Chapter 4

Exercise 4A

Proofs can be particularly challenging because it is often not clear what path will lead to success. Even the best mathematician will take several attempts at some problems before arriving at a successful proof. Indeed, some conjectures have remained unproven for centuries before a proof is finally found[1]. Because of this you should be particularly slow to reach for these worked solutions when you find the work difficult. Make sure you try several different approaches before taking even a peek here. The more you can do on your own or together with other students, the more satisfying it will be for you.

1. (a)

$$\begin{aligned} e+f &= 180^\circ \quad &\text{(angles in a st. line)}\\ f+g &= 180^\circ \quad &\text{(angles in a st. line)}\\ \therefore\ e+f &= f+g\\ \therefore\ e &= g\\ c &= e \quad &\text{(Alternate angles)}\\ \therefore\ c &= g \end{aligned}$$

□

(b)

$$\begin{aligned} e+f &= 180^\circ \quad &\text{(angles in a st. line)}\\ c &= e \quad &\text{(Alternate angles)}\\ \therefore\ c+f &= 180^\circ \end{aligned}$$

□

2. (a)

$$\begin{aligned} c+f &= 180^\circ \quad &\text{(cointerior angles)}\\ e+f &= 180^\circ \quad &\text{(angles in a st. line)}\\ \therefore\ c+f &= e+f\\ \therefore\ c &= e \end{aligned}$$

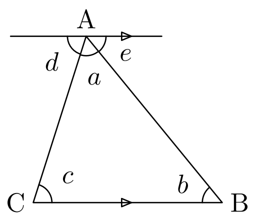
□

(b)

$$\begin{aligned} d+e &= 180^\circ \quad &\text{(cointerior angles)}\\ a+d &= 180^\circ \quad &\text{(angles in a st. line)}\\ \therefore\ a+d &= d+e\\ \therefore\ a &= e \end{aligned}$$

□

3.

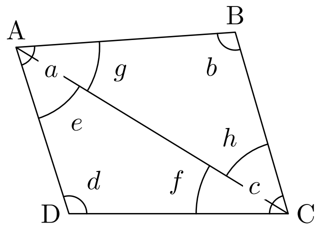


$$\begin{aligned} d+a+e &= 180^\circ \quad &\text{(angles in a st. line)}\\ d &= c \quad &\text{(alternate angles)}\\ \therefore\ c+a+e &= 180^\circ\\ e &= b \quad &\text{(alternate angles)}\\ \therefore\ c+a+b &= 180^\circ \end{aligned}$$

□

Now that this is proved, it can be used as a theorem in other proofs.

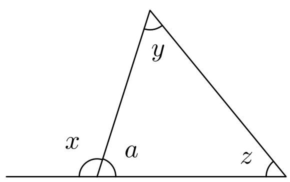
4.



$$\begin{aligned} a &= e+g\\ c &= f+h\\ d+e+f &= 180^\circ \quad &\text{(angles in }\triangle)\\ b+g+h &= 180^\circ \quad &\text{(angles in }\triangle)\\ \therefore\ (d+e+f)+(b+g+h) &= 180^\circ + 180^\circ\\ \therefore\ d+(e+g)+b+(f+h) &= 360^\circ\\ \therefore\ d+a+b+c &= 360^\circ \end{aligned}$$

□

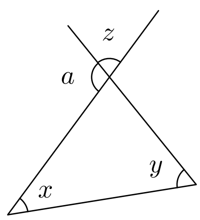
5.



$$\begin{aligned} a+y+z &= 180^\circ \quad &\text{(angles in }\triangle)\\ a+x &= 180^\circ \quad &\text{(angles in a st. line)}\\ \therefore\ a+x &= a+y+z\\ \therefore\ x &= y+z \end{aligned}$$

□

6.



$$\begin{aligned} a &= x+y \quad &\text{(ext. angle of }\triangle)\\ a+z &= 180^\circ \quad &\text{(angle in a st. line)}\\ \therefore\ x+y+z &= 180^\circ \end{aligned}$$

□

[1]Fermat's Last Theorem states that there are no positive integers a, b and c such that $a^n + b^n = c^n$ for n an integer greater than two. Originally proposed in 1637 it was finally proven in 1995 by British mathematician Andrew Wiles (although it had been proven for a number of special cases by various different people previously, including Fermat himself). The proof runs to well over 100 typewritten pages.

7. There are often many possible counter examples; citing any one will do to disprove a statement. The following are representative.
   (a) 2 is both prime and even.
   (b) A rectangle with length 2cm and width 1cm has four right angles but is not a square.
   (c) Rhombus ABCD has angles A and C 45° and angles B and D 135° and all four sides equal but is not a square.
   (d) If $n = 11$ then $n^2 - n + 11$ is a multiple of 11 and not prime.
   (e) 2^2 can not be expressed as the sum of two other square numbers. (The only possible candidates to add to 4 would be 0^2 and 1^2 and none of $0 + 0$, $0 + 1$ or $1 + 1$ yield 4.)
   (f) $x^2 + 4 = 0$ (i.e. $a = 1$, $b = 0$, $c = 4$) has no real solutions.

8. Any odd number can be represented as $2n+1$ for a suitably chosen integer n. Then

$$\begin{aligned}(2n+1)^2+3 &= 4n^2+2(2n)+1+3\\ &= 4n^2+4n+4\\ &= 4(n^2+n+1)\\ &= 4\times\text{an integer}\\ &= \text{a multiple of 4}\end{aligned}$$

□

9. Any odd number can be represented as $2n+1$ for a suitably chosen integer n. Then

$$\begin{aligned}&1+(2n+1)+(2n+1)^2+(2n+1)^3\\ &= 2n+2+4n^2+4n+1+8n^3+12n^2+6n+1\\ &= 8n^3+16n^2+12n+4\\ &= 4(2n^3+4n^2+3n+1)\\ &= 4\times\text{an integer}\\ &= \text{a multiple of 4}\end{aligned}$$

□

10. Any odd number can be represented as $2n-1$ for a suitably chosen integer n. The next consecutive odd number is

$$2n-1+2=2n+1$$

Then

$$\begin{aligned}(2n-1)(2n+1) &= 4n^2-1\\ &= (2n)^2-1\\ &= (\text{an even number})^2-1\\ &= \text{square of an an even no.}-1\end{aligned}$$

□

11. Given

$$\mathrm{T}_n = 3n-1$$

then the next consecutive term is

$$\mathrm{T}_{n+1} = 3(n+1)-1$$

then

$$\begin{aligned}\mathrm{T}_n+\mathrm{T}_{n+1} &= 3n-1+3(n+1)-1\\ &= 3n-1+3n+3-1\\ &= 6n+1\\ &= 2(3n)+1\\ &= 2(\text{an integer})+1\\ &= \text{an even number}+1\\ &= \text{an odd number}\end{aligned}$$

