Chapter 6

Exercise 6A

1. $$\begin{aligned} y + x\frac{\mathrm{d}y}{\mathrm{d}x} + 8 &= -2\frac{\mathrm{d}y}{\mathrm{d}x} \\ x\frac{\mathrm{d}y}{\mathrm{d}x} + 2\frac{\mathrm{d}y}{\mathrm{d}x} &= -y - 8 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{y+8}{x+2} \end{aligned}$$

2. $$\begin{aligned} y + x\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x} - 4 &= 6x \\ \frac{\mathrm{d}y}{\mathrm{d}x}(x+1) &= 6x - y + 4 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{6x - y + 4}{x+1} \end{aligned}$$

3. $$\begin{aligned} 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} - 2 &= 6xy + 3x^2\frac{\mathrm{d}y}{\mathrm{d}x} \\ 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} - 3x^2\frac{\mathrm{d}y}{\mathrm{d}x} &= 6xy + 2 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{6xy+2}{3y^2 - 3x^2} \end{aligned}$$

4. $$\begin{aligned} 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 6x^2y + 2x^3\frac{\mathrm{d}y}{\mathrm{d}x} + 5 \\ 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 2x^3\frac{\mathrm{d}y}{\mathrm{d}x} &= 6x^2y + 5 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{6x^2y+5}{2y - 2x^3} \end{aligned}$$

5. $$\begin{aligned} 10y\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x + 2y + 2x\frac{\mathrm{d}y}{\mathrm{d}x} - 3 \\ 10y\frac{\mathrm{d}y}{\mathrm{d}x} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x + 2y - 3 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x+2y-3}{10y - 2x} \end{aligned}$$

6. $$\begin{aligned} 1 + 6y\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x + 2y + 2x\frac{\mathrm{d}y}{\mathrm{d}x} \\ 6y\frac{\mathrm{d}y}{\mathrm{d}x} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x + 2y - 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x+2y-1}{6y - 2x} \end{aligned}$$

7. $$\begin{aligned} 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 9 \\ 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 9 - 2x \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{9-2x}{2y} \end{aligned}$$

8. $$\begin{aligned} 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 9\frac{\mathrm{d}y}{\mathrm{d}x} \\ 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 9\frac{\mathrm{d}y}{\mathrm{d}x} &= -2x \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-2x}{2y-9} \\ &= \frac{2x}{9-2y} \end{aligned}$$

9. $$\begin{aligned} 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 9y + 9x\frac{\mathrm{d}y}{\mathrm{d}x} \\ 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 9x\frac{\mathrm{d}y}{\mathrm{d}x} &= 9y - 2x \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{9y-2x}{2y-9x} \end{aligned}$$

10. $$\begin{aligned} 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 9y + 9x\frac{\mathrm{d}y}{\mathrm{d}x} + 1 + \frac{\mathrm{d}y}{\mathrm{d}x} \\ 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 9x\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{\mathrm{d}y}{\mathrm{d}x} &= 9y - 2x + 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{9y-2x+1}{2y-9x-1} \end{aligned}$$

11. $$\begin{aligned} \cos x - (\sin y)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (\sin y)\frac{\mathrm{d}y}{\mathrm{d}x} &= \cos x \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\cos x}{\sin x} \\ &= \frac{1}{\tan x} \\ &= \cot x \end{aligned}$$

12. $$\begin{aligned} 2x\cos y + x^2(-\sin y)\frac{\mathrm{d}y}{\mathrm{d}x} &= 10y + 10x\frac{\mathrm{d}y}{\mathrm{d}x} \\ -x^2\sin y\frac{\mathrm{d}y}{\mathrm{d}x} - 10x\frac{\mathrm{d}y}{\mathrm{d}x} &= 10y - 2x\cos y \\ \frac{\mathrm{d}y}{\mathrm{d}x}(x^2\sin y + 10x) &= 2x\cos y - 10y \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x\cos y - 10y}{x^2\sin y + 10x} \end{aligned}$$

13. $$\begin{aligned} 6 + y + x\frac{\mathrm{d}y}{\mathrm{d}x} + 2\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{6+y}{x+2} \\ &= -\frac{6+2}{-3+2} \\ &= 8 \end{aligned}$$

14. $$\begin{aligned} 6\frac{\mathrm{d}y}{\mathrm{d}x} + y + x\frac{\mathrm{d}y}{\mathrm{d}x} &= 3 \\ 6\frac{\mathrm{d}y}{\mathrm{d}x} + x\frac{\mathrm{d}y}{\mathrm{d}x} &= 3 - y \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3-y}{6+x} \\ &= \frac{3-2}{6+2} \\ &= \frac{1}{8} \end{aligned}$$

