Chapter 5

Exercise 5A

1. $\dfrac{d}{dx}\left(x^5 - x^2\right) = 5x^4 - 2x$

2. $$\begin{aligned}\frac{d}{dx}\left(3 + x^3\right) &= 0 + 3x^2\\ &= 3x^2\end{aligned}$$

3. $$\begin{aligned}\frac{d}{dx}(5 - \cos x) &= 0 - -\sin x\\ &= \sin x\end{aligned}$$

4. $$\begin{aligned}\frac{d}{dx}(\sin x - \cos x) &= \cos x - -\sin x\\ &= \cos x + \sin x\end{aligned}$$

5. $\dfrac{d}{dx}(\cos x - \sin x) = -\sin x - \cos x$

6. $$\begin{aligned}\frac{d}{dx}(x - \tan x) &= 1 - (1 + \tan^2 x)\\ &= -\tan^2 x\end{aligned}$$

7. $$\begin{aligned}\frac{d}{dx}((x+1)(2x-3)) &= 1(2x-3) + 2(x+1)\\ &= 2x - 3 + 2x + 2\\ &= 4x - 1\end{aligned}$$

8. $$\begin{aligned}\frac{d}{dx}\left(5x^2(1-5x)\right) &= 10x(1-5x) - 5(5x^2)\\ &= 10x - 50x^2 - 25x^2\\ &= 10x - 75x^2\end{aligned}$$

9. $$\begin{aligned}\frac{d}{dx}(6\sin x) &= (0)(\sin x) + (6)(\cos x)\\ &= 6\cos x\end{aligned}$$

10. $$\begin{aligned}\frac{d}{dx}(4\cos x) &= (0)(\cos x) + (4)(-\sin x)\\ &= -4\sin x\end{aligned}$$

11. $$\begin{aligned}\frac{d}{dx}(x\sin x) &= (1)(\sin x) + (x)(\cos x)\\ &= \sin x + x\cos x\end{aligned}$$

12. $$\begin{aligned}\frac{d}{dx}\left(x^2\cos x\right) &= (2x)(\cos x) + (x^2)(-\sin x)\\ &= 2x\cos x - x^2\sin x\end{aligned}$$

13. $$\begin{aligned}\frac{d}{dx}\left(\frac{x}{3x^2-1}\right) &= \frac{(1)(3x^2-1) - (x)(6x)}{(3x^2-1)^2}\\ &= \frac{3x^2 - 1 - 6x^2}{(3x^2-1)^2}\\ &= \frac{-3x^2-1}{(3x^2-1)^2}\\ &= -\frac{3x^2+1}{(3x^2-1)^2}\end{aligned}$$

14. $$\begin{aligned}\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right) &= \frac{(2x)(x^2-1) - (x^2+1)(2x)}{(x^2-1)^2}\\ &= \frac{(2x)(x^2-1-x^2-1)}{(x^2-1)^2}\\ &= \frac{(2x)(-2)}{(x^2-1)^2}\\ &= -\frac{4x}{(x^2-1)^2}\end{aligned}$$

15. $$\begin{aligned}\frac{d}{dx}\left(\frac{\cos x}{x}\right) &= \frac{(-\sin x)(x) - (\cos x)(1)}{x^2}\\ &= \frac{-x\sin x - \cos x}{x^2}\\ &= -\frac{\sin x}{x} - \frac{\cos x}{x^2}\end{aligned}$$

16. $$\begin{aligned}\frac{d}{dx}\left(\frac{\sin x}{x}\right) &= \frac{(\cos x)(x) - (\sin x)(1)}{x^2}\\ &= \frac{x\cos x - \sin x}{x^2}\\ &= \frac{\cos x}{x} - \frac{\sin x}{x^2}\end{aligned}$$

17. $$\begin{aligned}\frac{d}{dx}\left(\frac{x}{\sin x}\right) &= \frac{(1)(\sin x) - (x)(\cos x)}{\sin^2 x}\\ &= \frac{\sin x - x\cos x}{\sin^2 x}\end{aligned}$$

18. $$\begin{aligned}\frac{d}{dx}\left(\frac{x}{\cos x}\right) &= \frac{(1)(\cos x) - (x)(-\sin x)}{\cos^2 x}\\ &= \frac{\cos x + x\sin x}{\cos^2 x}\end{aligned}$$

19. $$\begin{aligned}\frac{dy}{dx} &= 6u\frac{du}{dx}\\ &= 6(x^2+1)(2x)\\ &= 12x(x^2+1)\end{aligned}$$

20. $$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2\sqrt{u}}\frac{du}{dx}\\ &= \frac{1}{2\sqrt{x^2-1}}(2x)\\ &= \frac{2x}{2\sqrt{x^2-1}}\\ &= \frac{x}{\sqrt{x^2-1}}\end{aligned}$$