□

12.
$$\begin{aligned}\mathrm{F}_{n+5} &= \mathrm{F}_{n+4}+\mathrm{F}_{n+3}\\ &= \mathrm{F}_{n+3}+\mathrm{F}_{n+2}+\mathrm{F}_{n+3}\\ &= 2\mathrm{F}_{n+3}+\mathrm{F}_{n+2}\\ &= 2(\mathrm{F}_{n+2}+\mathrm{F}_{n+1})+\mathrm{F}_{n+2}\\ &= 3\mathrm{F}_{n+2}+2\mathrm{F}_{n+1}\\ &= 3(\mathrm{F}_{n+1}+\mathrm{F}_n)+2\mathrm{F}_{n+1}\\ &= 5\mathrm{F}_{n+1}+3\mathrm{F}_n\end{aligned}$$

□

Exercise 4B

1. $$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= -\overrightarrow{OA} + \overrightarrow{OB}\\ \overrightarrow{CD} &= \overrightarrow{CO} + \overrightarrow{OD}\\ &= \overrightarrow{OC} + \overrightarrow{OD}\\ &= -h\overrightarrow{OA} + h\overrightarrow{OB}\\ &= h(-\overrightarrow{OA} + \overrightarrow{OB})\\ &= h\overrightarrow{AB}\end{aligned}$$

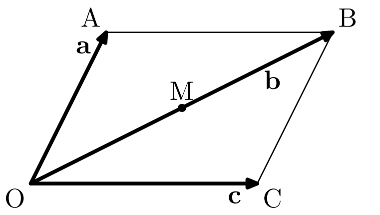
   Since $\overrightarrow{CD}$ is a scalar multiple of $\overrightarrow{AB}$, CD is parallel to AB. □

2. $$\begin{aligned}\overrightarrow{PQ} &= 0.5\overrightarrow{OA} + 0.5\overrightarrow{AB}\\ &= 0.5\mathbf{a} + 0.5(-\overrightarrow{OA} + \overrightarrow{OB})\\ &= 0.5\mathbf{a} - 0.5\mathbf{a} + 0.5\mathbf{b}\\ &= 0.5\mathbf{b}\\ \overrightarrow{SR} &= 0.5\overrightarrow{OC} + 0.5\overrightarrow{CB}\\ &= 0.5\mathbf{c} + 0.5(-\overrightarrow{OC} + \overrightarrow{OB})\\ &= 0.5\mathbf{c} - 0.5\mathbf{c} + 0.5\mathbf{b}\\ &= 0.5\mathbf{b}\\ \therefore\quad \overrightarrow{SR} &= \overrightarrow{PQ}\end{aligned}$$

   □

   This is sufficient to prove that PQRS is a parallelogram, since it shows that SR and PQ are both parallel and equal in length. However, since the question also asks us to find expressions for $\overrightarrow{QR}$ and $\overrightarrow{PS}$ ...

   $$\begin{aligned}\overrightarrow{QR} &= 0.5\overrightarrow{AB} + 0.5\overrightarrow{BC}\\ &= 0.5(-\overrightarrow{OA} + \overrightarrow{OB}) + 0.5(-\overrightarrow{OB} + \overrightarrow{OC})\\ &= -0.5\mathbf{a} + 0.5\mathbf{b} - 0.5\mathbf{b} + 0.5\mathbf{c}\\ &= 0.5\mathbf{c} - 0.5\mathbf{a}\\ \overrightarrow{PS} &= 0.5\overrightarrow{AO} + 0.5\overrightarrow{OC}\\ &= -0.5\mathbf{a} + 0.5\mathbf{c}\\ &= 0.5\mathbf{c} - 0.5\mathbf{a}\\ \therefore\quad \overrightarrow{PS} &= \overrightarrow{QR}\end{aligned}$$

3. 

   $$\begin{aligned}\overrightarrow{CM} &= \overrightarrow{CB} - 0.5\overrightarrow{OB}\\ &= \mathbf{a} - 0.5\mathbf{b}\\ \overrightarrow{CA} &= \overrightarrow{CB} - \overrightarrow{OB} + \overrightarrow{OA}\\ &= \mathbf{a} - \mathbf{b} + \mathbf{a}\\ &= 2\mathbf{a} - \mathbf{b}\\ \therefore\quad \overrightarrow{CA} &= 2\overrightarrow{CM}\end{aligned}$$

   Hence M lies on CA and is the midpoint of CA. □

4. $$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OA} + \overrightarrow{AM}\\ k\overrightarrow{OB} &= \overrightarrow{OA} + h\overrightarrow{AC}\\ \overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{OC}\\ \overrightarrow{AC} &= -\overrightarrow{OA} + \overrightarrow{OC}\\ \therefore\quad k(\overrightarrow{OA} + \overrightarrow{OC}) &= \overrightarrow{OA} + h(-\overrightarrow{OA} + \overrightarrow{OC})\\ k\overrightarrow{OA} + k\overrightarrow{OC} &= \overrightarrow{OA} - h\overrightarrow{OA} + h\overrightarrow{OC}\\ (h+k-1)\overrightarrow{OA} &= (h-k)\overrightarrow{OC}\\ h-k &= 0\\ h+k-1 &= 0\\ h &= k\\ 2h-1 &= 0\\ h = k &= 0.5\end{aligned}$$

   □

5. Refer to the diagram provided for question 4.

   $$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AM} + \overrightarrow{MB}\\ \overrightarrow{OC} &= \overrightarrow{OM} + \overrightarrow{MC}\\ \overrightarrow{OM} &= \overrightarrow{MB} &&\text{(since M bisects OB)}\\ \overrightarrow{AM} &= \overrightarrow{MC} &&\text{(since M bisects AC)}\\ \therefore\quad \overrightarrow{OC} &= \overrightarrow{MB} + \overrightarrow{AM}\\ &= \overrightarrow{AB}\end{aligned}$$

   Similarly

   $$\begin{aligned}\overrightarrow{OA} &= \overrightarrow{OM} + \overrightarrow{MA}\\ \overrightarrow{CB} &= \overrightarrow{CM} + \overrightarrow{MB}\\ \therefore\quad \overrightarrow{OA} &= \overrightarrow{MB} + \overrightarrow{CM}\\ &= \overrightarrow{CB}\end{aligned}$$

   ∴ OABC is a parallelogram. □

6. (a) i. $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b}$

   ii. $\overrightarrow{AC} = \overrightarrow{AO} + 0.5\overrightarrow{OB} = -\mathbf{a} + 0.5\mathbf{b}$

   iii. $\overrightarrow{AD} = 0.5\overrightarrow{AB} = -0.5\mathbf{a} + 0.5\mathbf{b}$

   iv. $$\begin{aligned}\overrightarrow{OD} &= \overrightarrow{OA} + \overrightarrow{AD}\\ &= \mathbf{a} - 0.5\mathbf{a} + 0.5\mathbf{b}\\ &= 0.5\mathbf{a} + 0.5\mathbf{b}\end{aligned}$$

   v. $\overrightarrow{OM} = \frac{2}{3}\overrightarrow{OD} = \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$

   vi. $$\begin{aligned}\overrightarrow{AM} &= \overrightarrow{AO} + \overrightarrow{OM}\\ &= -\mathbf{a} + \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}\\ &= -\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}\end{aligned}$$

   (b) M lies on AC such that AM:MC=2:1 if and only if $\overrightarrow{AM} = 2\overrightarrow{MC}$.
   To prove: $\overrightarrow{AM} = 2\overrightarrow{MC}$
   Proof:

R.H.S.:

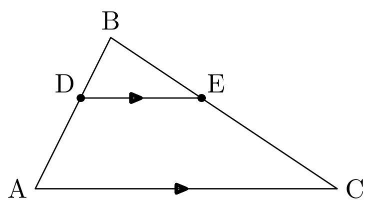
$$\begin{aligned}2\overrightarrow{\text{MC}} &= 2(\overrightarrow{\text{AC}} - \overrightarrow{\text{AM}})\\ &= 2\left(-\mathbf{a} + \frac{1}{2}\mathbf{b} - \left(-\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}\right)\right)\\ &= 2\left(-\frac{1}{3}\mathbf{a} + \frac{1}{6}\mathbf{b}\right)\\ &= -\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}\\ &= \overrightarrow{\text{AM}}\\ &= \text{L.H.S.}\end{aligned}$$

□

(c) M lies on BE such that BM:ME=2:1 if and only if $\overrightarrow{\text{BM}} = 2\overrightarrow{\text{ME}}$.
To prove: $\overrightarrow{\text{BM}} = 2\overrightarrow{\text{ME}}$
Proof:
R.H.S.:

$$\begin{aligned}2\overrightarrow{\text{ME}} &= 2(\overrightarrow{\text{BE}} - \overrightarrow{\text{BM}})\\ \overrightarrow{\text{BE}} &= \overrightarrow{\text{BO}} + \overrightarrow{\text{OE}}\\ &= \frac{1}{2}\mathbf{a} - \mathbf{b}\\ \overrightarrow{\text{BM}} &= \overrightarrow{\text{BO}} + \overrightarrow{\text{OM}}\\ &= -\mathbf{b} + \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}\\ &= \frac{1}{3}\mathbf{a} - \frac{2}{3}\mathbf{b}\\ 2\overrightarrow{\text{ME}} &= 2\left(\frac{1}{2}\mathbf{a} - \mathbf{b} - \left(\frac{1}{3}\mathbf{a} - \frac{2}{3}\mathbf{b}\right)\right)\\ &= 2\left(\frac{1}{6}\mathbf{a} - \frac{1}{3}\mathbf{b}\right)\\ &= \frac{1}{3}\mathbf{a} - \frac{2}{3}\mathbf{b}\\ &= \overrightarrow{\text{BM}}\\ &= \text{L.H.S.}\end{aligned}$$

□

7. 

$$\begin{aligned}\overrightarrow{\text{AB}} &= \mathbf{a}\\ \overrightarrow{\text{AC}} &= \mathbf{c}\\ \overrightarrow{\text{AD}} &= h\overrightarrow{\text{AB}}\\ &= h\mathbf{a}\\ \text{Let } \overrightarrow{\text{CE}} &= k\overrightarrow{\text{CE}}\\ \overrightarrow{\text{CB}} &= -\overrightarrow{\text{AC}} + \overrightarrow{\text{AB}}\\ &= \mathbf{a} - \mathbf{b}\end{aligned}$$

$$\begin{aligned}\therefore \quad \overrightarrow{\text{CE}} &= k(\mathbf{a} - \mathbf{b})\\ &= k\mathbf{a} - k\mathbf{b}\\ \overrightarrow{\text{DE}} &= -\overrightarrow{\text{AD}} + \overrightarrow{\text{AC}} + \overrightarrow{\text{CE}}\\ &= -h\mathbf{a} + \mathbf{b} + k\mathbf{a} - k\mathbf{b}\\ &= (k-h)\mathbf{a} + (1-k)\mathbf{b}\end{aligned}$$

But $\overrightarrow{\text{DE}}$ is parallel to $\overrightarrow{\text{AC}}$, so it must be a scalar multiple of **b**, so the component that is not parallel to **b** must be zero:

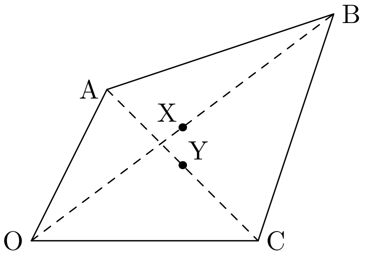
$$\begin{aligned}k - h &= 0\\ k &= h\\ \therefore \quad \overrightarrow{\text{CE}} &= h\overrightarrow{\text{CE}}\end{aligned}$$

□

8.

$$\begin{aligned}\overrightarrow{\text{OA}} + \overrightarrow{\text{AM}} &= \overrightarrow{\text{OM}}\\ \mathbf{a} + k\overrightarrow{\text{AC}} &= h\overrightarrow{\text{OD}}\\ \overrightarrow{\text{OD}} &= \overrightarrow{\text{OA}} + 0.5\overrightarrow{\text{AB}}\\ &= \mathbf{a} + 0.5(-\overrightarrow{\text{OA}} + \overrightarrow{\text{OB}})\\ &= \mathbf{a} + 0.5(\mathbf{b} - \mathbf{a})\\ &= 0.5\mathbf{a} + 0.5\mathbf{b}\\ \overrightarrow{\text{AC}} &= -\overrightarrow{\text{OA}} + 0.5\overrightarrow{\text{OB}}\\ &= 0.5\mathbf{b} - \mathbf{a}\\ \therefore \quad \mathbf{a} + k(0.5\mathbf{b} - \mathbf{a}) &= h(0.5\mathbf{a} + 0.5\mathbf{b}))\\ (1-k)\mathbf{a} + \frac{k}{2}\mathbf{b} &= \frac{h}{2}\mathbf{a} + \frac{h}{2}\mathbf{b}\\ (1-k)\mathbf{a} - \frac{h}{2}\mathbf{a} &= \frac{h}{2}\mathbf{b} - \frac{k}{2}\mathbf{b}\\ 1 - k - \frac{h}{2} &= 0\\ \text{and } \frac{h}{2} &= \frac{k}{2}\\ \therefore \quad h &= k\\ \therefore \quad 1 - k - \frac{k}{2} &= 0\\ \frac{3k}{2} &= 1\\ \therefore \quad h = k &= \frac{2}{3}\\ \overrightarrow{\text{BE}} &= -\overrightarrow{\text{OB}} + 0.5\overrightarrow{\text{OA}}\\ &= 0.5\mathbf{a} - \mathbf{b}\\ \overrightarrow{\text{BM}} &= -\overrightarrow{\text{OB}} + \frac{2}{3}\overrightarrow{\text{OD}}\\ &= -\mathbf{b} + \frac{2}{3}(0.5\mathbf{a} + 0.5\mathbf{b})\\ &= -\frac{2}{3}\mathbf{b} + \frac{1}{3}\mathbf{a}\\ &= \frac{2}{3}(0.5\mathbf{a} - \mathbf{b})\\ &= \frac{2}{3}\overrightarrow{\text{BE}}\end{aligned}$$

∴ the medians intersect at a point two thirds of the way along their length measured from the vertex. □

9. 