15.
$$\begin{aligned}3x^2 &= y + x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x}\\ 3x^2 - y &= \frac{\mathrm{d}y}{\mathrm{d}x}(x+2y)\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3x^2-y}{x+2y}\\ &= \frac{3(1)^2-(-3)}{1+2(-3)}\\ &= \frac{6}{-5}\\ &= -1.2\end{aligned}$$

16.
$$\begin{aligned}2y\frac{\mathrm{d}y}{\mathrm{d}x} + 3y + 3x\frac{\mathrm{d}y}{\mathrm{d}x} &= 4\\ 2y\frac{\mathrm{d}y}{\mathrm{d}x} + 3x\frac{\mathrm{d}y}{\mathrm{d}x} &= 4 - 3y\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4-3y}{2y+3x}\\ &= \frac{4-3(-4)}{2(-4)+3(1)}\\ &= \frac{16}{-5}\\ &= -3.2\end{aligned}$$

17.
$$\begin{aligned}2x + \frac{x\frac{\mathrm{d}y}{\mathrm{d}x} - y}{x^2} &= 2\frac{\mathrm{d}y}{\mathrm{d}x}\\ 2(1) + \frac{1\frac{\mathrm{d}y}{\mathrm{d}x} - 1}{1^2} &= 2\frac{\mathrm{d}y}{\mathrm{d}x}\\ 2 + \frac{\mathrm{d}y}{\mathrm{d}x} - 1 &= 2\frac{\mathrm{d}y}{\mathrm{d}x}\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= 1\\ y - y_1 &= m(x-x_1)\\ y - 1 &= 1(x-1)\\ y &= x\end{aligned}$$

18.
$$\begin{aligned}10x + \frac{1}{2\sqrt{xy}}(y + x\frac{\mathrm{d}y}{\mathrm{d}x}) &= 2y\frac{\mathrm{d}y}{\mathrm{d}x}\\ 10(4) + \frac{1}{2\sqrt{(4)(9)}}(9 + 4\frac{\mathrm{d}y}{\mathrm{d}x}) &= 2(9)\frac{\mathrm{d}y}{\mathrm{d}x}\\ 40 + \frac{1}{12}(9 + 4\frac{\mathrm{d}y}{\mathrm{d}x}) &= 18\frac{\mathrm{d}y}{\mathrm{d}x}\\ 480 + 9 + 4\frac{\mathrm{d}y}{\mathrm{d}x} &= 216\frac{\mathrm{d}y}{\mathrm{d}x}\\ 212\frac{\mathrm{d}y}{\mathrm{d}x} &= 489\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{489}{212}\end{aligned}$$

19.
$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 2xy + x^2\frac{\mathrm{d}y}{\mathrm{d}x}\\ &= 2xy + x^2(x^2y)\\ &= 2xy + x^4y\end{aligned}$$

20. Solve for $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$:
$$\begin{aligned}2x + 8y\frac{\mathrm{d}y}{\mathrm{d}x} - 2 + 6\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ 2x - 2 &= 0\\ x &= 1\\ (1)^2 + 4y^2 - 2(1) + 6y &= 17\\ 4y^2 + 6y + 1 - 2 - 17 &= 0\\ 4y^2 + 6y - 18 &= 0\\ 2y^2 + 3y - 9 &= 0\\ (2y-3)(y+3) &= 0\\ y &= 1.5\\ \text{or } y &= -3\end{aligned}$$
The points are $(1, 1.5)$ and $(1, -3)$.

21. Where the tangent is vertical, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined. We could find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ and then identify the points where this is undefined, but it may be simpler to instead find an expression for $\frac{\mathrm{d}x}{\mathrm{d}y}$. Where the tangent is vertical $\frac{\mathrm{d}x}{\mathrm{d}y} = 0$.
$$\begin{aligned}2x\frac{\mathrm{d}x}{\mathrm{d}y} + 2y - 4\frac{\mathrm{d}x}{\mathrm{d}y} + 6 &= 0\\ 2y + 6 &= 0\\ y &= -3\\ x^2 + (-3)^2 - 4x + 6(-3) + 12 &= 0\\ x^2 - 4x + 9 - 18 + 12 &= 0\\ x^2 - 4x + 3 &= 0\\ (x-3)(x-1) &= 0\\ x &= 3\\ \text{or } x &= 1\end{aligned}$$
The points are $(3, -3)$ and $(1, -3)$.