21. $$\begin{aligned}\frac{dy}{dx} &= (\cos u)\frac{du}{dx}\\ &= 6\cos(6x)\end{aligned}$$

22. $$\begin{aligned}\frac{dy}{dx} &= (-\sin u)\frac{du}{dx}\\ &= (-\sin(2x+3))(2)\\ &= -2\sin(2x+3)\end{aligned}$$

23. $\dfrac{dy}{dx} = 2\sin x\cos x$

24. $\frac{dy}{dx} = 3\sin^2 x \cos x$

25. $\frac{dy}{dx} = (5\cos^4 x)(-\sin x)$
$= -5\cos^4 x \sin x$

26. $\frac{dy}{dx} = (-\sin 3x)(3)$
$= -3\sin 3x$

27. $\frac{dy}{dx} = (\cos(3x-7))(3)$
$= 3\cos(3x-7)$

28. $\frac{dy}{dx} = \left(\cos(x^2-3)\right)(2x)$
$= 2x\cos(x^2-3)$

29. $\frac{dy}{dx} = -3(-\sin x)$
$= 3\sin x$

30. $\frac{dy}{dx} = 3 + 2(-\sin x)$
$= 3 - 2\sin x$

31. $\frac{dy}{dx} = 2\cos 2x$
(By this stage you should be getting to the point of being able to do these in a single step.)

32. $\frac{d}{dx}\tan 2x = \frac{2}{\cos^2 2x}$

33. $\frac{dy}{dx} = 2x + \sin x$

34. $$\begin{aligned}\frac{d}{dx}\frac{1+\sin x}{x^2} &= \frac{(\cos x)(x^2) - (1+\sin x)(2x)}{x^4}\\ &= \frac{x^2\cos x - 2x - 2x\sin x}{x^4}\\ &= \frac{x\cos x - 2 - 2\sin x}{x^3}\end{aligned}$$

35. $$\begin{aligned}\frac{d}{dx}\tan^2 x &= (2\tan x)(1+\tan^2 x)\\ &= 2\tan x + 2\tan^3 x\end{aligned}$$

36. $\frac{dy}{dx} = 3\cos x + 2\sin x$

37. $\frac{dy}{dx} = -3\sin 3x$

38. $\frac{dy}{dx} = -9\sin 9x$

39. $\frac{d}{dx}\tan 3x = \frac{3}{\cos^2 3x}$

40. $\frac{d}{dx}(\tan x + \tan 2x) = \frac{1}{\cos^2 x} + \frac{2}{\cos^2 2x}$

41. $$\begin{aligned}\frac{d}{dx}(3\cos 2x) &= 3(-2\sin 2x)\\ &= -6\sin 2x\end{aligned}$$

42. $$\begin{aligned}\frac{d}{dx}(5\sin 3x) &= 5(3\cos 3x)\\ &= 15\cos 3x\end{aligned}$$

43. $\frac{d}{dx}(2\sin 3x + 3\cos 2x) = 6\cos 3x - 6\sin 2x$

44. $$\begin{aligned}\frac{d}{dx}(\sin^5 x) &= (5\sin^4 x)(\cos x)\\ &= 5\sin^4 x\cos x\end{aligned}$$

45. $$\begin{aligned}\frac{d}{dx}(5\cos^2 x) &= (10\cos x)(-\sin x)\\ &= -10\cos x\sin x\end{aligned}$$

46. $$\begin{aligned}\frac{d}{dx}(-3\cos^4 x) &= (-12\cos^3 x)(-\sin x)\\ &= 12\cos^3 x\sin x\end{aligned}$$

47. $$\begin{aligned}\frac{d}{dx}(\cos^{0.5} x) &= (0.5\cos^{-0.5} x)(-\sin x)\\ &= -0.5\cos^{-0.5} x\sin x\\ &= -\frac{\sin x}{2\cos^{0.5} x}\end{aligned}$$

48. $$\begin{aligned}\frac{d}{dx}\sqrt{\sin x} &= (0.5\sin^{-0.5} x)(\cos x)\\ &= 0.5\sin^{-0.5} x\cos x\\ &= \frac{\cos x}{2\sqrt{\sin x}}\end{aligned}$$

49–57 You should be able to do these in a single step . . . no working required.

