Chapter 1

Exercise 1A

1. (a) $$\begin{aligned}AB &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}\\ &= \sqrt{4^2 + 3^2 - 2\times 4\times 3\cos(70^\circ - 160^\circ)}\\ &= \sqrt{25 - 24\cos(-90^\circ)}\\ &= \sqrt{25 - 24\times 0}\\ &= \sqrt{25}\\ &= 5\end{aligned}$$

   If we had recognised before we began that A and B form a right angle we could have simplified this by using Pythagoras rather than the cosine rule.

   (b) $$\begin{aligned}AB &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}\\ &= \sqrt{4^2 + 4^2 - 2\times 4\times 4\cos(310^\circ - 10^\circ)}\\ &= \sqrt{32 - 32\cos(300^\circ)}\\ &= \sqrt{32 - 32\times\frac{1}{2}}\\ &= \sqrt{16}\\ &= 4\end{aligned}$$

   (c) $$\begin{aligned}AB &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}\\ &= \sqrt{2^2 + 7^2 - 2\times 2\times 7\cos(225^\circ - 75^\circ)}\\ &= \sqrt{53 - 28\cos(150^\circ)}\\ &= \sqrt{53 - 28\times\frac{-\sqrt{3}}{2}}\\ &= \sqrt{53 + 14\sqrt{3}}\end{aligned}$$

   (d) $$\begin{aligned}AB &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}\\ &= \sqrt{3^2 + 2^2 - 2\times 3\times 2\cos(165^\circ - -150^\circ)}\\ &= \sqrt{13 - 12\cos(315^\circ)}\\ &= \sqrt{13 - 12\frac{\sqrt{2}}{2}}\\ &= \sqrt{13 - 6\sqrt{2}}\end{aligned}$$

2. (a) If we recognise that $\theta_1 - \theta_2$ is a right angle there is no need to use the cosine rule as Pythagoras will do.

   $$\begin{aligned}PQ &= \sqrt{r_1^2 + r_2^2}\\ &= \sqrt{5^2 + 12^2}\\ &= 13\end{aligned}$$

   (b) Here $\theta_2 - \theta_1 = \frac{3\pi}{4} - -\frac{3\pi}{4} = \frac{3\pi}{2}$ which also represents a right angle, so we can use Pythagoras again.

   $$\begin{aligned}PQ &= \sqrt{r_1^2 + r_2^2}\\ &= \sqrt{3^2 + 2^2}\\ &= \sqrt{13}\end{aligned}$$

   (c) $$\begin{aligned}PQ &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}\\ &= \sqrt{4^2 + 3^2 - 2\times 4\times 3\cos(\frac{\pi}{12} - -\frac{7\pi}{12})}\\ &= \sqrt{25 - 24\cos(\frac{8\pi}{12})}\\ &= \sqrt{25 - 24\cos(\frac{2\pi}{3})}\\ &= \sqrt{25 - 24\times -\frac{1}{2}}\\ &= \sqrt{25 + 12}\\ &= \sqrt{37}\end{aligned}$$

   (d) $$\begin{aligned}PQ &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}\\ &= \sqrt{3^2 + 4^2 - 2\times 3\times 4\cos(\frac{11\pi}{10} - \frac{\pi}{10})}\\ &= \sqrt{25 - 24\cos(\frac{10\pi}{10})}\\ &= \sqrt{25 - 24\cos(\pi)}\\ &= \sqrt{25 - 24\times -1}\\ &= \sqrt{25 + 24}\\ &= \sqrt{49}\\ &= 7\end{aligned}$$

   If we had recognised before we began that P and Q formed a straight angle we could simply have done

   $$\begin{aligned}PQ &= r_1 + r_2\\ &= 3 + 4\\ &= 7\end{aligned}$$

3. $$\begin{aligned}AB &= \sqrt{r_A^2 + r_B^2 - 2r_Ar_B\cos(\theta_B - \theta_A)}\\ &= \sqrt{2^2 + 3^2 - 2\times 2\times 3\cos(\frac{2\pi}{3} - \frac{\pi}{3})}\\ &= \sqrt{13 - 12\cos(\frac{\pi}{3})}\\ &= \sqrt{13 - 12\times\frac{1}{2}}\\ &= \sqrt{7}\end{aligned}$$

   $$\begin{aligned}AC &= \sqrt{r_A^2 + r_C^2 - 2r_Ar_C\cos(\theta_A - \theta_C)}\\ &= \sqrt{2^2 + (3\sqrt{2})^2 - 2\times 2\times 3\sqrt{2}\cos(\frac{\pi}{3} - \frac{\pi}{12})}\\ &= \sqrt{22 - 12\sqrt{2}\cos(\frac{\pi}{4})}\\ &= \sqrt{22 - 12\sqrt{2}\times\frac{1}{\sqrt{2}}}\\ &= \sqrt{10}\end{aligned}$$

   AC exceeds AB by $\sqrt{10} - \sqrt{7}$.