22.
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} - 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x + 1\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x+1}{1-3y^2}\end{aligned}$$
at $(1, 0)$:
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2(1)+1}{1-3(0)^2}\\ &= 3\\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{2(1-3y^2) - (2x+1)(-6y\frac{\mathrm{d}y}{\mathrm{d}x})}{(1-3y^2)^2}\\ &= \frac{2(1-3y^2) + 6y(2x+1)\frac{2x+1}{1-3y^2}}{(1-3y^2)^2}\\ &= \frac{2(1-3y^2)^2 + 6y(2x+1)^2}{(1-3y^2)^3}\end{aligned}$$
at $(1, 0)$:
$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{2(1-3(0)^2)^2 + 6(0)(2(1)+1)^2}{(1-3(0)^2)^3}\\ &= \frac{2(1)^2}{(1)^3}\\ &= 2\end{aligned}$$

23.
$$
\begin{aligned}
2x &= 2(\cos y)\frac{\mathrm{d}y}{\mathrm{d}x}\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x}{\cos y}
\end{aligned}
$$

at $\left(1, \frac{\pi}{6}\right)$:

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\cos\frac{\pi}{6}}\\
&= \frac{1}{\frac{\sqrt{3}}{2}}\\
&= \frac{2}{\sqrt{3}}\\
y - \frac{\pi}{6} &= \frac{2}{\sqrt{3}}(x-1)\\
6y - \pi &= \frac{12}{\sqrt{3}}(x-1)\\
&= 4\sqrt{3}(x-1)\\
6y &= 4\sqrt{3}x - 4\sqrt{3} + \pi
\end{aligned}
$$

24.
$$
\begin{aligned}
2y\frac{\mathrm{d}y}{\mathrm{d}x} - \sin x &= 3\frac{\mathrm{d}y}{\mathrm{d}x}\\
2y\frac{\mathrm{d}y}{\mathrm{d}x} - 3\frac{\mathrm{d}y}{\mathrm{d}x} &= \sin x\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\sin x}{2y-3}\\
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{(\cos x)(2y-3) - (\sin x)(2\frac{\mathrm{d}y}{\mathrm{d}x})}{(2y-3)^2}\\
&= \frac{(\cos x)(2y-3) - (2\sin x)\frac{\sin x}{2y-3}}{(2y-3)^2}\\
&= \frac{(2y-3)^2\cos x - 2\sin^2 x}{(2y-3)^3}\\
&= \frac{\cos x}{2y-3} - \frac{2\sin^2 x}{(2y-3)^3}
\end{aligned}
$$

25.
$$
\begin{aligned}
(2\cos y)\frac{\mathrm{d}y}{\mathrm{d}x} - 2x &= 2\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2+2x}{2\cos y}\\
&= \frac{1+x}{\cos y}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\cos y - (1+x)(-\sin y)\frac{\mathrm{d}y}{\mathrm{d}x}}{\cos^2 y}\\
&= \frac{\cos y - (1+x)(-\sin y)\frac{1+x}{\cos y}}{\cos^2 y}\\
&= \frac{\cos^2 y - (1+x)(-\sin y)(1+x)}{\cos^3 y}\\
&= \frac{\cos^2 y + (1+x)^2\sin y}{\cos^3 y}
\end{aligned}
$$

At $\left(-2, \frac{\pi}{6}\right)$:

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1+(-2)}{\cos\frac{\pi}{6}}\\
&= \frac{-1}{\frac{\sqrt{3}}{2}}\\
&= -\frac{2}{\sqrt{3}}\\
&= -\frac{2\sqrt{3}}{3}\\
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\cos^2\frac{\pi}{6} + (1+(-2))^2\sin\frac{\pi}{6}}{\cos^3\frac{\pi}{6}}\\
&= \frac{\frac{3}{4} + (1)\frac{1}{2}}{\frac{3\sqrt{3}}{8}}\\
&= \frac{\frac{5}{4}}{\frac{3\sqrt{3}}{8}}\\
&= \frac{10}{3\sqrt{3}}\\
&= \frac{10\sqrt{3}}{9}
\end{aligned}
$$

26.
$$
\begin{aligned}
6x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\
6x + 2y(-1) &= 0\\
6x - 2y &= 0\\
2y &= 6x\\
y &= 3x
\end{aligned}
$$

We need to find the points that satisfy this equation and also lie on the ellipse. You could solve this graphically or with technology or algebraically as follows:

$$
\begin{aligned}
3x^2 + (3x)^2 &= 9\\
3x^2 + 9x^2 &= 9\\
12x^2 &= 9\\
x^2 &= \frac{3}{4}\\
x &= \pm\frac{\sqrt{3}}{2}\\
y &= 3x\\
&= \pm\frac{3\sqrt{3}}{2}
\end{aligned}
$$

The points having gradient -1 are $\left(\frac{\sqrt{3}}{2}, \frac{3\sqrt{3}}{2}\right)$ and $\left(-\frac{\sqrt{3}}{2}, -\frac{3\sqrt{3}}{2}\right)$.