58. $$\begin{aligned}f'(x) &= 1\cos x + x(-\sin x)\\ &= \cos x - x\sin x\end{aligned}$$

59. $$\begin{aligned}f'(x) &= 2x\cos x + x^2(-\sin x)\\ &= 2x\cos x - x^2\sin x\end{aligned}$$

60. $$\begin{aligned}f'(x) &= 2\sin x + 2x(\cos x)\\ &= 2\sin x + 2x\cos x\end{aligned}$$

61. $$\begin{aligned}f'(x) &= (2\sin 3x)(3\cos 3x)\\ &= 6\sin 3x\cos 3x\end{aligned}$$

62. $$\begin{aligned}f'(x) &= (3\cos^2 2x)(-2\sin 2x)\\ &= -6\cos^2 2x\sin 2x\end{aligned}$$

63. (a) L.H.S:
$$\begin{aligned}\frac{d}{dx}\sec x &= \frac{d}{dx}\frac{1}{\cos x}\\ &= \frac{d}{dx}\cos^{-1} x\\ &= (-\cos^{-2} x)(-\sin x)\\ &= \frac{\sin x}{\cos^2 x}\\ &= \frac{1}{\cos x}\frac{\sin x}{\cos x}\\ &= \sec x\tan x\\ &= \text{R.H.S.}\end{aligned}$$
□

(b)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\csc x &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\sin x}\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1} x\\ &= (-\sin^{-2} x)(\cos x)\\ &= -\frac{\cos x}{\sin^2 x}\\ &= -\frac{1}{\sin x}\frac{\cos x}{\sin x}\\ &= -\csc x\cot x\\ &= \text{R.H.S.}\end{aligned}$$

□

(c)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\cot x &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\cos x}{\sin x}\\ &= \frac{(-\sin x)(\sin x)-(\cos x)(\cos x)}{\sin^2 x}\\ &= \frac{-\sin^2 x-\cos^2 x}{\sin^2 x}\\ &= -\frac{\sin^2 x+\cos^2 x}{\sin^2 x}\\ &= -\frac{1}{\sin^2 x}\\ &= -\csc^2 x\\ &= \text{R.H.S.}\end{aligned}$$

□

64. $$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos x$$
$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

65. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -2\sin 2x\\ -2\sin\left(2\times\frac{\pi}{6}\right) &= -2\sin\frac{\pi}{3}\\ &= -2\times\frac{\sqrt{3}}{2}\\ &= -\sqrt{3}\end{aligned}$$

66. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 2\cos x\cos x + 2\sin x(-\sin x)\\ &= 2\cos^2 x - 2\sin^2 x\\ &= 2\cos 2x\end{aligned}$$
$2\cos(2\times 0) = 2$
alternatively:
$$\begin{aligned}2\sin x\cos x &= \sin 2x\\ \frac{\mathrm{d}}{\mathrm{d}x}\sin 2x &= 2\cos 2x\\ 2\cos(2\times 0) &= 2\end{aligned}$$

67. $$\frac{\mathrm{d}y}{\mathrm{d}x} = 6\sin x\cos x$$
$6\sin\pi\cos\pi = 0$

68. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \cos x\\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\cos x\\ &= -\sin x\end{aligned}$$

69. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -5\sin 5x\\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}(-5\sin 5x)\\ &= -25\cos 5x\end{aligned}$$

70. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 6\cos 2x\\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}(6\cos 2x)\\ &= -12\sin 2x\end{aligned}$$

71. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \cos x - \sin x\\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}(\cos x - \sin x)\\ &= -\sin x - \cos x\end{aligned}$$

72. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 3\sin^2 x\cos x\\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}(3\sin^2 x\cos x)\\ &= (6\sin x\cos x)(\cos x) + (3\sin^2 x)(-\sin x)\\ &= 6\sin x\cos^2 x - 3\sin^3 x\\ &= 6\sin x(1-\sin^2 x) - 3\sin^3 x\\ &= 6\sin x - 6\sin^3 x - 3\sin^3 x\\ &= 6\sin x - 9\sin^3 x\end{aligned}$$

73. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -4\cos x\sin x\\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}(-4\cos x\sin x)\\ &= (4\sin x)(\sin x) + (-4\cos x)(\cos x)\\ &= 4\sin^2 x - 4\cos^2 x\\ &= -4(\cos^2 x - \sin^2 x)\\ &= -4\cos 2x\end{aligned}$$

74. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \sin x + x\cos x\\ \sin\frac{\pi}{2} + \frac{\pi}{2}\cos\frac{\pi}{2} &= 1 + 0\\ &= 1\\ y - y_1 &= m(x - x_1)\\ y - \frac{\pi}{2} &= 1(x - \frac{\pi}{2})\\ y &= x\end{aligned}$$

75. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 1 - 6\sin 2x\\ 1 - 6\sin(2\times 0) &= 1\\ y - y_1 &= m(x - x_1)\\ y - 3 &= 1(x - 0)\\ y &= x + 3\end{aligned}$$

76.
$$\begin{aligned}\frac{dy}{dx} &= 3(1+\tan^2 2x)(2)\\ &= 6+6\tan^2 2x\\ 6+6\tan^2(2\times\frac{\pi}{8}) &= 6+6\tan^2\frac{\pi}{4}\\ &= 6+6(1^2)\\ &= 12\\ y-y_1 &= m(x-x_1)\\ y-3 &= 12(x-\frac{\pi}{8})\\ y &= 12x-\frac{3\pi}{2}+3\end{aligned}$$

