Chapter 2

Exercise 2A

1. (a) $|z| = \sqrt{4^2 + (-3)^2}$
   $= 5$

   (b) $|z| = \sqrt{12^2 + 5^2}$
   $= 13$

   (c) $|z| = \sqrt{3^2 + 2^2}$
   $= \sqrt{13}$

   (d) $|z| = \sqrt{3^2 + (-2)^2}$
   $= \sqrt{13}$

   (e) $|z| = \sqrt{1^2 + 5^2}$
   $= \sqrt{26}$

   (f) $|z| = \sqrt{0^2 + 5^2}$
   $= 5$

2. Process:

   - Determine which quadrant z lies in on the Argand diagram by examining the sign of the real and imaginary parts
   - Use inverse tangent to determine the angle made with the real axis on an argand diagram.
   - Combine these to obtain the principal argument.

   (a) 1^{st} Quadrant: $\tan\theta = \frac{2}{2}$
   $\therefore \quad \arg z = \frac{\pi}{4}$

   (b) 4^{th} Quadrant: $\tan\theta = \frac{-2}{2}$
   $\therefore \quad \arg z = -\frac{\pi}{4}$

   (c) 2^{nd} Quadrant: $\tan\theta = \frac{2}{-2}$
   $\therefore \quad \arg z = \frac{3\pi}{4}$

   (d) 3^{rd} Quadrant: $\tan\theta = \frac{-2}{-2}$
   $\therefore \quad \arg z = -\frac{3\pi}{4}$

   (e) 2^{nd} Quadrant: $\tan\theta = \frac{2\sqrt{3}}{-2}$
   $\therefore \quad \arg z = \frac{2\pi}{3}$

   (f) 4^{th} Quadrant: $\tan\theta = \frac{-3\sqrt{3}}{3}$
   $\therefore \quad \arg z = -\frac{\pi}{3}$

3. There seems little point in showing working for these problems.
   z_1, z_2 and z_3: subtract 2π to obtain the principal argument.
   z_4: add 2π to obtain the principal argument.
   z_5 to z_{12}: read r from the magnitude shown on the diagram, and determine θ as the directed angle measured anticlockwise from the positive real axis.

4. There seems little point in showing working for these problems.
   Determine r by Pythagoras as for question 1.
   Determine θ as for question 2.

5. There seems little point in showing working for these problems. You should be able to do them in a single step.
   Determine the real and imaginary components by evaluating the trigonometric expressions exactly and then multiplying by r.

Exercise 2B

1. Read r from the magnitude of z and θ from the directed angle (converted to radians) measured anticlockwise from the positive real axis.

2–9 No working needed. You should be able to do these questions in a single step, converting to radians and adding or subtracting a multiple of 2π where necessary.

10. $7\operatorname{cis}\frac{\pi}{2} = 7\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right)$
    $= 7(0 + \mathrm{i})$
    $= 7\mathrm{i}$

11. $$5\operatorname{cis}\left(-\frac{\pi}{2}\right) = 5\left(\cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right)\right)$$
$$= 5(0 - \mathrm{i})$$
$$= -5\mathrm{i}$$

12. $$\operatorname{cis}\pi = \cos\pi + \mathrm{i}\sin\pi$$
$$= -1 + 0\mathrm{i}$$
$$= -1$$

13. $$3\operatorname{cis}2\pi = 3(\cos 2\pi + \mathrm{i}\sin 2\pi)$$
$$= 3(1 + 0\mathrm{i})$$
$$= 3$$

(With a little practice you may find you can do questions like these by simply sketching or visualising an Argand diagram.)

14. $$10\operatorname{cis}\frac{\pi}{4} = 10\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)$$
$$= 5\sqrt{2} + 5\sqrt{2}\mathrm{i}$$

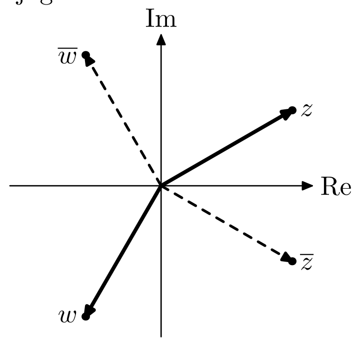
15. $$4\operatorname{cis}\frac{2\pi}{3} = 4\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right)$$
$$= -2 + 2\sqrt{3}\mathrm{i}$$

16. $$4\operatorname{cis}\left(-\frac{2\pi}{3}\right) = 4\left(\cos\left(-\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{3}\right)\right)$$
$$= -2 - 2\sqrt{3}\mathrm{i}$$

17. $$12\operatorname{cis}\left(-\frac{4\pi}{3}\right) = 12\operatorname{cis}\frac{2\pi}{3}$$
$$= 12\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right)$$
$$= -6 + 6\sqrt{3}\mathrm{i}$$

18–21 Do these questions in the same way as question 4 of exercise 2A (i.e. using Pythagoras to find r and inverse tangent to determine θ, adding or subtracting π for z in the second or third quadrant respectively).

22. (a) The conjugate is a reflection in the real axis:



(b) $\overline{z} = r_1\operatorname{cis}(-\alpha)$
$\overline{w} = r_2\operatorname{cis}(-\beta)$

23. No working needed.

24. No working needed.

25. First subtract 360° to get the principal argument, then multiply this by -1 to obtain the conjugate.

26. First add 360° to get the principal argument, then multiply this by -1 to obtain the conjugate.

27. No working needed.

28. No working needed.

29. No working needed.

30. First subtract 4π to get the principal argument, then multiply this by -1 to obtain the conjugate.

Exercise 2C

1. $$zw = (2 + 3\mathrm{i})(5 - 2\mathrm{i})$$
$$= 10 - 4\mathrm{i} + 15\mathrm{i} + 6$$
$$= 16 + 11\mathrm{i}$$

2. $$zw = (3 + 2\mathrm{i})(-1 + 2\mathrm{i})$$
$$= -3 + 6\mathrm{i} - 2\mathrm{i} - 4$$
$$= -7 + 4\mathrm{i}$$