Exercise 6B

1. (a) $\frac{dx}{dt} = 6\cos 2t$

   (b) $\frac{dy}{dt} = -10\sin 5t$

   (c) $$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\ &= -10\sin 5t\frac{1}{6\cos 2t}\\ &= -\frac{5\sin 5t}{3\cos 2t}\end{aligned}$$

2. (a) $$\begin{aligned}\frac{dx}{dt} &= 2\sin t\cos t\\ &= \sin 2t\end{aligned}$$

   (b) $\frac{dy}{dt} = -3\sin 3t$

   (c) $$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\ &= -3\sin 3t\frac{1}{\sin 2t}\\ &= -\frac{3\sin 3t}{\sin 2t}\end{aligned}$$

3. $$\begin{aligned}\frac{dx}{dt} &= 3\\ \frac{dy}{dt} &= 2t\\ \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\ &= \frac{2t}{3}\end{aligned}$$

4. $$\begin{aligned}\frac{dx}{dt} &= 2t\\ \frac{dy}{dt} &= 3\\ \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\ &= \frac{3}{2t}\end{aligned}$$

5. $$\begin{aligned}\frac{dx}{dt} &= 15t^2\\ \frac{dy}{dt} &= 2t+2\\ \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\ &= \frac{2t+2}{15t^2}\end{aligned}$$

6. $$\begin{aligned}\frac{dx}{dt} &= 6t+6\\ \frac{dy}{dt} &= -\frac{1}{(t+1)^2}\\ \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\ &= -\frac{1}{(t+1)^2}\frac{1}{6(t+1)}\\ &= -\frac{1}{6(t+1)^3}\end{aligned}$$

7. $$\begin{aligned}\frac{dx}{dt} &= 2t\\ \frac{dy}{dt} &= 2(t-1)\\ \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\ &= \frac{2(t-1)}{2t}\\ &= \frac{t-1}{t}\\ &= 1-\frac{1}{t}\end{aligned}$$

8. $$\begin{aligned}\frac{dx}{dt} &= \frac{(t-1)-t}{(t-1)^2}\\ &= -\frac{1}{(t-1)^2}\\ \frac{dy}{dt} &= -2\frac{1}{(t+1)^2}\\ \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\ &= -2\frac{1}{(t+1)^2}(-(t-1)^2)\\ &= \frac{2(t-1)^2}{(t+1)^2}\end{aligned}$$

9. $$\begin{aligned}\frac{dx}{dt} &= 2t\\ \frac{dy}{dt} &= 3t^2\\ \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\ &= \frac{3t^2}{2t}\\ &= \frac{3t}{2}\end{aligned}$$

   at $t = -1$:
   $$\frac{dy}{dx} = -\frac{3}{2}$$

10. $$\begin{aligned}\frac{dx}{dt} &= -\frac{1}{(t+1)^2}\\ \frac{dy}{dt} &= 2t\\ \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\ &= -2t(t+1)^2 \text{at } t = 2:\\ \frac{dy}{dx} &= -2\times 2(2+1)^2\\ &= -36\end{aligned}$$

11.
$$\begin{aligned}
\frac{dx}{dt} &= 4t+3\\
\frac{dy}{dt} &= 3t^2-12\\
\frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\
&= \frac{3t^2-12}{4t+3}\\
\frac{dy}{dx} &= 0\\
\frac{3t^2-12}{4t+3} &= 0\\
3t^2-12 &= 0\\
t^2-4 &= 0\\
t^2 &= 4\\
t &= \pm 2
\end{aligned}$$

for $t=+2$:
$$\begin{aligned}
x &= 2(2)^2+3(2)\\
&= 14\\
y &= (2)^3-12(2)\\
&= -16
\end{aligned}$$

for $t=-2$:
$$\begin{aligned}
x &= 2(-2)^2+3(-2)\\
&= 2\\
y &= (-2)^3-12(-2)\\
&= 16
\end{aligned}$$

The points on the curve where $\frac{dy}{dx}=0$ are $(14,-16)$ and $(2,16)$.

12.
$$\begin{aligned}
\frac{dx}{dt} &= -3\sin t\\
\frac{dy}{dt} &= 5\cos t\\
\frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\
&= \frac{5\cos t}{-3\sin t}\text{at } t=\frac{\pi}{6}:\\
\frac{dy}{dx} &= \frac{5\cos\frac{\pi}{6}}{-3\sin\frac{\pi}{6}}\\
&= \frac{\frac{5\sqrt{3}}{2}}{\frac{-3}{2}}\\
&= -\frac{5\sqrt{3}}{3}\\
y-5\sin\frac{\pi}{6} &= -\frac{5\sqrt{3}}{3}\left(x-3\cos\frac{\pi}{6}\right)\\
y-\frac{5}{2} &= -\frac{5\sqrt{3}}{3}\left(x-\frac{3\sqrt{3}}{2}\right)\\
y &= -\frac{5\sqrt{3}}{3}\left(x-\frac{3\sqrt{3}}{2}\right)+\frac{5}{2}\\
&= -\frac{5\sqrt{3}x}{3}+\frac{15}{2}+\frac{5}{2}\\
&= -\frac{5\sqrt{3}x}{3}+10
\end{aligned}$$