4. (a) Cartesian coordinates of A:

$$(4\cos 10°, 4\sin 10°)$$

Cartesian coordinates of B:

$$(3\cos 130°, 3\sin 130°)$$

$$\begin{aligned}
\text{AB} &= \sqrt{\begin{matrix}(4\cos 10° - 3\cos 130°)^2 \\ +(4\sin 10° - 3\sin 130°)^2\end{matrix}} \\
&= \sqrt{\begin{matrix}16\cos^2 10° - 24\cos 10°\cos 130° + 9\cos^2 130° \\ +16\sin^2 10° - 24\sin 10°\sin 130° + 9\sin^2 130°\end{matrix}} \\
&= \sqrt{16 - 24(\cos 10°\cos 130° + \sin 10°\sin 130°) + 9} \\
&= \sqrt{25 - 24\cos(10° - 130°)} \\
&= \sqrt{25 - 24\cos(-120°)} \\
&= \sqrt{25 - 24 \times -\frac{1}{2}} \\
&= \sqrt{25 + 12} \\
&= \sqrt{37}
\end{aligned}$$

(b)

$$\begin{aligned}
\text{AB} &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)} \\
&= \sqrt{4^2 + 3^2 - 2 \times 4 \times 3\cos(130° - 10°)} \\
&= \sqrt{25 - 24\cos(120°)} \\
&= \sqrt{25 - 24 \times -\frac{1}{2}} \\
&= \sqrt{25 + 12} \\
&= \sqrt{37}
\end{aligned}$$

5. (a) Cartesian coordinates of P:

$$(3\sqrt{2}\cos\frac{4\pi}{5}, 3\sqrt{2}\sin\frac{4\pi}{5})$$

Cartesian coordinates of Q:

$$\begin{aligned}
&(4\cos -\frac{19\pi}{20}, 4\sin -\frac{19\pi}{20}) \\
=&(4\cos\frac{19\pi}{20}, -4\sin\frac{19\pi}{20})
\end{aligned}$$

$$\begin{aligned}
\text{PQ} &= \sqrt{\begin{matrix}(3\sqrt{2}\cos\frac{4\pi}{5} - 4\cos\frac{19\pi}{20})^2 \\ +(3\sqrt{2}\sin\frac{4\pi}{5} - -4\sin\frac{19\pi}{20})^2\end{matrix}} \\
&= \sqrt{\begin{matrix}(3\sqrt{2}\cos\frac{4\pi}{5} - 4\cos\frac{19\pi}{20})^2 \\ +(3\sqrt{2}\sin\frac{4\pi}{5} + 4\sin\frac{19\pi}{20})^2\end{matrix}} \\
&= \sqrt{\begin{matrix}18\cos^2\frac{4\pi}{5} - 24\sqrt{2}\cos\frac{4\pi}{5}\cos\frac{19\pi}{20} + 16\cos^2\frac{19\pi}{20} \\ +18\sin^2\frac{4\pi}{5} + 24\sqrt{2}\sin\frac{4\pi}{5}\sin\frac{19\pi}{20} + 16\sin^2\frac{19\pi}{20}\end{matrix}} \\
&= \sqrt{18 - 24\sqrt{2}(\cos\frac{4\pi}{5}\cos\frac{19\pi}{20} - \sin\frac{4\pi}{5}\sin\frac{19\pi}{20}) + 16} \\
&= \sqrt{34 - 24\sqrt{2}\cos(\frac{4\pi}{5} + \frac{19\pi}{20})} \\
&= \sqrt{34 - 24\sqrt{2}\cos(\frac{16\pi}{20} + \frac{19\pi}{20})} \\
&= \sqrt{34 - 24\sqrt{2}\cos(\frac{35\pi}{20})} \\
&= \sqrt{34 - 24\sqrt{2}\cos(\frac{7\pi}{4})} \\
&= \sqrt{34 - 24\sqrt{2} \times \frac{1}{\sqrt{2}}} \\
&= \sqrt{34 - 24} \\
&= \sqrt{10}
\end{aligned}$$

(b)

$$\begin{aligned}
\text{PQ} &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)} \\
&= \sqrt{(3\sqrt{2})^2 + 4^2 - 2 \times 3\sqrt{2} \times 4\cos(\frac{4\pi}{5} - -\frac{19\pi}{20})} \\
&= \sqrt{18 + 16 - 24\sqrt{2}\cos(\frac{16\pi}{20} + \frac{19\pi}{20})} \\
&= \sqrt{34 - 24\sqrt{2}\cos(\frac{35\pi}{20})} \\
&= \sqrt{34 - 24\sqrt{2}\cos(\frac{7\pi}{4})} \\
&= \sqrt{34 - 24\sqrt{2} \times \frac{1}{\sqrt{2}}} \\
&= \sqrt{34 - 24} \\
&= \sqrt{10}
\end{aligned}$$

Exercise 1B

Questions 1 to 11 need no working. Refer to the solutions in Sadler.