3. $$zw = 3 \times 5\operatorname{cis}(60^\circ + 20^\circ)$$
$$= 15\operatorname{cis}80^\circ$$

4. $$zw = 3 \times 3\operatorname{cis}(120^\circ + 150^\circ)$$
$$= 9\operatorname{cis}270^\circ$$
$$= 9\operatorname{cis}(-90^\circ)$$

5. $$zw = 3 \times 3\operatorname{cis}(30^\circ - 80^\circ)$$
$$= 9\operatorname{cis}(-50^\circ)$$

6. $$zw = 5 \times 2\operatorname{cis}\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$$
$$= 10\operatorname{cis}\frac{7\pi}{12}$$

7. $$\begin{aligned} zw &= 4 \times 2 \operatorname{cis}\left(\frac{\pi}{4} - \frac{4\pi}{4}\right) \\ &= 8 \operatorname{cis}\left(-\frac{\pi}{2}\right) \end{aligned}$$

8. $$\begin{aligned} zw &= 2 \times 1 \left(\cos\left(50^\circ + 60^\circ\right) + \mathrm{i}\sin\left(50^\circ + 60^\circ\right)\right) \\ &= 2\left(\cos 110^\circ + \mathrm{i}\sin 110^\circ\right) \end{aligned}$$

9. $$\begin{aligned} zw &= 2 \times 3 \left(\cos\left(170^\circ + 150^\circ\right) + \mathrm{i}\sin\left(170^\circ + 150^\circ\right)\right) \\ &= 6\left(\cos 320^\circ + \mathrm{i}\sin 320^\circ\right) \\ &= 6\left(\cos(-40^\circ) + \mathrm{i}\sin(-40^\circ)`\right) \end{aligned}$$

10. $$\begin{aligned} \frac{z}{w} &= \frac{6-3\mathrm{i}}{3-4\mathrm{i}} \\ &= \frac{(6-3\mathrm{i})(3+4\mathrm{i})}{(3-4\mathrm{i})(3+4\mathrm{i})} \\ &= \frac{18+24\mathrm{i}-9\mathrm{i}+12}{9+16} \\ &= \frac{30+15\mathrm{i}}{25} \\ &= 1.2+0.6\mathrm{i} \end{aligned}$$

11. $$\begin{aligned} \frac{z}{w} &= \frac{-6+3\mathrm{i}}{-3+4\mathrm{i}} \\ &= \frac{-1(6-3\mathrm{i})}{-1(3-4\mathrm{i})} \\ &= 1.2+0.6\mathrm{i} \end{aligned}$$

12. $$\begin{aligned} \frac{z}{w} &= \frac{8}{2}\operatorname{cis}(60^\circ - 40^\circ) \\ &= 4\operatorname{cis} 20^\circ \end{aligned}$$

13. $$\begin{aligned} \frac{z}{w} &= \frac{5}{1}\operatorname{cis}(120^\circ - 150^\circ) \\ &= 5\operatorname{cis}(-30^\circ) \end{aligned}$$

14. $$\begin{aligned} \frac{z}{w} &= \frac{3}{3}\operatorname{cis}(-150^\circ - 80^\circ) \\ &= \operatorname{cis}(-230^\circ) \\ &= \operatorname{cis}(-230^\circ + 360^\circ) \\ &= \operatorname{cis} 130^\circ \end{aligned}$$

15. $$\begin{aligned} \frac{z}{w} &= \frac{2}{2}\operatorname{cis}\left(\frac{3\pi}{5} - \frac{2\pi}{5}\right) \\ &= \operatorname{cis}\frac{\pi}{5} \end{aligned}$$

16. $$\begin{aligned} \frac{z}{w} &= \frac{4}{2}\operatorname{cis}\left(\frac{\pi}{4} - -\frac{3\pi}{4}\right) \\ &= 2\operatorname{cis}\pi \end{aligned}$$

17. $$\begin{aligned} \frac{z}{w} &= \frac{5}{2}\left(\cos\left(\frac{3\pi}{4} - \frac{\pi}{2}\right) + \mathrm{i}\sin\left(\frac{3\pi}{4} - \frac{\pi}{2}\right)\right) \\ &= 2.5\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right) \end{aligned}$$

18. $$\begin{aligned} \frac{z}{w} &= \frac{2}{5}\left(\cos(50^\circ - 50^\circ) + \mathrm{i}\sin(50^\circ - 50^\circ)\right) \\ &= 0.4\left(\cos 0 + \mathrm{i}\sin 0\right) \end{aligned}$$

19. From z, zw has been rotated 40° and scaled by 2, so
$$w = 2\operatorname{cis} 40^\circ$$

20. From z, zw has been rotated $180-30-50 = 100^\circ$ and scaled by 3, so
$$w = 3\operatorname{cis} 100^\circ$$

21. From z, zw has been rotated -90° and scaled by 2, so
$$w = 2\operatorname{cis}(-90^\circ)$$

22. From z, zw has been rotated $180 - 110 + 50 = 120^\circ$ and scaled by 2, so
$$w = 2\operatorname{cis} 120^\circ$$

23. From z, zw has been rotated $180 - 110 + 90 = 160^\circ$ and scaled by 1, so
$$w = \operatorname{cis} 160^\circ$$

24. From z, zw has been rotated -140° and scaled by 2, so
$$w = 2\operatorname{cis}(-140^\circ)$$

25. From z, $\frac{z}{w}$ has been rotated -120° and scaled by 1, so
$$w = \operatorname{cis} 120^\circ$$

26. From z, $\frac{z}{w}$ has been rotated -80° and scaled by $\frac{1}{2}$, so
$$w = 2\operatorname{cis} 80^\circ$$

27. From z, $\frac{z}{w}$ has been rotated 100° and scaled by $\frac{1}{2}$, so
$$w = 2\operatorname{cis}(-100^\circ)$$

28. (a) $$\begin{aligned} 2z &= 2(6\operatorname{cis} 40^\circ) \\ &= 12\operatorname{cis} 40^\circ \end{aligned}$$