To prove:

$$\overrightarrow{OA} + \overrightarrow{BA} + \overrightarrow{OC} + \overrightarrow{BC} = 4\overrightarrow{XY}$$

R.H.S.

$$\begin{aligned}4\overrightarrow{XY} &= \overrightarrow{XY} + \overrightarrow{XY} + \overrightarrow{XY} + \overrightarrow{XY}\\ &= 0.5\overrightarrow{BO} + \overrightarrow{OA} + 0.5\overrightarrow{AC}\\ &\quad - 0.5\overrightarrow{BO} + \overrightarrow{BA} + 0.5\overrightarrow{AC}\\ &\quad + 0.5\overrightarrow{BO} + \overrightarrow{OC} - 0.5\overrightarrow{AC}\\ &\quad - 0.5\overrightarrow{BO} + \overrightarrow{BC} - 0.5\overrightarrow{AC}\\ &= \overrightarrow{OA} + \overrightarrow{BA} + \overrightarrow{OC} + \overrightarrow{BC}\\ &= \text{L.H.S.}\end{aligned}$$

□

This proof depends on expressing $\overrightarrow{XY}$ in four different ways. It may not be obvious at first that this is going to work, but after you've done enough of these you sometimes get a hunch about an approach that will pay off. Practice is the key.

10. (a) $\mathbf{a} \cdot \mathbf{c} = 0$ (because they are perpendicular).
    (b) $\overrightarrow{AC} = -\mathbf{a} + \mathbf{c}$
    $\overrightarrow{OB} = \mathbf{a} + \mathbf{c}$
    (c) To prove: $|\overrightarrow{AC}| = |\overrightarrow{OB}|$ First consider the L.H.S.:

$$\begin{aligned}|\overrightarrow{AC}| &= \sqrt{(AC)^2}\\ &= \sqrt{\overrightarrow{AC} \cdot \overrightarrow{AC}}\\ &= \sqrt{(-\mathbf{a}+\mathbf{c}) \cdot (-\mathbf{a}+\mathbf{c})}\\ &= \sqrt{\mathbf{a} \cdot \mathbf{a} - 2\mathbf{a} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{c}}\\ &= \sqrt{\mathbf{a} \cdot \mathbf{a} - 2 \times 0 + \mathbf{c} \cdot \mathbf{c}}\\ &= \sqrt{\mathbf{a} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{c}}\end{aligned}$$

Now consider the R.H.S.:

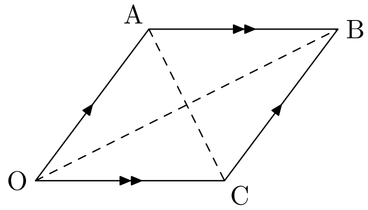
$$\begin{aligned}|\overrightarrow{OB}| &= \sqrt{(OB)^2}\\ &= \sqrt{\overrightarrow{OB} \cdot \overrightarrow{OB}}\\ &= \sqrt{(\mathbf{a}+\mathbf{c}) \cdot (\mathbf{a}+\mathbf{c})}\\ &= \sqrt{\mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{c}}\\ &= \sqrt{\mathbf{a} \cdot \mathbf{a} + 2 \times 0 + \mathbf{c} \cdot \mathbf{c}}\\ &= \sqrt{\mathbf{a} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{c}}\\ &= \text{L.H.S.}\end{aligned}$$

□

11. (a) $\overrightarrow{AB} = -\mathbf{a} + \mathbf{b}$
    (b) To prove: $(AB)^2 = (OA)^2 + (OB)^2$

$$\begin{aligned}(OA)^2 &= \mathbf{a} \cdot \mathbf{a}\\ (OB)^2 &= \mathbf{b} \cdot \mathbf{b}\\ (AB)^2 &= (-\mathbf{a}+\mathbf{b}) \cdot (-\mathbf{a}+\mathbf{b})\\ &= \mathbf{a} \cdot \mathbf{a} - 2\mathbf{a} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{c}\\ &= \mathbf{a} \cdot \mathbf{a} + 2 \times 0 + \mathbf{c} \cdot \mathbf{c}\\ &= \mathbf{a} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{c}\\ &= (OA)^2 + (OB)^2\end{aligned}$$

□

12. 

$$\begin{aligned}\text{Let } \overrightarrow{OA} &= \mathbf{a}\\ \text{and } \overrightarrow{OC} &= \mathbf{c}\\ \overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{AB}\\ &= \mathbf{a} + \mathbf{c}\\ \overrightarrow{CA} &= \overrightarrow{OA} - \overrightarrow{OC}\\ &= \mathbf{a} - \mathbf{c}\\ |\mathbf{a}| - |\mathbf{c}| &= \mathbf{a} \cdot \mathbf{a} - \mathbf{c} \cdot \mathbf{c}\\ &= (\mathbf{a}+\mathbf{c}) \cdot (\mathbf{a}-\mathbf{c})\\ &= \overrightarrow{OB} \cdot \overrightarrow{CA}\end{aligned}$$

If the perpendiculars are parallel, then

$$\begin{aligned}\overrightarrow{OB} \cdot \overrightarrow{CA} &= 0\\ \therefore \quad |\mathbf{a}| - |\mathbf{c}| &= 0\\ |\mathbf{a}| &= |\mathbf{c}|\end{aligned}$$

□

13. (a) $\overrightarrow{AC} = -\mathbf{a} + \mathbf{c}$

$$\begin{aligned}\overrightarrow{BD} &= \mathbf{a} + \frac{1}{2}\overrightarrow{AC}\\ &= \mathbf{a} + \frac{1}{2}(-\mathbf{a}+\mathbf{c})\\ &= \frac{1}{2}(\mathbf{a}+\mathbf{c})\end{aligned}$$

(b)

$$\begin{aligned}\overrightarrow{AC} \cdot \overrightarrow{BD} &= (-\mathbf{a}+\mathbf{c}) \cdot \frac{1}{2}(\mathbf{a}+\mathbf{c})\\ &= \frac{1}{2}(-\mathbf{a} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{c})\\ &= \frac{1}{2}(-(|\mathbf{a}|)^2 + (|\mathbf{c}|)^2)\end{aligned}$$

but $|\mathbf{a}| = |\mathbf{c}|$ because the triangle is isosceles, so

$$\begin{aligned}\overrightarrow{AC} \cdot \overrightarrow{BD} &= \frac{1}{2}(-(|\mathbf{c}|)^2 + (|\mathbf{c}|)^2)\\ &= 0\end{aligned}$$

$\therefore$ AC and BD are perpendicular and $\angle BDA$ is a right angle. □

14. (a) $\overrightarrow{CB} = \overrightarrow{CO} + \overrightarrow{OB}$
$= -\mathbf{c} + \mathbf{b}$
$\overrightarrow{AO} = \overrightarrow{OB}$
$= \mathbf{b}$
$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$
$= \mathbf{b} + \mathbf{c}$

(b) $\overrightarrow{AC} \cdot \overrightarrow{CB} = (\mathbf{b} + \mathbf{c}) \cdot (\mathbf{b} - \mathbf{c})$
$= \mathbf{b} \cdot \mathbf{b} - \mathbf{c} \cdot \mathbf{c}$
$= (|\mathbf{b}|)^2 - (|\mathbf{c}|)^2$
but $|\mathbf{b}| = |\mathbf{c}|$ because they are both radii of the circle, so
$\overrightarrow{AC} \cdot \overrightarrow{CB} = (|\mathbf{c}|)^2 - (|\mathbf{c}|)^2$
$= 0$
$\therefore$ AC and CB are perpendicular and ∠ACB is a right angle. □