77. (a)
$$\begin{aligned}f'(x) &= 2\cos 2x\\ f'(\pi/6) &= 2\cos(\pi/3)\\ &= 1\end{aligned}$$
(b)
$$\begin{aligned}f''(x) &= -4\sin 2x\\ f''(\pi/6) &= -4\sin(\pi/3)\\ &= -2\sqrt{3}\end{aligned}$$

78. (a)
$$\begin{aligned}f'(x) &= 2\sin x\cos x\\ &= \sin 2x\\ f'(\pi/6) &= \sin(\pi/3)\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$
(b)
$$\begin{aligned}f''(x) &= 2\cos 2x\\ f''(\pi/6) &= 2\cos(\pi/3)\\ &= 1\end{aligned}$$

79. (a)
$$\begin{aligned}f'(x) &= 12\sin^2 x\cos x\\ f'(2) &= 12\sin^2(2)\cos(2)\\ &= -4.13 \text{ (2d.p.)}\end{aligned}$$
(b)
$$\begin{aligned}f''(x) &= (24\sin x\cos x)(\cos x)\\ &\quad + (12\sin^2 x)(-\sin x)\\ &= 24\sin x\cos^2 x - 12\sin^3 x\\ &= 24\sin x(1-\sin^2 x) - 12\sin^3 x\\ &= 24\sin x - 24\sin^3 x - 12\sin^3 x\\ &= 24\sin x - 36\sin^3 x\\ f''(2) &= 24\sin(2) - 36\sin^3(2)\\ &= -5.24 \text{ (2d.p.)}\end{aligned}$$

80. Because our limits are defined in terms of radians, it is necessary to do a conversion when working in degrees:
$$\begin{aligned}\sin x^\circ &= \sin\left(\frac{\pi}{180}x\right)\\ \frac{dy}{dx} &= \cos\left(\frac{\pi}{180}x\right)\left(\frac{\pi}{180}\right)\end{aligned}$$
converting back to degrees
$$= \frac{\pi}{180}\cos x^\circ$$

81. The length of the rectangle as drawn is $20\cos\theta$ and the breadth is $20\sin\theta$ so the area is given by
$$\begin{aligned}A &= (20\cos\theta)(20\sin\theta)\\ &= 400\sin\theta\cos\theta\\ &= 200(2\sin\theta\cos\theta)\\ &= 200\sin 2\theta\\ \frac{dA}{d\theta} &= 400\cos 2\theta\end{aligned}$$
Set the derivative to zero to find the maximum:
$$\begin{aligned}400\cos 2\theta &= 0\\ 2\theta &= \frac{\pi}{2}\end{aligned}$$
which implies that the diagonals are perpendicular, hence the rectangle is a square. □
$$\begin{aligned}A &= 200\sin\frac{\pi}{2}\\ &= 200\text{cm}^2\end{aligned}$$
For side length, just take the square root:
$$\begin{aligned}l &= \sqrt{200}\\ &= 10\sqrt{2}\end{aligned}$$

82.
$$\begin{aligned}A &= \frac{1}{2}(8)(10)\sin 0.1t\\ &= 40\sin 0.1t\\ \frac{dA}{dt} &= 4\cos 0.1t\end{aligned}$$
(a)
$$\begin{aligned}\frac{dA}{dt} &= 4\cos(0.1\times 1)\\ &= 3.98\text{cm}^2/\text{s}\end{aligned}$$
(b)
$$\begin{aligned}\frac{dA}{dt} &= 4\cos(0.1\times 5)\\ &= 3.51\text{cm}^2/\text{s}\end{aligned}$$
(c)
$$\begin{aligned}\frac{dA}{dt} &= 4\cos(0.1\times 10)\\ &= 2.16\text{cm}^2/\text{s}\end{aligned}$$
(d)
$$\begin{aligned}\frac{dA}{dt} &= 4\cos(0.1\times 20)\\ &= -1.66\text{cm}^2/\text{s}\end{aligned}$$

83. (a) The maximum value of x is 5 and occurs when $3t=\frac{\pi}{2}$, i.e. $t=\frac{\pi}{6}$.

(b)
$$\begin{aligned}5\sin 3t &= 2.5\\ \sin 3t &= 0.5\\ 3t &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}\\ t &= \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}\end{aligned}$$
(c)
$$\begin{aligned}\frac{dx}{dt} &= 15\cos 3t\\ &= 15\cos(3\times 0.6)\\ &= -3.4\end{aligned}$$

(d) $\frac{d^2x}{dt^2} = \frac{d}{dt}15\cos 3t$
$= -45\sin 3t$
$= -9(5\sin 3t)$
$= -9x$ $\square$ $k = -9$.