(b) $$\begin{aligned} 3w &= 3(2\operatorname{cis} 30^\circ) \\ &= 6\operatorname{cis} 30^\circ \end{aligned}$$

(c) $$\begin{aligned} zw &= 6 \times 2\operatorname{cis}(40^\circ + 30^\circ) \\ &= 12\operatorname{cis} 70^\circ \end{aligned}$$

(d) $$\begin{aligned} wz &= zw \\ &= 12\operatorname{cis} 70^\circ \end{aligned}$$

(e) $$\begin{aligned} iz &= 6\operatorname{cis}(40^\circ + 90^\circ) \\ &= 6\operatorname{cis} 130^\circ \end{aligned}$$

(f) $$\begin{aligned} iw &= 2\operatorname{cis}(30^\circ + 90^\circ) \\ &= 2\operatorname{cis} 120^\circ \end{aligned}$$

(g) $$\begin{aligned} \frac{w}{z} &= \frac{2}{6}\operatorname{cis}(30^\circ - 40^\circ) \\ &= \frac{1}{3}\operatorname{cis}(-10^\circ) \end{aligned}$$

(h) $$\begin{aligned} \frac{1}{z} &= \frac{1}{6}\operatorname{cis}(0 - 40^\circ) \\ &= \frac{1}{6}\operatorname{cis}(-40^\circ) \end{aligned}$$

29. (a) $$\begin{aligned} zw &= 8 \times 4\operatorname{cis}(\frac{2\pi}{3} + \frac{3\pi}{4}) \\ &= 32\operatorname{cis}\frac{17\pi}{12} \\ &= 32\operatorname{cis}\left(-\frac{7\pi}{12}\right) \end{aligned}$$

(b) $wz = zw$
$$= 32\operatorname{cis}\left(-\frac{7\pi}{12}\right)$$

(c) $\frac{w}{z} = \frac{4}{8}\operatorname{cis}(\frac{3\pi}{4} - \frac{2\pi}{3})$
$$= 0.5\operatorname{cis}\frac{\pi}{12}$$

(d) $\frac{z}{w} = \frac{8}{4}\operatorname{cis}(\frac{2\pi}{3} - \frac{3\pi}{4})$
$$= 2\operatorname{cis}\left(-\frac{\pi}{12}\right)$$

(e) $\overline{z} = 8\operatorname{cis}\left(-\frac{2\pi}{3}\right)$

(f) $\overline{w} = 4\operatorname{cis}\left(-\frac{3\pi}{4}\right)$

(g) $\frac{1}{z} = \frac{1}{8}\operatorname{cis}\left(0 - \frac{2\pi}{3}\right)$
$$= 0.125\operatorname{cis}\left(-\frac{2\pi}{3}\right)$$

(h) $\frac{\mathrm{i}}{w} = \frac{1}{4}\operatorname{cis}\left(\frac{\pi}{2} - \frac{3\pi}{4}\right)$
$$= 0.25\operatorname{cis}\left(-\frac{\pi}{4}\right)$$

Exercise 2D

1-5 No working required.

6. First rewrite as $z : |z - (3 - 3\mathrm{i})| = 3$

7. Think of this as points equidistant between $(-8 + 0\mathrm{i})$ and $(0 + 4\mathrm{i})$. Plot those points on the Argand diagram, and the locus is the perpendicular bisector of the line segment between them.

8. Think of this as points equidistant between $(-2 - 3\mathrm{i})$ and $(4 - \mathrm{i})$. Plot those points on the Argand diagram, and the locus is the perpendicular bisector of the line segment between them.

9. Since $x = \operatorname{Re}(z)$, $z : \operatorname{Re}(z) = 5$ is the vertical line $x = 5$.

10. Since $y = \operatorname{Im}(z)$, $z : \operatorname{Im}(z) = -4$ is the horizontal line $y = -4$.

11.
$$\begin{aligned} \theta &= \arg z \\ \tan\theta &= \frac{y}{x} \\ \therefore \quad \tan\frac{\pi}{3} &= \frac{y}{x} \\ y &= \sqrt{3}x \; : \; x \geq 0 \end{aligned}$$
We need to specify $x \geq 0$ to exclude the 4^{th} quadrant.

12.
$$\begin{aligned} \theta &= \arg z \\ \tan\theta &= \frac{y}{x} \\ \therefore \quad \tan\left(-\frac{\pi}{3}\right) &= \frac{y}{x} \\ y &= -\sqrt{3}x \; : \; x \geq 0 \end{aligned}$$
We need to specify $x \geq 0$ to exclude the 2^{nd} quadrant.

13. Simply substitute x for $\operatorname{Re}(z)$ and y for $\operatorname{Im}(z)$ to give $x + y = 6$.

14.
$$\begin{aligned} |z| &= 6 \\ \sqrt{x^2 + y^2} &= 6 \\ x^2 + y^2 &= 36 \end{aligned}$$

15. This is a circular region centred at $(0, 4)$ having radius 3. You should be able to produce a Cartesian equation from this description, or you can do it algebraically:
$$\begin{aligned} |z - 4\mathrm{i}| &\leq 3 \\ |x + (y - 4)\mathrm{i}| &\leq 3 \\ \sqrt{x^2 + (y - 4)^2} &\leq 3 \\ x^2 + (y - 4)^2 &\leq 9 \end{aligned}$$
You have to be careful with inequalities like this because multiplying both sides of the equation by a negative changes the direction of the inequality. It can be particularly tricky when you square both sides of an inequation as we have in the last step here: you have to make sure you know with certainty that both sides are positive. For example, $a < 2$ is not the same as $a^2 < 4$. We are safe here, however, since the square root on the left hand side is always positive and so is 3 on the right hand side.

