cos x}{240}\\
&\qquad-\frac{-100\cos^3 x+75\cos x}{240}\\
&\quad=\frac{-48\cos^5 x+160\cos^3 x-240\cos x)}{240}\\
&\quad=-\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x
\end{aligned}$$

as required.