13. (a)
$$\begin{aligned}
\frac{dx}{dt} &= 4\cos t\\
\frac{dy}{dt} &= 4\cos 2t\\
\frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\
&= \frac{4\cos 2t}{4\sin t}\\
&= \frac{\cos 2t}{\cos t}
\end{aligned}$$

(b) At $t=\frac{\pi}{6}$:
$$\begin{aligned}
x &= 4\sin\frac{\pi}{6}\\
&= 2\\
y &= 2\sin\frac{\pi}{3}\\
&= \sqrt{3}\\
\frac{dy}{dx} &= \frac{\cos\frac{\pi}{3}}{\cos\frac{\pi}{6}}\\
&= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\\
&= \frac{1}{\sqrt{3}}\\
&= \frac{\sqrt{3}}{3}
\end{aligned}$$

(c)
$$\begin{aligned}
\frac{\cos 2t}{\cos t} &= 0\\
\cos 2t &= 0\\
2t &= \frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2}\\
t &= \frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}
\end{aligned}$$

14. (a)
$$\begin{aligned}
\frac{dy}{dt} &= 1-\frac{2}{t^2}\\
\frac{dx}{dt} &= 2+\frac{1}{t^2}\\
\frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx}\\
&= \frac{1-\frac{2}{t^2}}{2+\frac{1}{t^2}}\\
&= \frac{t^2-2}{2t^2+1}
\end{aligned}$$

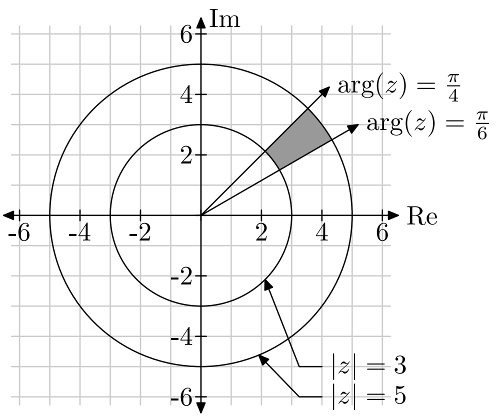
(b)
$$\begin{aligned}
\frac{d^2y}{dx^2} &= \dot{d}t\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}\\
&= \left(\frac{2t(2t^2+1)-(t^2-2)4t}{(2t^2+1)^2}\right)\frac{1}{2+\frac{1}{t^2}}\\
&= \left(\frac{2t(2t^2+1)-2t(2t^2-4)}{(2t^2+1)^2}\right)\frac{t^2}{2t^2+1}\\
&= \left(\frac{2t(2t^2+1-2t^2+4)}{(2t^2+1)^2}\right)\frac{t^2}{2t^2+1}\\
&= \frac{2t^3(5)}{(2t^2+1)^3}\\
&= \frac{10t^3}{(2t^2+1)^3}
\end{aligned}$$

Miscellaneous Exercise 6

1. $$\begin{aligned} 6\operatorname{cis}\frac{3\pi}{4} &= 6\cos\frac{3\pi}{4} + 6\mathrm{i}\sin\frac{3\pi}{4} \\ &= -3\sqrt{2} + 3\sqrt{2}\mathrm{i} \end{aligned}$$

2. $$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\cos x(1-\sin x) - (-\cos x)(1+\sin x)}{(1-\sin x)^2} \\ &= \frac{\cos x(1-\sin x + 1 + \sin x}{(1-\sin x)^2} \\ &= \frac{2\cos x}{(1-\sin x)^2} \end{aligned}$$

3. $a = 3$ (read from the radius of the inner circle)
   $b = 5$ (read from the radius of the outer circle)
   $c = 3$ (when $\theta = \pi$, $r \approx 9.5 \approx 3\pi$)
   At point A:
   $$3\theta = 3$$
   coordinates are $(3, 1)$
   At point B:
   $$3\theta = 5$$
   coordinates are $\left(5, \frac{5}{3}\right)$

4. 

5. (a) $$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2(3-2x) - (2x+1)(-2)}{(3-2x)^2} \\ &= \frac{6-4x+4x+2}{(3-2x)^2} \\ &= \frac{8}{(3-2x)^2} \end{aligned}$$

   (b) $$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 3\sin^2(2x+1)\cos(2x+1)(2) \\ &= 6\sin^2(2x+1)\cos(2x+1) \end{aligned}$$

   (c) $$\begin{aligned} 6xy + 3x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} &= 5 \\ 3x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} &= 5 - 6xy \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{5-6xy}{3(x^2+y^2)} \end{aligned}$$

   (d) $$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= 2t+3 \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 4t^3 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} \\ &= \frac{4t^3}{2t+3} \end{aligned}$$

6. $$\begin{aligned} \mathrm{f}'(x) &= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\,\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\ &= \lim_{h\to 0}\frac{(x+h)-(x)}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} \\ &= \frac{1}{\sqrt{x+0}+\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