16. This is a circle radius 4 centred at $(2, 3)$.
$$\begin{aligned} |z - (2 + 3\mathrm{i})| &= 4 \\ |(x - 2) + (y - 3)\mathrm{i}| &= 4 \\ (x - 2)^2 + (y - 3)^2 &= 16 \end{aligned}$$

17. This is a circle radius 4 centred at $(2, -3)$.

$$|z - 2 + 3\mathrm{i}| = 4$$
$$|(x-2) + (y+3)\mathrm{i}| = 4$$
$$(x-2)^2 + (y+3)^2 = 16$$

18. This is a line defined by the locus of points equidistant from $(2, 0)$ and $(6, 0)$, i.e. the vertical line $x = 4$. Showing this algebraically:

$$|z - 2| = |z - 6|$$
$$|(x-2) + y\mathrm{i}| = |(x-6) + y\mathrm{i}|$$
$$(x-2)^2 + y^2 = (x-6)^2 + y^2$$
$$x^2 - 4x + 4 + y^2 = x^2 - 12x + 36 + y^2$$
$$-4x + 4 = -12x + 36$$
$$8x = 32$$
$$x = 4$$

19. This is a line defined by the locus of points equidistant from $(0, 6)$ and $(2, 0)$:

$$|z - 6\mathrm{i}| = |z - 2|$$
$$|x + (y-6)\mathrm{i}| = |(x-2) + y\mathrm{i}|$$
$$x^2 + (y-6)^2 = (x-2)^2 + y^2$$
$$x^2 + y^2 - 12y + 36 = x^2 - 4x + 4 + y^2$$
$$-12y + 36 = -4x + 4$$
$$4x - 12y = -32$$
$$x - 3y = -8$$

20. This is a line defined by the locus of points equidistant from $(2, 1)$ and $(4, -5)$:

$$|z - (2 + \mathrm{i})| = |z - (4 - 5\mathrm{i})|$$
$$|(x-2) + (y-1)\mathrm{i}| = |(x-4) + (y+5)\mathrm{i}|$$
$$(x-2)^2 + (y-1)^2 = (x-4)^2 + (y+5)^2$$
$$x^2 - 4x + 4 + y^2 - 2y + 1 = x^2 - 8x + 16 + y^2 + 10y + 25$$
$$4x - 12y = 36$$
$$x - 3y = 9$$

21. This is a doughnut-shaped region centred at the origin, including all points on or inside the 5-unit circle and on or outside the 3-unit circle. For the Cartesian equation, substitute $\sqrt{x^2 + y^2}$ for $|z|$.

22. This is the region in the first quadrant bounded below by the line $z = \frac{\pi}{6}$ and above by the line $z = \frac{\pi}{3}$.

$$\begin{array}{rcl} \frac{\pi}{6} \leq & \theta & \leq \frac{\pi}{3} \\ \tan\frac{\pi}{6} \leq & \tan\theta & \leq \tan\frac{\pi}{3} \\ \frac{1}{\sqrt{3}} \leq & \frac{y}{x} & \sqrt{3} \\ \frac{x}{\sqrt{3}} \leq & y & \sqrt{3}x \quad : \quad x \geq 0 \end{array}$$

Note that the second line above (where we take the tangent) is only valid because the tangent function is "strictly increasing" in the first quadrant, so $a > b$ implies $\tan a > \tan b$. This is only true for functions that have positive gradient in the given domain.

The step taken in the last step (multiplication by x) is only valid because we restrict $x \geq 0$. (If x was negative, this step would change the direction of the inequalities.

An alternative, possibly simpler, approach to this question would be to first observe that the specified region lies above (or on) the line $y = \frac{x}{\sqrt{3}}$ and below (or on) the line $y = \sqrt{3}x$, (again using $\tan\theta = \frac{y}{x}$ to obtain these equations). This is essentially doing the same thing as the above, but it's perhaps a more intuitive way of looking at it.

23. We have a circle centred at $(-3, 3)$ having a radius of 2.

    (a) The minimum value of $\mathrm{Im}(z)$ is $3 - 2 = 1$.

    (b) The maximum value of $\mathrm{Re}(z)$ is $-3+2 = -1$ and the minimum is $-3 - 2 = -5$ so the maximum value of $|\,\mathrm{Re}(z)|$ is $|-5| = 5$.

    (c) The distance between the origin and the centre of the circle is $\sqrt{(-3)^2 + 3^2} = 3\sqrt{2}$. The point on the circle nearest the origin is 2 units nearer, so the minimum $|z|$ is $3\sqrt{2} - 2$.

    (d) The point on the circle furthest from the origin is 2 units further away than the centre, so the maximum $|z|$ is $3\sqrt{2} + 2$.

    (e) The locus of $\overline{z}$ is the reflection in the real axis of the locus of z. The point on this image furthest from the origin is the same distance from the origin as the corresponding point on the original: $3\sqrt{2} + 2$.

24. We have a circle centred at $(4, 3)$ having a radius of 2.

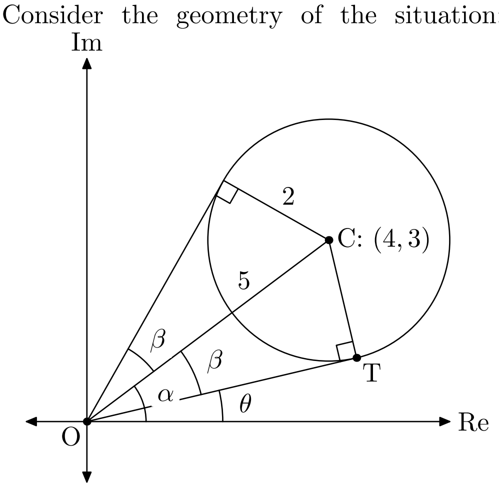
    (a) The minimum value of $\mathrm{Im}(z)$ is $3 - 2 = 1$.

    (b) The maximum value of $\mathrm{Re}(z)$ is $4 + 2 = 6$.

    (c) The distance between the origin and the centre of the circle is $\sqrt{4^2 + 3^2} = 5$. The point on the circle furthest from the origin is 2 units further, so the maximum $|z|$ is $5 + 2 = 7$.

    (d) The point on the circle nearest the origin is 2 units nearer than the centre, so the minimum $|z|$ is $5 - 2 = 3$.