15.
$$\begin{aligned} \overrightarrow{BC} &= -\mathbf{b} + \mathbf{c} \\ \mathbf{a} \cdot \overrightarrow{BC} &= 0 \\ \therefore \quad \mathbf{a} \cdot (-\mathbf{b} + \mathbf{c}) &= 0 \\ -\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} &= 0 \\ \mathbf{a} \cdot \mathbf{c} &= \mathbf{a} \cdot \mathbf{b} \end{aligned}$$

$$\begin{aligned} \overrightarrow{AC} &= -\mathbf{a} + \mathbf{c} \\ \mathbf{b} \cdot \overrightarrow{AC} &= 0 \\ \therefore \quad \mathbf{b} \cdot (-\mathbf{a} + \mathbf{c}) &= 0 \\ -\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} &= 0 \\ \mathbf{b} \cdot \mathbf{c} &= \mathbf{a} \cdot \mathbf{b} \end{aligned}$$

$$\begin{aligned} \overrightarrow{AB} &= -\mathbf{a} + \mathbf{b} \\ \mathbf{c} \cdot \overrightarrow{AB} &= \mathbf{c} \cdot (-\mathbf{a} + \mathbf{b}) \\ &= -\mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c} \\ &= -\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b} \\ &= 0 \end{aligned}$$

$\therefore \overrightarrow{CF}$ is perpendicular to $\overrightarrow{AB}$. □

Exercise 4C

1. If the number is even, we can represent it as $2n$, for a suitably chosen integer value n.
$$(2n)^2 = 4n^2$$
which is a multiple of 4, and hence is even.
If the number is odd, we can represent it as $2n+1$ for a suitably chosen integer value n.
$$\begin{aligned} (2n+1)^2 &= 4n^2 + 4n + 1 \\ &= 4(n^2 + n) + 1 \end{aligned}$$
which is one more than a multiple of 4, and hence is odd. □

2. There are five possible cases. The number is
   - a multiple of 5;
   - one more than a multiple of 5;
   - two more than a multiple of 5;
   - three more than a multiple of 5; or
   - four more than a multiple of 5.

   Considering each of these exhaustively:
   - If the number is a multiple of 5:
   $$\begin{aligned} (5n)^2 &= 25n^2 \\ &= 5(5n^2) \end{aligned}$$
   The square is a multiple of 5.
   - If the number is one more than a multiple of 5:
   $$\begin{aligned} (5n+1)^2 &= 25n^2 + 10n + 1 \\ &= 5(5n^2 + 2n) + 1 \end{aligned}$$
   The square is 1 more than a multiple of 5.
   - If the number is two more than a multiple of 5:
   $$\begin{aligned} (5n+2)^2 &= 25n^2 + 20n + 4 \\ &= 5(5n^2 + 4n) + 4 \end{aligned}$$
   The square is 4 more than a multiple of 5.
   - If the number is three more than a multiple of 5:
   $$\begin{aligned} (5n+3)^2 &= 25n^2 + 30n + 9 \\ &= 5(5n^2 + 4n + 1) + 4 \end{aligned}$$
   The square is 4 more than a multiple of 5.
   - If the number is four more than a multiple of 5:
   $$\begin{aligned} (5n+4)^2 &= 25n^2 + 80n + 16 \\ &= 5(5n^2 + 4n + 3) + 1 \end{aligned}$$
   The square is 1 more than a multiple of 5.

   □

3. The the number is

   - a multiple of 3;
   - one more than a multiple of 3; or
   - two more than a multiple of 3.

   Considering each of these exhaustively:

   - If the number is a multiple of 3:
     $$\begin{aligned}(3n)^3 &= 27n^3\\ &= 9(3n^3)\end{aligned}$$
     The cube is a multiple of 9.
   - If the number is one more than a multiple of 9:
     $$\begin{aligned}(3n+1)^3 &= (9n^2+6n+1)(3n+1)\\ &= 27n^3+9n^2+18n^2+6n+3n+1\\ &= 27n^3+27n^2+9n+1\\ &= 9(3n^3+3n^3+n)+1\end{aligned}$$
     The cube is 1 more than a multiple of 9.
   - If the number is two more than a multiple of 3:
     $$\begin{aligned}(3n+2)^3 &= (9n^2+12n+4)(3n+2)\\ &= 27n^3+18n^2+36n^2+24n+12n+8\\ &= 27n^3+54n^2+36n+9-1\\ &= 9(3n^3+6n^3+4n+1)-1\end{aligned}$$
     The cube is 1 less than a multiple of 9.

   □

4. - Suppose T_n is even, i.e. $T_n = 2x$ for some integer x, then
     $$\begin{aligned}T_{n+1} &= 3T_n+2\\ &= 3(2x)+2\\ &= 2(3x+1)\end{aligned}$$
     Hence T_{n+1} is also even.
   - Suppose T_n is odd, i.e. $T_n = 2x+1$ for some integer x, then
     $$\begin{aligned}T_{n+1} &= 3T_n+2\\ &= 3(2x+1)+2\\ &= 6x+5\\ &= 2(3x+2)+1\end{aligned}$$
     Hence T_{n+1} is also odd.

   $\therefore T_{n+1}$ has the same parity as T_n. □

5. Consider $x^5 - x = x(x-1)(x+1)(x^2+1)$

   - If $x = 5n$ then $x^5 - x$ has $x = 5n$ as a factor, so it is a multiple of 5.
   - If $x = 5n+1$ then $x^5 - x$ has $(x-1) = (5n+1)-1 = 5n$ as a factor, so it is a multiple of 5.
   - If $x = 5n+2$ then
     $$\begin{aligned}x^2+1 &= (5n+2)^2+1\\ &= 25n^2+20n+5\\ &= 5(5n^2+4n+1)\end{aligned}$$
     Hence as $x^5 - x$ has $(x^2+1) = 5(5n^2+4n+1)$ as a factor, it is a multiple of 5.
   - If $x = 5n+3$ then
     $$\begin{aligned}x^2+1 &= (5n+3)^2+1\\ &= 25n^2+30n+10\\ &= 5(5n^2+6n+2)\end{aligned}$$
     Hence as $x^5 - x$ has $(x^2+1) = 5(5n^2+6n+2)$ as a factor, it is a multiple of 5.
   - If $x = 5n+4$ then $x^5 - x$ has $(x+1) = (5n+4)+1 = 5n+5 = 5(n+1)$ as a factor, so it is a multiple of 5.

   Hence, $x^5 - x$ for $x > 1$ is always a multiple of 5. □

   As one or other of x and $x-1$ is even, $x^5 - x$ always has 2 as a factor. Since it has both 2 and 5 as factors, it is always a multiple of 10.

   If x is odd, both $x-1$ and $x+1$ are even, so $x^5 - x$ is a multiple of $2 \times 2 \times 5 = 20$.