7. $$\left(\begin{pmatrix}-1\\-10\\4\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\-1\end{pmatrix}\right)\cdot\begin{pmatrix}2\\4\\-1\end{pmatrix} = 5$$
   $$\begin{aligned} 2(-1+2\lambda) + 4(-10+3\lambda) - 1(4-\lambda) &= 5 \\ -2+4\lambda-40+12\lambda-4+\lambda &= 5 \\ 17\lambda - 46 &= 5 \\ 17\lambda &= 51 \\ \lambda &= 3 \end{aligned}$$
   $$\begin{aligned} \mathbf{r} &= \begin{pmatrix}-1\\-10\\4\end{pmatrix} + 3\begin{pmatrix}2\\3\\-1\end{pmatrix} \\ \mathbf{r} &= \begin{pmatrix}5\\-1\\1\end{pmatrix} \end{aligned}$$

8. $$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= -5\sin t \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 5\cos t \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} \\ &= \frac{5\cos t}{-5\sin t} \\ &= -\frac{\cos t}{\sin t} \end{aligned}$$
   At $t = \frac{2\pi}{3}$:

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{\cos\frac{2\pi}{3}}{\sin\frac{2\pi}{3}} \\
&= -\frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} \\
&= \frac{1}{\sqrt{3}} \\
y - 5\sin\frac{2\pi}{3} &= \frac{1}{\sqrt{3}}(x - 5\cos\frac{2\pi}{3}) \\
\sqrt{3}y - \frac{15}{2} &= x + \frac{5}{2} \\
\sqrt{3}y &= x + \frac{5}{2} + \frac{15}{2} \\
\sqrt{3}y &= x + 10
\end{aligned}$$

9. (a) $\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} = \frac{d}{dx}\sin x = \cos x$

   (b) $\lim_{h\to 0}\frac{\cos 3(x+h)-\cos 3(x)}{h} = \frac{d}{dx}\cos 3x = -3\sin 3x$

   (c) $\lim_{h\to 0}\frac{\tan(x+h)-\tan(x)}{h} = \frac{d}{dx}\tan x = 1 - \tan^2 x$

   (d) $\lim_{h\to 0}\frac{(x+h)^2-x^2}{h} = \frac{d}{dx}x^2 = 2x$

   (e) $\lim_{h\to 0}\frac{(x+h)^3-x^3}{h} = \frac{d}{dx}x^3 = 3x^2$

   (f) $\lim_{h\to 0}\frac{\sin^2(x+h)-\sin^2(x)}{h} = \frac{d}{dx}\sin^2 x = 2\sin x\cos x = \sin 2x$

10. (a)
$$\begin{aligned}
4y^3\frac{dy}{dx} &= 4x^3 - 4y - 4x\frac{dy}{dx} \\
4y^3\frac{dy}{dx} + 4x\frac{dy}{dx} &= 4x^3 - 4y \\
\frac{dy}{dx} &= \frac{4x^3-4y}{4y^3+4x} \\
&= \frac{x^3-y}{y^3+1}
\end{aligned}$$

   (b)
$$\begin{aligned}
\frac{dy}{dx} &= \frac{(2)^3-(1)}{(1)^3+(2)} \\
&= \frac{7}{3}
\end{aligned}$$

11. (a) Any positive or negative scalar multiple of $(2\mathbf{i} + \mathbf{j} - 3\mathbf{k})$ is a correct answer.

   (b) Any vector that has a zero dot product with $(2\mathbf{i} + \mathbf{j} - 3\mathbf{k})$ is a correct answer. One simple way to find an example perpendicular is to make one of the components zero, then swap the other two and change the sign of one of them. For example, if we zero the $\mathbf{k}$ component, swap the $\mathbf{i}$ and $\mathbf{j}$ components and negate the new $\mathbf{i}$ component we get $(-\mathbf{i} + 2\mathbf{j} + 0\mathbf{k})$ and $(2\mathbf{i} + 1\mathbf{j}) - 3 \cdot (-\mathbf{i} + 2\mathbf{j} + 0\mathbf{k}) = -2 + 2 + 0 = 0$.

12. Start with defining P and Q in terms of A and B and vice versa:

$$\begin{aligned}
P &= A + B \\
Q &= A - B \\
P + Q &= A + B + A - B \\
&= 2A \\
\therefore\quad A &= \frac{P+Q}{2} \\
P - Q &= A + B - A + B \\
&= 2B \\
\therefore\quad B &= \frac{P-Q}{2}
\end{aligned}$$

$$\begin{aligned}
\sin P + \sin Q &= \sin(A+B) + \sin(A-B) \\
&= \sin A\cos B - \cos A\sin B \\
&\quad + \sin A\cos B + \cos A\sin B \\
&= 2\sin A\cos B \\
&= 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2} \\
\sin P - \sin Q &= \sin(A+B) - \sin(A-B) \\
&= \sin A\cos B - \cos A\sin B \\
&\quad - \sin A\cos B - \cos A\sin B \\
&= 2\cos A\sin B \\
&= 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}
\end{aligned}$$