    (e) Consider the geometry of the situation:



The minimum value of $\arg z$ corresponds to the angle for the lower tangent, T.

$$\begin{aligned}\tan\alpha &= \frac{3}{4}\\ \alpha &= 0.644\\ \sin\beta &= \frac{2}{5}\\ \beta &= 0.412\\ \theta &= \alpha - \beta\\ &= 0.23\end{aligned}$$

(f) The maximum value of $\arg z$ corresponds to the angle for the upper tangent.

$$\begin{aligned}\theta &= \alpha + \beta\\ &= 1.06\end{aligned}$$

Miscellaneous Exercise 2

1. (a) $$\begin{aligned}z + w &= 3 + 2 + (-4 + 3)\text{i}\\ &= 5 - \text{i}\end{aligned}$$

   (b) $$\begin{aligned}z - w &= 3 - 2 + (-4 - 3)\text{i}\\ &= 1 - 7\text{i}\end{aligned}$$

   (c) $$\begin{aligned}zw &= (3 - 4\text{i})(2 + 3\text{i})\\ &= 6 + 9\text{i} - 8\text{i} + 12\\ &= 18 + \text{i}\end{aligned}$$

   (d) $$\begin{aligned}z^2 &= (3 - 4\text{i})^2\\ &= 9 - 24\text{i} - 16\\ &= -7 - 24\text{i}\end{aligned}$$

   (e) $$\begin{aligned}\frac{z}{w} &= \frac{3 - 4\text{i}}{2 + 3\text{i}}\\ &= \frac{(3 - 4\text{i})(2 - 3\text{i})}{(2 + 3\text{i})(2 - 3\text{i})}\\ &= \frac{6 - 9\text{i} - 8\text{i} - 12}{4 + 9}\\ &= \frac{-6 - 17\text{i}}{13}\\ &= -\frac{6}{13} - \frac{17}{13}\text{i}\end{aligned}$$

   (f) $$\begin{aligned}\frac{w}{z} &= \frac{2 + 3\text{i}}{3 - 4\text{i}}\\ &= \frac{(2 + 3\text{i})(3 + 4\text{i})}{(3 - 4\text{i})(3 + 4\text{i})}\\ &= \frac{6 + 8\text{i} + 9\text{i} - 12}{9 + 16}\\ &= \frac{-6 + 17\text{i}}{25}\\ &= -\frac{6}{25} + \frac{17}{25}\text{i}\end{aligned}$$

2. (a) $\overrightarrow{\text{AB}} = \mathbf{c}$

   (b) $\overrightarrow{\text{AD}} = \frac{1}{4}\overrightarrow{\text{AB}} = \frac{1}{4}\mathbf{c}$

   (c) $\overrightarrow{\text{DB}} = \frac{3}{4}\overrightarrow{\text{AB}} = \frac{3}{4}\mathbf{c}$

   (d) $$\begin{aligned}\overrightarrow{\text{DE}} &= \overrightarrow{\text{DB}} + \overrightarrow{\text{BE}}\\ &= \frac{3}{4}\mathbf{c} + \frac{1}{2}\mathbf{c}\\ &= \frac{5}{4}\mathbf{c}\end{aligned}$$

   (e) $$\begin{aligned}\overrightarrow{\text{OB}} &= \overrightarrow{\text{OA}} + \overrightarrow{\text{AB}}\\ &= \mathbf{a} + \mathbf{c}\end{aligned}$$

   (f) $$\begin{aligned}\overrightarrow{\text{OD}} &= \overrightarrow{\text{OA}} + \overrightarrow{\text{AD}}\\ &= \mathbf{a} + \frac{1}{4}\mathbf{c}\end{aligned}$$

   (g) $$\begin{aligned}\overrightarrow{\text{CE}} &= \overrightarrow{\text{CB}} + \overrightarrow{\text{BE}}\\ &= \mathbf{a} + \frac{1}{2}\mathbf{c}\end{aligned}$$

   (h) $$\begin{aligned}\overrightarrow{\text{OE}} &= \overrightarrow{\text{OA}} + \overrightarrow{\text{AE}}\\ &= \mathbf{a} + \frac{3}{2}\mathbf{c}\end{aligned}$$

   alternatively

   $$\begin{aligned}\overrightarrow{\text{OE}} &= \overrightarrow{\text{OC}} + \overrightarrow{\text{CE}}\\ &= \mathbf{c} + \mathbf{a} + \frac{1}{2}\mathbf{c}\\ &= \mathbf{a} + \frac{3}{2}\mathbf{c}\end{aligned}$$

3. (a) $$\begin{aligned}r &= \sqrt{(-3)^2 + (-3\sqrt{3})^2}\\ &= \sqrt{9 + 27}\\ &= 6\\ \tan\theta &= \frac{-3\sqrt{3}}{-3}\\ \theta &= -\frac{2\pi}{3} \qquad \text{(third quadrant)}\\ -3 - 3\sqrt{3}\text{i} &= 6\,\text{cis}\left(-\frac{2\pi}{3}\right)\end{aligned}$$

(b) $$\begin{aligned} &8\,\mathrm{cis}\left(-\frac{5\pi}{6}\right)\\ &= 8\left(\cos\left(-\frac{5\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right)\\ &= 8\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}\mathrm{i}\right)\\ &= -4\sqrt{3}-4\mathrm{i}\end{aligned}$$

4. (a) $(2\cos\frac{\pi}{2}, 2\sin\frac{\pi}{2}) = (0,2)$
   (b) $(5\cos\pi, 5\sin\pi) = (-5,0)$
   (c) $(4\cos\frac{-3\pi}{4}, 4\sin\frac{-3\pi}{4}) = (-2\sqrt{2}, -2\sqrt{2})$

5. $$\begin{aligned} z &= \sqrt{1^2+1^2}\,\mathrm{cis}\tan^{-1}\frac{1}{1}\\ &= \sqrt{2}\,\mathrm{cis}\frac{\pi}{4}\\ w &= \sqrt{(-1)^2+1^2}\,\mathrm{cis}\tan^{-1}\frac{1}{-1}\\ &= \sqrt{2}\,\mathrm{cis}\frac{3\pi}{4}\\ zw &= \sqrt{2}\sqrt{2}\,\mathrm{cis}\left(\frac{\pi}{4}+\frac{3\pi}{4}\right)\\ &= 2\,\mathrm{cis}\,\pi\\ \frac{z}{w} &= \frac{\sqrt{2}}{\sqrt{2}}\,\mathrm{cis}\left(\frac{\pi}{4}-\frac{3\pi}{4}\right)\\ &= \mathrm{cis}\left(-\frac{\pi}{2}\right)\end{aligned}$$

6. Show that the variable part of each expression is a scalar multiple of that of the other, i.e. $(2\mathbf{i}-7\mathbf{j}) = -1(-2\mathbf{i}+7\mathbf{j})$.