   If x is even, $x-1$ and $x+1$ are both odd. For x^2+1:
   $$\begin{aligned}x^2+1 &= (2n)^2+1\\ &= 4n^2+1\end{aligned}$$
   which is also odd, so $x^5 - x$ has only one factor of 2, and so is not a multiple of 20. We can check this with an example. If $x = 2$,
   $$\begin{aligned}x^5 - x &= 2^5 - 2\\ &= 32-2\\ &= 30\end{aligned}$$
   which is not a multiple of 20.

6. - If $x = 7n$ then $x^7 - x$ has $x = 7n$ as a factor, so it is a multiple of 7.
   - If $x = 7n+1$ then $x^7 - x$ has $(x-1) = (7n+1)-1 = 7n$ as a factor, so it is a multiple of 7.
   - If $x = 7n+2$ then
     $$\begin{aligned}x^2+x+1 &= (7n+2)^2+(7n+2)+1\\ &= 49n^2+28n+4+7n+2+1\\ &= 49n^2+35n+7\\ &= 7(7n^2+5n+1)\end{aligned}$$
     Hence as $x^7 - x$ has $(x^2+x+1) = 7(7n^2+5n+1)$ as a factor, it is a multiple of 7.

- If $x = 7n + 3$ then

$$\begin{aligned} x^2 - x + 1 &= (7n+3)^2 - (7n+3) + 1 \\ &= 49n^2 + 42n + 9 - 7n - 3 + 1 \\ &= 49n^2 + 35n + 7 \\ &= 7(7n^2 + 5n + 1) \end{aligned}$$

  Hence as $x^7 - x$ has $(x^2 - x + 1) = 7(7n^2 + 5n + 1)$ as a factor, it is a multiple of 7.

- If $x = 7n + 4$ then

$$\begin{aligned} x^2 + x + 1 &= (7n+4)^2 + (7n+4) + 1 \\ &= 49n^2 + 56n + 16 + 7n + 4 + 1 \\ &= 49n^2 + 56n + 21 \\ &= 7(7n^2 + 8n + 3) \end{aligned}$$

  Hence as $x^7 - x$ has $(x^2 + x + 1) = 7(7n^2 + 8n + 3)$ as a factor, it is a multiple of 7.

- If $x = 7n + 5$ then

$$\begin{aligned} x^2 - x + 1 &= (7n+5)^2 - (7n+5) + 1 \\ &= 49n^2 + 70n + 25 - 7n - 5 + 1 \\ &= 49n^2 + 63n + 21 \\ &= 7(7n^2 + 9n + 3) \end{aligned}$$

  Hence as $x^7 - x$ has $(x^2 - x + 1) = 7(7n^2 + 9n + 3)$ as a factor, it is a multiple of 7.

- If $x = 7n + 6$ then $x^7 - x$ has $(x + 1) = (7n+6)+1 = 7n+7 = 7(n+1)$ as a factor, so it is a multiple of 7.

Hence, $x^7 - x$ for $x > 1$ is always a multiple of 7. □

Exercise 4D

1. Suppose the triangle is right angled. Then the hypotenuse is the longest side, 10cm. By Pythagoras's theorem, the length of the hypotenuse is given by

$$h^2 = 8^2 + 9^2 = 145$$

   but $10^2 = 100$: a contradiction.

   Therefore a triangle with sides 8cm, 9cm and 10cm is not right angled. □

2. Suppose that the lines intersect at point P.

$$\begin{aligned} \text{P} &= 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda\,(4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \\ \text{and } \text{P} &= 8\mathbf{i} + \mu\,(2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \end{aligned}$$

   Equating corresponding components:

$$\begin{aligned} 2 + 4\lambda &= 8 + 2\mu \\ 3 - 2\lambda &= \mu \\ -1 + 3\lambda &= -3\mu \end{aligned}$$

$$\begin{aligned} 2 + 4\lambda &= 8 + 2\mu \\ 6 - 4\lambda &= 2\mu \\ 8 &= 8 + 4\mu \\ \mu &= 0 \\ 3 - 2\lambda &= 0 \\ \lambda &= 1.5 \end{aligned}$$

   From $\mu = 0$ we get

$$\begin{aligned} \text{P} &= 8\mathbf{i} + 0\,(2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \\ &= 8\mathbf{i} \end{aligned}$$

   From $\lambda = 1.5$ we get

$$\begin{aligned} \text{P} &= 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + 1.5\,(4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \\ &= 8\mathbf{i} + 3.5\mathbf{k} \end{aligned}$$

   resulting in a contradiction.

   Therefore the lines do not intersect. □

3. Suppose that there exist integers p and q such that $6p + 10q = 151$.

   $6p + 10q = 2(3p + 5q)$ is even. But 151 is odd: a contradiction.

   Therefore no such integer p and q exist. □

4. Suppose that for some positive real a and b

$$\begin{aligned} \frac{a}{b} + \frac{b}{a} &< 2 \\ \frac{a^2 + b^2}{ab} &< 2 \\ a^2 + b^2 &< 2ab \\ a^2 - 2ab + b^2 &< 0 \\ (a - b)^2 &< 0 \end{aligned}$$

   But this requires the square of a real number to be negative: a contradiction.

Therefore no such positive real a and b exist. $\square$

(Note that the step taken in the third line above is only valid because we know ab is positive. If ab was negative then the inequality would change direction.)

5. Suppose the triangle is right angled. Then

$$\begin{aligned}(3x)^2 + (4x+5)^2 &= (5x+4)^2 \\ 9x^2 + 16x^2 + 40x + 25 &= 25x^2 + 40x + 16 \\ 25x^2 + 40x + 25 &= 25x^2 + 40x + 16 \\ 25 &= 16\end{aligned}$$

Therefore the triangle can not be right angled for any value of x. $\square$

6. Suppose that $\log_2 5$ is rational. Then

$$\log_2 5 = \frac{a}{b}$$

for integer a and b with no common factors.

$$\begin{aligned}2^{\frac{a}{b}} &= 5 \\ 2^a &= 5^b\end{aligned}$$

2^a is an integer power of 2, and hence is even. But 5^b is an integer power of an odd number, and hence is odd: a contradiction.

Therefore $\log_2 5$ is irrational. $\square$

7. Suppose there exists some smallest positive rational number $\frac{a}{b}$ where a and b are positive integers.

Because this is the smallest rational number, every other positive rational is greater than it.

$$\begin{aligned}\text{Then}\quad \frac{a}{b+1} &> \frac{a}{b} \\ ab &> a(b+1) \\ ab &> ab + a \\ 0 &> a\end{aligned}$$

But this is a contradiction, since we specified a and b both positive.

Therefore there is no smallest rational number greater than zero. $\square$

8. Suppose that there are a finite number of primes. Since every finite set of numbers must have a largest member, there is a largest prime, a, (a very large number, since it is larger than all known primes).

Now consider the number b obtained from the product of all the primes:

$$b = 2 \times 3 \times 5 \times 7 \times 11 \times \ldots \times a$$

Now consider the number $b+1$. This is clearly larger than a. Because b has every prime as a factor, $b+1$ will have a remainder of 1 when divided by every prime number. This means that the only factors of $b+1$ will be 1 and itself. Therefore b is a prime number larger than a: a contradiction.