84. $\frac{d}{d\theta}3\sin\theta + 4\cos\theta = 3\cos\theta - 4\sin\theta$
$3\cos\theta - 4\sin\theta = 0$
$4\sin\theta = 3\cos\theta$
$\tan\theta = \frac{3}{4}$
$\theta = 0.64353 \sin\theta + 4\cos\theta \quad = 5$

(This can be confirmed as a maximum, if necessary, either graphically or by evaluating points on either side, or using the second derivative test.)

Miscellaneous Exercise 5

1. Draw and shade a circle centred at $3 + \mathrm{i}$ having radius 3.

2. Proof by exhaustion:
Because $(-n)^2 = n^2$ it will be sufficient to prove this for non-negative integers.

Any non-negatve integer can be represented as $n = 10t + u$ where t is a natural number and u is a single digit.

Hence any square can be represented as

$$\begin{aligned} n^2 &= (10t + u)^2 \\ &= 100t^2 + 20ut + u^2 \\ &= 10(10t^2 + 2ut) + u^2 \end{aligned}$$

Because $10(10t^2 + 2ut)$ is a multiple of 10, it has a zero in the units digit, so the units digit of n^2 is determined solely by the units digit of u^2.

The possible units digit of any square number can hence be determined exhaustively:

u	0	1	2	3	4	5	6	7	8	9
u^2	0	1	4	9	6	5	6	9	4	1

(where the tens digit of u^2 has been discarded).

Thus the only possible last digits of any square number are 0, 1, 4, 5, 6 and 9. It is not possible to obtain a last digit of 2, 3, 7 or 8. $\square$

3. $$\begin{aligned} r &= \sqrt{(-\sqrt{3})^2 + 1^2} \\ &= 2 \end{aligned}$$

$$\tan\theta = \frac{1}{-\sqrt{3}}$$

$$\theta = \frac{5\pi}{6} \text{ (second quadrant)}$$

$$(-\sqrt{3} + \mathrm{i}) = 2\,\mathrm{cis}\,\frac{5\pi}{6}$$

4. (a) $$\begin{aligned} \sin x &= \frac{\cos x}{\sqrt{3}} \\ \tan x &= \frac{1}{\sqrt{3}} \\ x &= \frac{\pi}{6}, \frac{7\pi}{6} \end{aligned}$$

(b) $$\begin{aligned} \sin^2 x + (1 + \cos x)(\cos x) &= 0.5 \\ \sin^2 x + \cos x + \cos^2 x &= 0.5 \\ 1 + \cos x &= 0.5 \\ \cos x &= -0.5 \\ x &= \pm\frac{2\pi}{3} \end{aligned}$$

(c) $$\begin{aligned} \sin x(2\cos x - \sin x) &= \cos^2 x \\ 2\sin x\cos x - \sin^2 x &= \cos^2 x \\ \sin 2x &= \sin^2 x + \cos^2 x \\ &= 1 \\ 2x &= \frac{\pi}{2}, -\frac{3\pi}{2} \\ x &= \frac{\pi}{4}, -\frac{3\pi}{4} \end{aligned}$$

5. $$\begin{aligned} \frac{dy}{dx} &= \frac{(-\sin x)(x) - (\cos x)(1)}{x^2} \\ &= -\frac{x\sin x + \cos x}{x^2} \end{aligned}$$

At $x = \frac{\pi}{2}$,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}}{\left(\frac{\pi}{2}\right)^2}\\ &= -\frac{\frac{\pi}{2}+0}{\left(\frac{\pi}{2}\right)^2}\\ &= -\frac{1}{\frac{\pi}{2}}\\ &= -\frac{2}{\pi}\\ y - y_1 &= m(x - x_1)\\ y - 0 &= -\frac{2}{\pi}\left(x - \frac{\pi}{2}\right)\\ y &= -\frac{2}{\pi}x + 1\end{aligned}$$

6. Equate **i** and **j** components and solve for μ and λ:

$$\begin{aligned}3 + 4\lambda &= 2 + 3\mu\\ 2 + \lambda &= 1 + \mu\\ -6 - 3\lambda &= -3 - 3\mu\\ -3 + \lambda &= -1\\ \lambda &= 2\\ \mu &= 3\end{aligned}$$

Now see whether this solution works for the **k** components:

$$\begin{aligned}-1 + 3\lambda &= 1 + 2\mu\\ -1 + 3(2) &\neq 1 + 2(3)\end{aligned}$$

The lines do not intersect.