7. Show that the variable parts are perpendicular vectors, using the scalar product, i.e. $(6\mathbf{i}-4\mathbf{j})\cdot(2\mathbf{i}+3\mathbf{j}) = 12-12 = 0$.

8. (a) The only scalar multiples of non-parallel, non-zero vectors that equates them is zero, so $p = q = 0$.
   (b) $p-3=0$ so $p=3$, and $q=0$.
   (c) $p+2=0$ so $p=-2$, and $q-1=0$ so $q=1$.
   (d) $$\begin{aligned} p\mathbf{a}+2\mathbf{b} &= 3\mathbf{a}-q\mathbf{b}\\ p\mathbf{a}-3\mathbf{a} &= -q\mathbf{b}-2\mathbf{b}\\ (p-3)\mathbf{a} &= (-q-2)\mathbf{b}\\ p &= 3\\ q &= -2\end{aligned}$$
   (e) $$\begin{aligned} p\mathbf{a}+q\mathbf{a}+p\mathbf{b}-2q\mathbf{b} &= 3\mathbf{a}+6\mathbf{b}\\ (p+q-3)\mathbf{a} &= (6-p+2q)\mathbf{b}\\ p+q &= 3\\ p-2q &= 6\\ p &= 4\\ q &= -1\end{aligned}$$
   (f) $$\begin{aligned} p\mathbf{a}+2\mathbf{a}-2p\mathbf{b} &= \mathbf{b}+5q\mathbf{b}-q\mathbf{a}\\ (p+2+q)\mathbf{a} &= (1+5q+2p)\mathbf{b}\\ p+q &= -2\\ 2p+5q &= -1\\ -2p-2q &= 4\\ 3q &= 3\\ q &= 1\\ p &= -3\end{aligned}$$

9. $$\begin{aligned} 3\mathbf{i}+\mathbf{j}+\lambda(7\mathbf{i}-5\mathbf{j}) &= 5\mathbf{i}-6\mathbf{j}+\mu(4\mathbf{i}-\mathbf{j})\\ (3+7\lambda-5-4\mu)\mathbf{i} &= (-6-\mu-1+5\lambda)\mathbf{j}\\ (7\lambda-4\mu-2)\mathbf{i} &= (5\lambda-\mu-7)\mathbf{j}\\ 7\lambda-4\mu &= 2\\ 5\lambda-\mu &= 7\\ -20\lambda+4\mu &= -28\\ -13\lambda &= -26\\ \lambda &= 2\\ \mathbf{r} &= 3\mathbf{i}+\mathbf{j}+2(7\mathbf{i}-5\mathbf{j})\\ &= 17\mathbf{i}-9\mathbf{j}\end{aligned}$$

10. (a) $\frac{\mathrm{d}}{\mathrm{d}x}x^3 = 3x^2$
    (b) $\frac{\mathrm{d}}{\mathrm{d}x}6x^5 = 30x^4$
    (c) $\frac{\mathrm{d}}{\mathrm{d}x}(x^2-7x+3) = 2x-7$
    (d) $\frac{\mathrm{d}}{\mathrm{d}x}((x+4)(x+2)) = \frac{\mathrm{d}}{\mathrm{d}x}(x^2+6x+8) = 2x+6$ (You could use the product rule here if you preferred.)
    (e) $\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^3+x}{x} = \frac{\mathrm{d}}{\mathrm{d}x}(x^2+1) = 2x$ (You could use the quotient rule here if you preferred, but this approach is probably simpler.)
    (f) $\frac{\mathrm{d}}{\mathrm{d}x}(2x+3)^3 = 3(2x+3)(2) = 12x+18$

11. $$\begin{aligned} 4\pi &= 6\theta\\ \theta &= \frac{2\pi}{3}\\ \text{polar coordinates}: &\quad \left(4\pi, \frac{2\pi}{3}\right)\\ x &= 4\pi\cos\frac{2\pi}{3}\\ &= -2\pi\\ y &= 4\pi\sin\frac{2\pi}{3}\\ &= 2\sqrt{3}\pi\\ \text{Cartesian coordinates}: &\quad (-2\pi, 2\sqrt{3}\pi)\end{aligned}$$

12. (a)
$$\begin{aligned} p(2\mathbf{i}+4\mathbf{j})+q(5\mathbf{i}-3\mathbf{j}) &= -9\mathbf{i}+21\mathbf{j}\\ 2p+5q &= -9\\ 4p-3q &= 21\\ -4p-10q &= 18\\ -13q &= 39\\ q &= -3\\ 2p+5(-3) &= -9\\ 2p &= 6\\ p &= 3\\ \therefore\quad -9\mathbf{i}+21\mathbf{j} &= 3\mathbf{a}-3\mathbf{b} \end{aligned}$$

(b)
$$\begin{aligned} p(2\mathbf{i}+4\mathbf{j})+q(5\mathbf{i}-3\mathbf{j}) &= 4\mathbf{i}-18\mathbf{j}\\ 2p+5q &= 4\\ 4p-3q &= -18\\ -4p-10q &= -8\\ -13q &= -26\\ q &= 2\\ 2p+5(2) &= 4\\ 2p &= -6\\ p &= -3\\ \therefore\quad 4\mathbf{i}-18\mathbf{j} &= -3\mathbf{a}+2\mathbf{b} \end{aligned}$$