Therefore there is an infinite number of primes. $\square$

Miscellaneous Exercise 4

1. $$\begin{aligned}\cos(-x) &= \cos(0-x) \\ &= \cos 0 \cos x + \sin 0 \sin x \\ &= 1 \cos x + 0 \sin x \\ &= \cos x\end{aligned}$$ $\square$

2. $$\begin{aligned}\cos\left(\frac{\pi}{2} - x\right) &= \sin\frac{\pi}{2}\cos x - \cos\frac{\pi}{2}\sin x \\ &= 1\cos x - 0 \sin x \\ &= \cos x\end{aligned}$$ $\square$

3. (a) $$\begin{aligned}z &= 2\operatorname{cis}\frac{\pi}{6} \\ &= 2\cos\frac{\pi}{6} + 2\mathrm{i}\sin\frac{\pi}{6} \\ &= \sqrt{3} + \mathrm{i}\end{aligned}$$

(b) $$\begin{aligned}w &= -1 - \sqrt{3}\mathrm{i} \\ &= \sqrt{(-1)^2 + (-\sqrt{3})^2}\operatorname{cis}\left(\tan^{-1}\frac{-\sqrt{3}}{-1}\right) \\ &= 2\operatorname{cis}\frac{-2\pi}{3} \quad \text{(3rd quadrant)}\end{aligned}$$

(c) $$\begin{aligned}zw &= \left(2\operatorname{cis}\frac{\pi}{6}\right)\left(2\operatorname{cis}\frac{-2\pi}{3}\right) \\ &= 4\operatorname{cis}\left(\frac{\pi}{6} - \frac{2\pi}{3}\right) \\ &= 4\operatorname{cis}\frac{-\pi}{2} \\ &= 4\cos\frac{-\pi}{2} + 4\mathrm{i}\sin\frac{-\pi}{2} \\ &= -4\mathrm{i}\end{aligned}$$

(d) $$\frac{z}{w} = \frac{2\,\text{cis}\,\frac{\pi}{6}}{2\,\text{cis}\,\frac{-2\pi}{3}}$$
$$= \text{cis}\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)$$
$$= \text{cis}\,\frac{5\pi}{6}$$
$$= \cos\frac{5\pi}{6} + \text{i}\sin\frac{5\pi}{6}$$
$$= -\frac{\sqrt{3}}{2} + \frac{1}{2}\text{i}$$

4. (a)
$$\overrightarrow{\text{OB}} = \mathbf{a} + \mathbf{c}$$
$$\overrightarrow{\text{OB}} \cdot \overrightarrow{\text{OA}} = (\mathbf{a} + \mathbf{c}) \cdot \mathbf{a} = a^2 + \mathbf{a} \cdot \mathbf{c}$$
$$\overrightarrow{\text{OB}} \cdot \overrightarrow{\text{OC}} = (\mathbf{a} + \mathbf{c}) \cdot \mathbf{c} = c^2 + \mathbf{a} \cdot \mathbf{c}$$
□

(b) Since OA=OC then $a^2 = c^2$ and it follows from the above that $\overrightarrow{\text{OB}} \cdot \overrightarrow{\text{OA}} = \overrightarrow{\text{OB}} \cdot \overrightarrow{\text{OC}}$
$$(\text{OB})(\text{OA})\cos\alpha = (\text{OB})(\text{OC})\cos\beta$$
$$\cos\alpha = \cos\beta$$
$$\alpha = \beta$$
(since both α and β are between 0 and 180°) □

5. L.H.S.:
$$(1 + \sin \text{A} \cos \text{A})(\sin \text{A} - \cos \text{A})$$
$$= \sin \text{A} - \cos \text{A} + \sin^2 \text{A} \cos \text{A} - \sin \text{A} \cos^2 \text{A}$$
$$= \sin \text{A} - \cos \text{A} + (1 - \cos^2 \text{A}) \cos \text{A} - \sin \text{A}(1 - \sin^2 \text{A})$$
$$= \sin \text{A} - \cos \text{A} + \cos \text{A} - \cos^3 \text{A} - \sin \text{A} + \sin^3 \text{A}$$
$$= -\cos^3 \text{A} + \sin^3 \text{A}$$
$$= \sin^3 \text{A} - \cos^3 \text{A}$$
$$= \text{R.H.S.}$$
□

6. L.H.S.:
$$\cos^4\theta - \sin^4\theta = (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)$$
$$= 1(\cos^2\theta - \sin^2\theta)$$
$$= (1 - \sin^2\theta) - \sin^2\theta$$
$$= 1 - 2\sin^2\theta$$
$$= \text{R.H.S.}$$
□

7. $$\frac{\text{d}y}{\text{d}x} = \lim_{h\to 0} \frac{(x+h)^3 + (x+h) - (x^3 + x)}{h}$$
$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 + x + h - x^3 - x}{h}$$
$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 + h}{h}$$
$$= \lim_{h\to 0}(3x^2 + 3xh + h^2 + 1)$$
$$= 3x^2 + 1$$

8. There are infinite possible correct solutions. First find any vector perpendicular to the one given (for example by setting one component to zero and then swapping the other two, changing the sign of one) such as:
$$\begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}$$
then scale this vector so that it has unit magnitude:
$$\frac{1}{\sqrt{0^2 + 1^2 + (-4)^2}}\begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} = \frac{1}{\sqrt{17}}\begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}$$

9. The set describes the locus of a circle radius 3 centred $(4 + 4\text{i})$ on the Argand plane.
   (a) The minimum possible value of $\text{Im}(z)$ is $4 - 3 = 1$.
   (b) The maximum possible value of $\text{Re}(z)$ is $4 + 3 = 7$.
   (c) The distance between the origin and the circle's centre is $|4 + 4\text{i}| = 4\sqrt{2}$. Since this is greater than 3, the minimum possible value of $|z|$ is $4\sqrt{2} - 3$.
   (d) Similarly the maximum possible value of $|z|$ is $4\sqrt{2} + 3$.
   (e) Since $\bar{z}$ is just the reflection of z in the real axis, the maximum value of $|\bar{z}|$ is equal to the maximum value of $|z|$, i.e. $4\sqrt{2} + 3$.