7. Point A:

$$(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) + 1(-\mathbf{i} + 3\mathbf{j} + \mathbf{k}) = \mathbf{i} + 6\mathbf{j}$$

Point B:

$$(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) + 5(-\mathbf{i} + 3\mathbf{j} + \mathbf{k}) = -3\mathbf{i} + 18\mathbf{j} + 4\mathbf{k}$$

$\overrightarrow{\mathrm{AC}} : \overrightarrow{\mathrm{BA}} = -1 : 4$ means $\overrightarrow{\mathrm{AC}} : \overrightarrow{\mathrm{AB}} = -1 : -4 = 1 : 4$ so the position vector of C is

$$\begin{aligned}\overrightarrow{\mathrm{OC}} &= \overrightarrow{\mathrm{OA}} + \frac{1}{4}\overrightarrow{\mathrm{AB}}\\ &= \mathbf{i} + 6\mathbf{j} + \frac{1}{4}\left((-3\mathbf{i} + 18\mathbf{j} + 4\mathbf{k}) - (\mathbf{i} + 6\mathbf{j})\right)\\ &= \mathbf{i} + 6\mathbf{j} + \frac{1}{4}(-4\mathbf{i} + 12\mathbf{j} + 4\mathbf{k})\\ &= \mathbf{i} + 6\mathbf{j} + (-\mathbf{i} + 3\mathbf{j} + \mathbf{k})\\ &= 9\mathbf{j} + \mathbf{k}\end{aligned}$$

8. To prove:

$$z_1\overline{z_1} = |z_1|^2$$

Proof:
LHS:

$$\begin{aligned}z_1 + \overline{z_1} &= (a + b\mathrm{i})(a - b\mathrm{i})\\ &= a^2 - ab\mathrm{i} + ab\mathrm{i} - b^2\mathrm{i}^2\\ &= a^2 + b^2\\ &= |a + b\mathrm{i}|^2\\ &= |z_1|^2\\ &= \text{RHS}\end{aligned}$$

□

To prove:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

Proof:
LHS:

$$\begin{aligned}\overline{z_1 + z_2} &= \overline{a + b\mathrm{i} + c + d\mathrm{i}}\\ &= \overline{a + c + (b + d)\mathrm{i}}\\ &= a + c - (b + d)\mathrm{i}\\ &= a - b\mathrm{i} + c - d\mathrm{i}\\ &= \overline{a + b\mathrm{i}} + \overline{c + d\mathrm{i}}\\ &= \overline{z_1} + \overline{z_2}\\ &= \text{RHS}\end{aligned}$$

□

To prove:

$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$$

Proof:
LHS:

$$\begin{aligned}\overline{z_1 z_2} &= \overline{(a + b\mathrm{i})(c + d\mathrm{i})}\\ &= \overline{ac + ad\mathrm{i} + bc\mathrm{i} - bd}\\ &= \overline{ac - bd + (ad + bc)\mathrm{i}}\\ &= ac - bd - (ad + bc)\mathrm{i}\\ &= ac - bd - ad\mathrm{i} - bc\mathrm{i}\\ &= a(c - d\mathrm{i}) - b\mathrm{i}(c - d\mathrm{i})\\ &= (a - b\mathrm{i})(c - d\mathrm{i})\\ &= \overline{(a + b\mathrm{i})}\overline{(c + d\mathrm{i})}\\ &= \overline{z_1}\,\overline{z_2}\\ &= \text{RHS}\end{aligned}$$

□

To prove:

$$|z_1 z_2| = |z_1||z_2|$$

Proof:

LHS:

$$\begin{aligned}|z_1 z_2| &= |(a+b\mathrm{i})(c+d\mathrm{i})| \\ &= |ac+ad\mathrm{i}+bc\mathrm{i}-bd| \\ &= |ac-bd+(ad+bc)\mathrm{i}| \\ &= \sqrt{(ac-bd)^2+(ad+bc)^2} \\ &= \sqrt{\begin{array}{c}a^2c^2-2abcd+b^2d^2\\+a^2d^2+2abcd+b^2c^2\end{array}} \\ &= \sqrt{a^2(c^2+d^2)+b^2(d^2+c^2)} \\ &= \sqrt{(a^2+b^2)(c^2+d^2)} \\ &= \sqrt{(a^2+b^2)}\sqrt{(c^2+d^2)} \\ &= |z_1||z_2| \\ &= \text{RHS}\end{aligned}$$

□

To prove:

$$|z_1 \div z_2| = |z_1| \div |z_2|$$

Proof:
LHS:

$$\begin{aligned}|z_1 \div z_2| &= \left|\frac{a+b\mathrm{i}}{c+d\mathrm{i}}\frac{c-d\mathrm{i}}{c-d\mathrm{i}}\right| \\ &= \left|\frac{ac+bd+(-ad+bc)\mathrm{i}}{c^2+d^2}\right| \\ &= \frac{|ac+bd+(-ad+bc)\mathrm{i}|}{c^2+d^2} \\ &= \frac{\sqrt{(ac+bd)^2+(-ad+bc)^2}}{c^2+d^2} \\ &= \frac{\sqrt{\begin{array}{c}a^2c^2+2abcd+b^2d^2\\+a^2d^2-2abcd+b^2c^2\end{array}}}{c^2+d^2} \\ &= \frac{\sqrt{a^2(c^2+d^2)+b^2(d^2+c^2)}}{c^2+d^2} \\ &= \frac{\sqrt{(a^2+b^2)(c^2+d^2)}}{c^2+d^2} \\ &= \frac{\sqrt{(a^2+b^2)}\sqrt{(c^2+d^2)}}{c^2+d^2} \\ &= \frac{\sqrt{(a^2+b^2)}}{\sqrt{(c^2+d^2)}} \\ &= |z_1| \div |z_2| \\ &= \text{RHS}\end{aligned}$$