(c)
$$\begin{aligned} p(2\mathbf{i}+4\mathbf{j})+q(5\mathbf{i}-3\mathbf{j}) &= -7\mathbf{i}+12\mathbf{j}\\ 2p+5q &= -7\\ 4p-3q &= 12\\ -4p-10q &= 14\\ -13q &= 26\\ q &= -2\\ 2p+5(-2) &= -7\\ 2p &= 3\\ p &= 1.5\\ \therefore\quad -7\mathbf{i}+12\mathbf{j} &= 1.5\mathbf{a}-2\mathbf{b} \end{aligned}$$

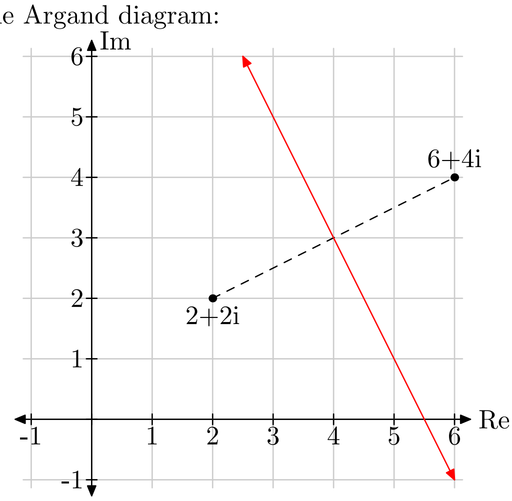
(d)
$$\begin{aligned} p(2\mathbf{i}+4\mathbf{j})+q(5\mathbf{i}-3\mathbf{j}) &= -34\mathbf{i}+23\mathbf{j}\\ 2p+5q &= -34\\ 4p-3q &= 23\\ -4p-10q &= 68\\ -13q &= 91\\ q &= -7\\ 2p+5(-7) &= -34\\ 2p &= 1\\ p &= 0.5\\ \therefore\quad -34\mathbf{i}+23\mathbf{j} &= 0.5\mathbf{a}-7\mathbf{b} \end{aligned}$$

13. $\sin^2 \mathrm{A}\cos \mathrm{A} = (1-\cos^2 \mathrm{A})\cos \mathrm{A} = \cos \mathrm{A} - \cos^3 \mathrm{A}$

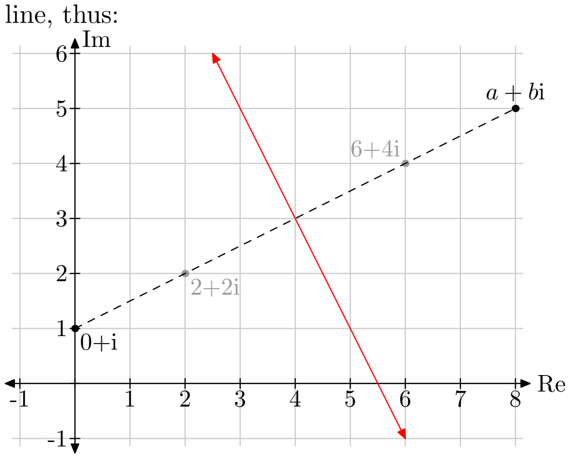
14.
$$\begin{aligned} \sin 2\theta\cos\theta &= (2\sin\theta\cos\theta)\cos\theta\\ &= 2\sin\theta\cos^2\theta\\ &= 2\sin\theta(1-\sin^2\theta)\\ &= 2\sin\theta-2\sin^3\theta \end{aligned}$$

15.
$$\begin{aligned} 2\cos^2 x &= 2-\sin x\\ 2(1-\sin^2 x) &= 2-\sin x\\ 2-2\sin^2 x &= 2-\sin x\\ -2\sin^2 x &= -\sin x \end{aligned}$$
$$\begin{aligned} 2\sin x &= 1 \qquad\text{or}\qquad \sin x = 0\\ \sin x &= 0.5 \qquad\qquad x\in\{0,\pi,2\pi\}\\ x &\in\left\{\frac{\pi}{6},\frac{5\pi}{6}\right\} \end{aligned}$$
Solutions are $0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, 2\pi$.

16. From the first set of points given we can obtain the Argand diagram:

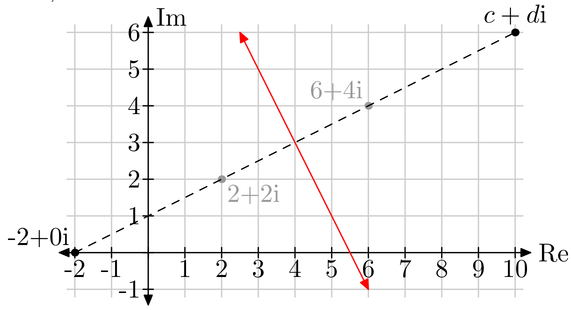


From the second set we see that the line is the set of points equidistant from $(0+\mathrm{i})$ and $(a+b\mathrm{i})$, hence $(a+b\mathrm{i})$ is the reflection of $(0+\mathrm{i})$ in the line, thus:



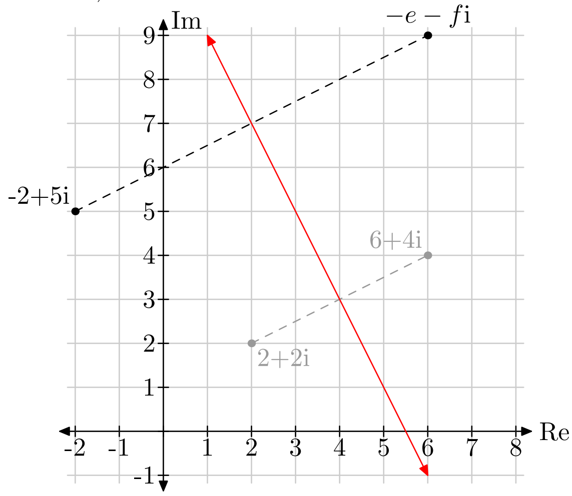
giving us $a=8$ and $b=5$.