10. $$\mathbf{r}_\text{F} = \frac{1}{2}(\mathbf{r}_\text{A} + \mathbf{r}_\text{B})$$
$$= \frac{1}{2}(2\mathbf{i} + 6\mathbf{j} - 4\mathbf{k})$$
$$= \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$$
The radius of the sphere is AF:
$$\text{AF} = |\mathbf{r}_\text{F} - \mathbf{r}_\text{A}|$$
$$= |\,(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) - (-\mathbf{i} + 6\mathbf{j} - 8\mathbf{k})\,|$$
$$= |\,(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})\,|$$
$$= \sqrt{2^2 + (-3)^2 + 6^2}$$
$$= 7$$
CF, DF and EF are all also equal to the radius of the sphere, i.e. 7.
$$\text{CF} = 7$$
$$|\mathbf{r}_\text{F} - \mathbf{r}_\text{C}| = 7$$
$$|\,(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) - (7\mathbf{i} + c\mathbf{k})| = 7$$
$$|-6\mathbf{i} + 3\mathbf{j} + (-2 - c)\mathbf{k}| = 7$$
$$(-6)^2 + 3^2 + (-2 - c)^2 = 49$$
$$(-2 - c)^2 = 4$$
$$-2 - c = \pm 2$$
$$2 + c = \mp 2$$
$$c = 0$$
(rejecting the solution $c = -4$ because we are given that c is non-negative.)

$$\begin{aligned}
\text{DF} &= 7 \\
|\mathbf{r}_\text{F} - \mathbf{r}_\text{D}| &= 7 \\
|(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) - (-\mathbf{i} + d\mathbf{j} - 5\mathbf{k})| &= 7 \\
|2\mathbf{i} + (3-d)\mathbf{j} + 3\mathbf{k}| &= 7 \\
2^2 + (3-d)^2 + 3^2 &= 49 \\
(3-d)^2 &= 36 \\
3 - d &= \pm 6 \\
d &= 3 \mp 6 \\
d &= 9
\end{aligned}$$

(rejecting the negative solution again.)

$$\begin{aligned}
\text{DE} &= 7 \\
|\mathbf{r}_\text{F} - \mathbf{r}_\text{E}| &= 7 \\
|(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) - (4\mathbf{i} + \mathbf{j} + e\mathbf{k})| &= 7 \\
|-3\mathbf{i} + 2\mathbf{j} + (-2-e)\mathbf{k}| &= 7 \\
(-3)^2 + 2^2 + (-2-e)^2 &= 49 \\
(-2-e)^2 &= 36 \\
-2 - e &= \pm 6 \\
e &= -2 \mp 6 \\
e &= 4
\end{aligned}$$

(rejecting the negative solution again.)

11.
$$\begin{aligned}
\overrightarrow{\text{AB}} &= \mathbf{r}_\text{B} - \mathbf{r}_\text{A} \\
&= \begin{pmatrix} 2200 \\ 4000 \\ 600 \end{pmatrix} - \begin{pmatrix} 600 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1600 \\ 4000 \\ 600 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
{}_\text{A}\mathbf{v}_\text{B} &= \mathbf{v}_\text{A} - \mathbf{v}_\text{B} \\
&= \begin{pmatrix} -196 \\ 213 \\ 18 \end{pmatrix} - \begin{pmatrix} -240 \\ 100 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 44 \\ 113 \\ 18 \end{pmatrix}
\end{aligned}$$

For the missiles to intercept,

$$\begin{aligned}
\overrightarrow{\text{AB}} &= t\,{}_\text{A}\mathbf{v}_\text{B} \\
\begin{pmatrix} 1600 \\ 4000 \\ 600 \end{pmatrix} &= t \begin{pmatrix} 44 \\ 113 \\ 18 \end{pmatrix}
\end{aligned}$$

i components:

$$\begin{aligned}
1600 &= 44t \\
t &= \frac{400}{11}
\end{aligned}$$

k components:

$$\begin{aligned}
600 &= 18t \\
t &= \frac{100}{3}
\end{aligned}$$

Because $\overrightarrow{\text{AB}}$ is not a scalar multiple of ${}_\text{A}\mathbf{v}_\text{B}$ the missiles will not intercept.

To find how much it misses by (i.e. the minimum distance), let P be the point of closest approach.

$$\begin{aligned}
\overrightarrow{\text{BP}} &= \overrightarrow{\text{BA}} + \overrightarrow{\text{AP}} \\
&= -\overrightarrow{\text{AB}} + t\,{}_\text{A}\mathbf{v}_\text{B} \\
&= -\begin{pmatrix} 1600 \\ 4000 \\ 600 \end{pmatrix} + t \begin{pmatrix} 44 \\ 113 \\ 18 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{\text{BP}} \cdot {}_\text{A}\mathbf{v}_\text{B} &= 0 \\
\left(-\begin{pmatrix} 1600 \\ 4000 \\ 600 \end{pmatrix} + t \begin{pmatrix} 44 \\ 113 \\ 18 \end{pmatrix} \right) \cdot \begin{pmatrix} 44 \\ 113 \\ 18 \end{pmatrix} &= 0 \\
-1600 \times 44 - 4000 \times 113 - 600 \times 18 & \\
+t(44^2 + 113^2 + 18^2) &= 0 \\
-533200 + 15029t &= 0 \\
t &= 35.478
\end{aligned}$$

$$\begin{aligned}
|\overrightarrow{\text{BP}}| &= \left| -\begin{pmatrix} 1600 \\ 4000 \\ 600 \end{pmatrix} + t \begin{pmatrix} 44 \\ 113 \\ 18 \end{pmatrix} \right| \\
|\overrightarrow{\text{BP}}| &= \left| \begin{pmatrix} -38.96 \\ 9.02 \\ 38.61 \end{pmatrix} \right| \\
&= 55.59\text{m}
\end{aligned}$$

The missile misses by about 56m.

After 20 seconds the positions are now

$$\begin{aligned}
\mathbf{r}_\text{A} &= \begin{pmatrix} 600 \\ 0 \\ 0 \end{pmatrix} + 20 \begin{pmatrix} -196 \\ 213 \end{pmatrix} 18 \\
&= \begin{pmatrix} -3320 \\ 4260 \\ 360 \end{pmatrix} \\
\mathbf{r}_\text{B} &= \begin{pmatrix} 2200 \\ 4000 \\ 600 \end{pmatrix} + 20 \begin{pmatrix} -240 \\ 100 \end{pmatrix} 0 \\
&= \begin{pmatrix} -2600 \\ 6000 \\ 600 \end{pmatrix} \\
\overrightarrow{\text{AB}} &= \mathbf{r}_\text{B} - \mathbf{r}_\text{A} \\
&= \begin{pmatrix} -2600 \\ 6000 \\ 600 \end{pmatrix} - \begin{pmatrix} -3320 \\ 4260 \\ 360 \end{pmatrix} \\
&= \begin{pmatrix} 720 \\ 1740 \\ 240 \end{pmatrix}
\end{aligned}$$

In order to intercept 15 seconds later:

$$\begin{aligned}
15\,{}_{\mathrm{A}}\mathbf{v}_{\mathrm{B}} &= \overrightarrow{\mathrm{AB}}\\
{}_{\mathrm{A}}\mathbf{v}_{\mathrm{B}} &= \frac{1}{15}\begin{pmatrix}720\\1740\\240\end{pmatrix}\\
&= \begin{pmatrix}48\\116\\16\end{pmatrix}\\
\mathbf{v}_{\mathrm{A}} - \mathbf{v}_{\mathrm{B}} &= \begin{pmatrix}48\\116\\16\end{pmatrix}\\
\mathbf{v}_{\mathrm{A}} &= \begin{pmatrix}48\\116\\16\end{pmatrix} + \mathbf{v}_{\mathrm{B}}\\
&= \begin{pmatrix}48\\116\\16\end{pmatrix} + \begin{pmatrix}-240\\100\\0\end{pmatrix}\\
&= \begin{pmatrix}-192\\216\\16\end{pmatrix}\ \mathrm{m/s}
\end{aligned}$$