□

13. $\lambda\begin{pmatrix}2\\3\\1\end{pmatrix} + \mu\begin{pmatrix}1\\4\\-1\end{pmatrix} + \eta\begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}-6\\5\\-3\end{pmatrix}$

If we look at the $\mathbf{j}$ and $\mathbf{k}$ components, we can solve for λ and μ, then solve for η using the $\mathbf{i}$ component.

$$\begin{aligned}
3\lambda + 4\mu &= 5 \\
\lambda - \mu &= -3 \\
4\lambda - 4\mu &= -12 \\
7\lambda &= -7 \\
\lambda &= -1 \\
-1 - \mu &= -3 \\
\mu &= 2 \\
2(-1) + 1(2) + 2\eta &= -6 \\
\eta &= -3 \\
\begin{pmatrix}-6\\5\\-3\end{pmatrix} &= -\mathbf{a} + 2\mathbf{b} - 3\mathbf{c}
\end{aligned}$$

14. $$\begin{aligned}\overrightarrow{AB} &= (3\mathbf{p}-\mathbf{q})-(2\mathbf{p}+\mathbf{q})\\ &= \mathbf{p}-2\mathbf{q}\\ \overrightarrow{AC} &= (6\mathbf{p}-7\mathbf{q})-(2\mathbf{p}+\mathbf{q})\\ &= 4\mathbf{p}-8\mathbf{q}\\ &= 4\overrightarrow{AB}\end{aligned}$$

$\therefore$ A, B and C are collinear. □

Since $\overrightarrow{AB}$ is one quarter of $\overrightarrow{AC}$ it follows that

$$\begin{aligned}\overrightarrow{AB} : \overrightarrow{BC} &= 1 : 3\\ \text{and } \overrightarrow{AB} : \overrightarrow{AC} &= 1 : 4\end{aligned}$$

15. (a) $$\begin{aligned}\cos\theta &= \frac{(\mathbf{i}-2\mathbf{k})\cdot(2\mathbf{i}+3\mathbf{j}-\mathbf{k})}{|\,(\mathbf{i}-2\mathbf{k})\,||\,(2\mathbf{i}+3\mathbf{j}-\mathbf{k})\,|}\\ \cos\theta &= \frac{2+2}{\sqrt{5}\sqrt{14}}\\ \theta &= 1.07\end{aligned}$$

(b) $$\begin{aligned}&-2\mathbf{i}+3\mathbf{j}+8\mathbf{k}+\lambda(\mathbf{i}-2\mathbf{k})\\ &\qquad = 5\mathbf{i}+6\mathbf{j}-3\mathbf{k}+\mu(2\mathbf{i}+3\mathbf{j}-\mathbf{k})\end{aligned}$$

j components:

$$\begin{aligned}3 &= 6+3\mu\\ \mu &= -1\end{aligned}$$

i components:

$$\begin{aligned}-2+\lambda &= 5+2\mu\\ \lambda &= 7+2(-1)\\ \lambda &= 5\end{aligned}$$

Lines intersect if these values also satisfy the **k** components:

$$\begin{aligned}8-2(5) &= -3-(-1)\\ -2 &= -2\end{aligned}$$

Therefore the lines intersect.

$$\begin{aligned}\mathrm{P} &= -2\mathbf{i}+3\mathbf{j}+8\mathbf{k}+5(\mathbf{i}-2\mathbf{k})\\ &= 3\mathbf{i}+3\mathbf{j}-2\mathbf{k}\end{aligned}$$

(c) $$\begin{aligned}\overrightarrow{AB} &= (3\mathbf{i}+\mathbf{j}+\mathbf{k})-(2\mathbf{i}-\mathbf{j}+3\mathbf{k})\\ &= \mathbf{i}+2\mathbf{j}-2\mathbf{k}\\ \mathbf{r}\cdot(\mathbf{i}+2\mathbf{j}-2\mathbf{k}) &= (3\mathbf{i}+3\mathbf{j}-2\mathbf{k})\cdot(\mathbf{i}+2\mathbf{j}-2\mathbf{k})\\ &= 3+6+4\\ \mathbf{r}\cdot(\mathbf{i}+2\mathbf{j}-2\mathbf{k}) &= 13\end{aligned}$$

16. (a) $$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= 1-\frac{1}{t^2}\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 2t+2\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}\\ &= \frac{2t+2}{1-\frac{1}{t^2}}\\ &= \frac{2t^2(t+1)}{t^2-1}\\ &= \frac{2t^2(t+1)}{(t+1)(t-1)}\\ &= \frac{2t^2}{t-1}\end{aligned}$$