□

9. To prove:

$$\sin \mathrm{A} \sin 2\mathrm{A} = 2\cos \mathrm{A} - 2\cos^3 \mathrm{A}$$

Proof:

LHS:

$$\begin{aligned}\sin \mathrm{A}\sin 2\mathrm{A} &= \sin \mathrm{A}(2\sin \mathrm{A}\cos \mathrm{A}) \\ &= 2\sin^2 \mathrm{A}\cos \mathrm{A} \\ &= 2(1-\cos^2 \mathrm{A})\cos \mathrm{A} \\ &= 2\cos \mathrm{A} - 2\cos^3 \mathrm{A} \\ &= \text{RHS}\end{aligned}$$

□

10. To prove:

$$\sin 3x = 3\sin x - 4\sin^3 x$$

Proof:
LHS:

$$\begin{aligned}\sin 3x &= \sin(2x+x) \\ &= \sin 2x\cos x + \cos 2x \sin x \\ &= (2\sin x\cos)x\cos x + (1-2\sin^2 x)\sin x \\ &= 2\sin x\cos^2 x + \sin x - 2\sin^3 x \\ &= 2\sin x(1-\sin^2 x) + \sin x - 2\sin^3 x \\ &= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x \\ &= 3\sin x - 4\sin^3 x \\ &= \text{RHS}\end{aligned}$$

□

11. (a) $\overrightarrow{\mathrm{ED}} = \overrightarrow{\mathrm{EA}} + \frac{1}{2}\overrightarrow{\mathrm{AC}}$

$$\begin{aligned}&= -h\mathbf{c} + \frac{1}{2}(-\mathbf{a}+\mathbf{c}) \\ &= \frac{1}{2}\mathbf{c} - h\mathbf{c} - \frac{1}{2}\mathbf{a}\end{aligned}$$

(b) $\overrightarrow{\mathrm{DF}} = \overrightarrow{\mathrm{CF}} + \frac{1}{2}\overrightarrow{\mathrm{AC}}$

$$\begin{aligned}&= -k\mathbf{c} + \frac{1}{2}(-\mathbf{a}+\mathbf{c}) \\ &= \frac{1}{2}\mathbf{c} - k\mathbf{c} - \frac{1}{2}\mathbf{a}\end{aligned}$$

(c) i.

$$\begin{aligned}\overrightarrow{\mathrm{ED}} &= m\overrightarrow{\mathrm{DF}} \\ \frac{1}{2}\mathbf{c} - h\mathbf{c} - \frac{1}{2}\mathbf{a} &= m\left(\frac{1}{2}\mathbf{c} - k\mathbf{c} - \frac{1}{2}\mathbf{a}\right) \\ \left(\frac{m}{2} - \frac{1}{2}\right)\mathbf{a} &= \left(\frac{m}{2} - mk - \frac{1}{2} + h\right)\mathbf{c} \\ \frac{m}{2} - \frac{1}{2} &= 0 \\ m &= 1\end{aligned}$$

□

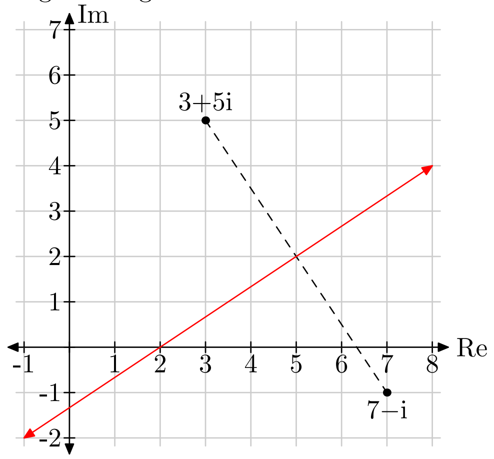
ii. Taking the right hand side of the third line above and substuting for m:

$$\begin{aligned}\frac{m}{2} - mk - \frac{1}{2} + h &= 0 \\ \frac{1}{2} - k - \frac{1}{2} + h &= 0 \\ -k + h &= 0 \\ h &= k\end{aligned}$$

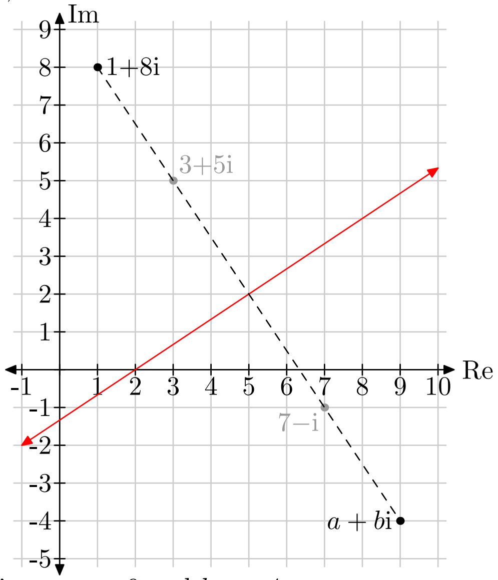
□

12.