From the third set we see that the line is the set of points equidistant from $(-2+0\mathrm{i})$ and $(c+d\mathrm{i})$, hence $(c+d\mathrm{i})$ is the reflection of $(-2+0\mathrm{i})$ in the line, thus:



giving us $c = 10$ and $d = 6$.

From the fourth set we see that the line is the set of points equidistant from $(-2+5\mathrm{i})$ and $(-e-f\mathrm{i})$, hence $(-e-f\mathrm{i})$ is the reflection of $(-2+5\mathrm{i})$ in the line, thus:



giving us $e = -6$ and $f = -9$.

17. $\begin{pmatrix}14\\7\end{pmatrix} + t\begin{pmatrix}-8\\6\end{pmatrix} = \begin{pmatrix}-4\\1\end{pmatrix} + t\begin{pmatrix}4\\10\end{pmatrix}$

$\mathbf{i}$ components:

$$\begin{aligned}14 - 8t &= -4 + 4t\\ 12t &= 18\\ t &= 1.5\end{aligned}$$

$\mathbf{j}$ components:

$$7 + 6(1.5) = 1 + 10(1.5)$$

Since the value of $t = 1.5$ satisfies both $\mathbf{i}$ and $\mathbf{j}$ components, a collision occurs at 12:30pm + 1.5 hours, i.e. at 2pm. The location of the collision is given by

$$\begin{aligned}\mathbf{r} &= \begin{pmatrix}14\\7\end{pmatrix} + 1.5\begin{pmatrix}-8\\6\end{pmatrix}\\ &= \begin{pmatrix}14\\7\end{pmatrix} + \begin{pmatrix}-12\\9\end{pmatrix}\\ &= \begin{pmatrix}2\\16\end{pmatrix}\text{km}\end{aligned}$$

18. (a) $$\begin{aligned}{}_{\mathrm{A}}\mathbf{v}_{\mathrm{B}} &= \mathbf{v}_{\mathrm{A}} - \mathbf{v}_{\mathrm{B}}\\ &= 13\mathbf{i} + 2\mathbf{j} - (5\mathbf{i} + 8\mathbf{j})\\ &= 8\mathbf{i} - 6\mathbf{j}\end{aligned}$$

$$\begin{aligned}\overrightarrow{\mathrm{BA}} &= \mathbf{r}_{\mathrm{A}} - \mathbf{r}_{\mathrm{B}}\\ &= (12\mathbf{i} + 15\mathbf{j}) - (19\mathbf{i} + 16\mathbf{j})\\ &= -7\mathbf{i} - \mathbf{j}\end{aligned}$$

$$\begin{aligned}\overrightarrow{\mathrm{BP}} &= \overrightarrow{\mathrm{BA}} + t\,{}_{\mathrm{A}}\mathbf{v}_{\mathrm{B}}\\ &= -7\mathbf{i} - \mathbf{j} + t(8\mathbf{i} - 6\mathbf{j})\\ &= (-7 + 8t)\mathbf{i} + (-1 - 6t)\mathbf{j}\end{aligned}$$

$$\begin{aligned}\overrightarrow{\mathrm{BP}} \cdot {}_{\mathrm{A}}\mathbf{v}_{\mathrm{B}} &= 0\\ ((-7 + 8t)\mathbf{i} + (-1 - 6t)\mathbf{j}) \cdot (8\mathbf{i} - 6\mathbf{j}) &= 0\\ 8(-7 + 8t) - 6(-1 - 6t) &= 0\\ -56 + 64t + 6 + 36t &= 0\\ 100t - 50 &= 0\\ t &= 0.5\end{aligned}$$

$$\begin{aligned}\left|\overrightarrow{\mathrm{BP}}\right| &= |(-7 + 8 \times 0.5)\mathbf{i} + (-1 - 6 \times 0.5)\mathbf{j}|\\ &= |-3\mathbf{i} - 4\mathbf{j}|\\ &= 5\text{km}\end{aligned}$$

The minimum distance is 0.5 km and occurs at 3:30pm.

(b) The position of the ships at time t is:

$$\begin{aligned}\mathbf{r}_{\mathrm{A}}(t) &= 12\mathbf{i} + 15\mathbf{j} + t(13\mathbf{i} + 2\mathbf{j})\\ &= (12 + 13t)\mathbf{i} + (15 + 2t)\mathbf{j}\\ \mathbf{r}_{\mathrm{B}}(t) &= 19\mathbf{i} + 16\mathbf{j} + t(5\mathbf{i} + 8\mathbf{j})\\ &= (19 + 5t)\mathbf{i} + (16 + 8t)\mathbf{j}\end{aligned}$$

The displacement $\overrightarrow{\mathrm{AB}}$ at time t is given by

$$\begin{aligned}\overrightarrow{\mathrm{AB}}(t) &= \mathbf{r}_{\mathrm{B}}(t) - \mathbf{r}_{\mathrm{A}}(t)\\ &= (19 + 5t)\mathbf{i} + (16 + 8t)\mathbf{j}\\ &\qquad - ((12 + 13t)\mathbf{i} + (15 + 2t)\mathbf{j})\\ &= (7 - 8t)\mathbf{i} + (1 + 6t)\mathbf{j}\\ d &= |\overrightarrow{\mathrm{AB}}|\\ d^2 &= (7 - 8t)^2 + (1 + 6t)^2\\ \frac{\mathrm{d}}{\mathrm{d}t}d^2 &= 2(7 - 8t)(-8) + 2(1 + 6t)(6)\\ &= -112 + 128t + 12 + 72t\\ &= 200t - 100\end{aligned}$$

At the turning point,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}d^2 &= 0\\ 200t - 100 &= 0\\ t &= 0.5\end{aligned}$$

$$\begin{aligned}d &= \sqrt{(7 - 8 \times 0.5)^2 + (1 + 6 \times 0.5)^2}\\ &= \sqrt{3^2 + 4^2}\\ &= 5\text{km}\end{aligned}$$

The minimum distance is 0.5 km and occurs at 3:30pm.