(b) $$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{4t(t-1)\frac{\mathrm{d}t}{\mathrm{d}x}-(2t^2)\frac{\mathrm{d}t}{\mathrm{d}x}}{(t-1)^2}\\ &= \frac{4t^2-4t-2t^2}{(t-1)^2}\frac{\mathrm{d}t}{\mathrm{d}x}\\ &= \frac{2t^2-4t}{(t-1)^2}\frac{1}{1-\frac{1}{t^2}}\\ &= \frac{2t(t-2)}{(t-1)^2}\frac{t^2}{t^2-1}\\ &= \frac{2t^3(t-2)}{(t-1)^2(t+1)(t-1)}\\ &= \frac{2t^3(t-2)}{(t-1)^3(t+1)}\end{aligned}$$

17. (a) $$\begin{aligned}\overrightarrow{AC} &= -\mathbf{a}+\mathbf{c}\\ \overrightarrow{GE} &= \mathbf{a}+0.5\overrightarrow{AC}\\ &= \mathbf{a}+0.5(-\mathbf{a}+\mathbf{c})\\ &= 0.5(\mathbf{a}+\mathbf{c})\\ \overrightarrow{GE}\cdot\overrightarrow{AC} &= 0\\ 0.5(\mathbf{a}+\mathbf{c})\cdot 0.5(-\mathbf{a}+\mathbf{c}) &= 0\\ (\mathbf{a}+\mathbf{c})\cdot(-\mathbf{a}+\mathbf{c}) &= 0\\ -\mathbf{a}^2+\mathbf{c}^2 &= 0\\ \mathbf{a}^2 &= \mathbf{c}^2\end{aligned}$$

□

(b) $$\begin{aligned}\overrightarrow{BC} &= -\mathbf{b}+\mathbf{c}\\ \overrightarrow{GD} &= \mathbf{b}+0.5\overrightarrow{BC}\\ &= \mathbf{b}+0.5(-\mathbf{b}+\mathbf{c})\\ &= 0.5(\mathbf{b}+\mathbf{c})\\ \overrightarrow{GD}\cdot\overrightarrow{BC} &= 0\\ 0.5(\mathbf{b}+\mathbf{c})\cdot 0.5(-\mathbf{b}+\mathbf{c}) &= 0\\ (\mathbf{b}+\mathbf{c})\cdot(-\mathbf{b}+\mathbf{c}) &= 0\\ -\mathbf{b}^2+\mathbf{c}^2 &= 0\\ \mathbf{b}^2 &= \mathbf{c}^2\end{aligned}$$

□

(c)
$$\begin{aligned}
\overrightarrow{\text{AB}} &= -\mathbf{a} + \mathbf{b} \\
\overrightarrow{\text{GF}} &= \mathbf{a} + 0.5\overrightarrow{\text{AB}} \\
&= \mathbf{a} + 0.5(-\mathbf{a} + \mathbf{b} \\
&= 0.5(\mathbf{a} + \mathbf{b}) \\
\overrightarrow{\text{GF}} \cdot \overrightarrow{\text{AB}} &= 0.5(\mathbf{a} + \mathbf{b}) \cdot 0.5(-\mathbf{a} + \mathbf{b}) \\
&= 0.25(\mathbf{a} + \mathbf{b}) \cdot (-\mathbf{a} + \mathbf{b}) \\
&= -\mathbf{a}^2 + \mathbf{b}^2 \\
&= -\mathbf{c}^2 + \mathbf{c}^2 \\
&= 0
\end{aligned}$$

$\therefore$ GF is perpendicular to AB. □

18. Let P be the point on the line closest to A.

$$\begin{aligned}
\overrightarrow{\text{AP}} &= \overrightarrow{\text{OP}} - \overrightarrow{\text{OA}} \\
&= -3\mathbf{i} + 2\mathbf{j} + 7\mathbf{k} + \mu\,(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) - (-5\mathbf{i} + 10\mathbf{j} - 3\mathbf{k}) \\
&= 2\mathbf{i} - 8\mathbf{j} + 10\mathbf{k} + \mu\,(\mathbf{i} - \mathbf{j} + 2\mathbf{k})
\end{aligned}$$

AP is perpendicular to line L:

$$\begin{aligned}
\overrightarrow{\text{AP}} \cdot (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) &= 0 \\
(2\mathbf{i} - 8\mathbf{j} + 10\mathbf{k} + \mu\,(\mathbf{i} - \mathbf{j} + 2\mathbf{k})) \cdot (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) &= 0 \\
2 + 8 + 20 + \mu(1 + 1 + 4) &= 0 \\
30 + 6\mu &= 0 \\
\mu &= -5
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{\text{AP}} &= 2\mathbf{i} - 8\mathbf{j} + 10\mathbf{k} + (-5)\,(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \\
&= -3\mathbf{i} - 3\mathbf{j} \\
|\overrightarrow{\text{AP}}| &= \sqrt{(-3)^2 + (-3)^2} \\
&= 3\sqrt{2}
\end{aligned}$$