13. From the first set of points given we can obtain the Argand diagram:

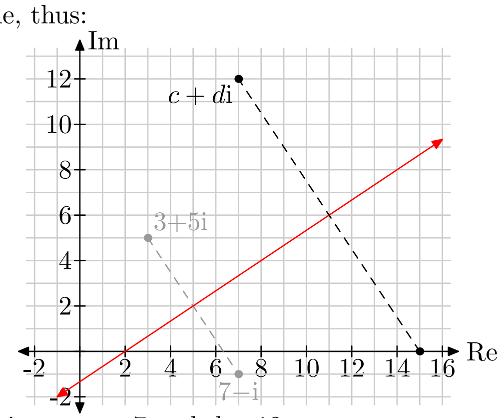


From the second set we see that the line is the set of points equidistant from $(1+8\mathrm{i})$ and $(a+b\mathrm{i})$, hence $(a+b\mathrm{i})$ is the reflection of $(1+8\mathrm{i})$ in the line, thus:



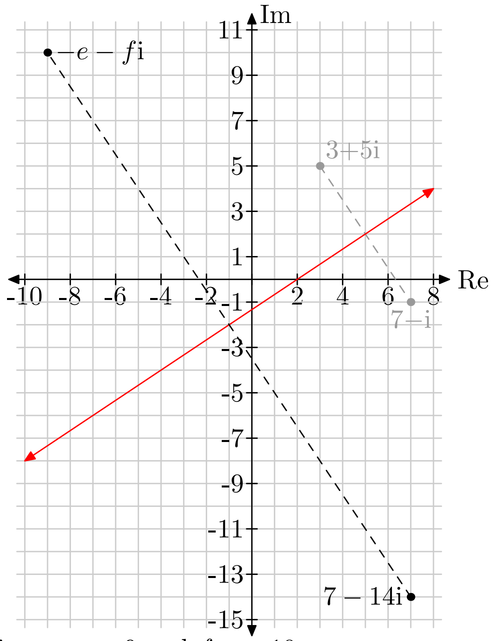
giving us $a = 9$ and $b = -4$.

From the third set we see that the line is the set of points equidistant from $(15+0\mathrm{i})$ and $(c+d\mathrm{i})$, hence $(c+d\mathrm{i})$ is the reflection of $(-2+0\mathrm{i})$ in the line, thus:



giving us $c = 7$ and $d = 12$.

From the fourth set we see that the line is the set of points equidistant from $(7-14\mathrm{i})$ and $(-e-f\mathrm{i})$, hence $(-e-f\mathrm{i})$ is the reflection of $(7-14\mathrm{i})$ in the line, thus:



giving us $e = 9$ and $f = -10$.

14. The dog will cause the light to switch on if the line along which it is walking intersects a sphere of radius 6m centred at the light. The equation of such a sphere is

$$\left|\mathbf{r} - \begin{pmatrix}-1\\8\\5\end{pmatrix}\right| = 6$$

and the dog will trigger the light if there exists a real solution to this when we substitute the expression for $\mathbf{r}$ from the line into this equation.

$$\left|\begin{pmatrix}-2\\0\\0\end{pmatrix} + \lambda -111 - \begin{pmatrix}-1\\8\\5\end{pmatrix}\right| = 6$$

$$\left|\begin{pmatrix}-1\\-8\\-5\end{pmatrix} + \lambda -111\right| = 6$$

$$(-1-\lambda)^2 + (-8+\lambda)^2 + (-5+\lambda)^2 = 6^2$$

$$1 + 2\lambda + \lambda^2 + 64 - 16\lambda + \lambda^2 + 25 - 10\lambda + \lambda^2 = 36$$

$$3\lambda^2 - 24\lambda + 54 = 0$$

$$\lambda^2 - 8\lambda + 18 = 0$$

We know this will have real solutions only if the discriminant (the bit in the square root in the quadratic formula) is not negative, i.e.

$$(-8)^2 - 4 \times 1 \times 18 \geq 0$$

but in fact

$$(-8)^2 - 4 \times 1 \times 18 = -8$$

so the quadratic has no solution and the dog will not trigger the light.

There are at least two other ways you might have approached this problem.

- using scalar product ideas and vectors to find the minimum distance
- by finding an expression for the distance between the dog and the light as a function of lambda and determining its mimimum using calculus or other methods.