12. (a) For the given point $(0, \frac{3\pi}{2})$,

$$\begin{aligned}
r &= \frac{3\pi}{2} \\
\theta &= \frac{\pi}{2} \\
\therefore \quad r &= 3\theta \\
k &= 3
\end{aligned}$$

For point A

$$\begin{aligned}
\theta &= \pi \\
r &= 3\theta \\
&= 3\pi \\
x &= -3\pi \\
y &= 0
\end{aligned}$$

Point A has Cartesian coordinates $(-3\pi, 0)$.

(b) For the given point $(-5\pi, 0)$,

$$\begin{aligned} r &= 5\pi \\ \theta &= \pi \\ \therefore \quad r &= 5\theta \\ k &= 5 \end{aligned}$$

For point B

$$\begin{aligned} \theta &= \frac{\pi}{2} \\ r &= 5\theta \\ &= \frac{5\pi}{2} \\ x &= 0 \\ y &= \frac{5\pi}{2} \end{aligned}$$

Point B has Cartesian coordinates $(0, \frac{5\pi}{2})$.

(c) For the given point $(0, 3)$,

$$\begin{aligned} r &= 3 \\ \theta &= \frac{\pi}{2} \\ \therefore \quad r &= \frac{3}{\pi/2}\theta \\ &= \frac{6}{\pi}\theta \\ k &= \frac{6}{\pi} \end{aligned}$$

For point C

$$\begin{aligned} \theta &= \pi \\ r &= \frac{6}{\pi}\theta \\ &= 6 \\ x &= -6 \\ y &= 0 \end{aligned}$$

Point C has Cartesian coordinates $(-6, 0)$.

(d) For the given point $(-10, 0)$,

$$\begin{aligned} r &= 10 \\ \theta &= \pi \\ \therefore \quad r &= \frac{10}{\pi}\theta \\ k &= \frac{10}{\pi} \end{aligned}$$

For point D

$$\begin{aligned} \theta &= \frac{\pi}{2} \\ r &= \frac{10}{\pi}\theta \\ &= 5 \\ x &= 0 \\ y &= 5 \end{aligned}$$

Point D has Cartesian coordinates $(0, 5)$.

Note regarding the extension work: these polar graphs are collectively called petal plots (for obvious reasons).

Miscellaneous Exercise 1

1–3 No working required.

4. (a) $2\mathbf{a} = 6\mathbf{i} - 2\mathbf{j}$

(b) $$\begin{aligned} \frac{|\mathbf{a}|}{|\mathbf{b}|}\mathbf{b} &= \frac{\sqrt{10}}{\sqrt{20}}(2\mathbf{i} + 4\mathbf{j}) \\ &= \frac{2}{\sqrt{2}}(\mathbf{i} + 2\mathbf{j}) \\ &= \sqrt{2}(\mathbf{i} + 2\mathbf{j}) \end{aligned}$$

(c) $$\begin{aligned} |d\mathbf{i} - 9\mathbf{j}| &= |3\mathbf{a}| \\ d^2 + 9^2 &= 9^2 + 3^2 \\ d &= \pm 3 \end{aligned}$$

(d) $\mathbf{a} \cdot \mathbf{b} = 3 \times 2 + -1 \times 4 = 2$

(e) $$\begin{aligned} \cos\theta &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \\ &= \frac{2}{\sqrt{10}\sqrt{20}} \\ &= \frac{\sqrt{2}}{10} \\ \theta &= \cos^{-1}\frac{\sqrt{2}}{10} \\ &\approx 82^\circ \end{aligned}$$

5.
$$\begin{aligned}\mathbf{p}\cdot\mathbf{q} &= 0\\ a(\mathbf{i}+\mathbf{j})\cdot(2\mathbf{i}-b\mathbf{j}) &= 0\\ 2a-ab &= 0\\ b &= 2\end{aligned}$$
(reject $a=0$ since $\mathbf{p}$ is non zero.)

$$\begin{aligned}a\sqrt{1^2+1^2} &= \sqrt{2^2+(-b)^2}\\ 2a^2 &= 4+b^2\\ &= 4+2^2\\ a^2 &= 4\\ a &= \pm 2\end{aligned}$$

$$\begin{aligned}(2\mathbf{i}-b\mathbf{j})-3(c\mathbf{i}+d\mathbf{j}) &= 23\mathbf{i}-5\mathbf{j}\\ (2-3c)\mathbf{i}+(-2-3d)\mathbf{j} &= 23\mathbf{i}-5\mathbf{j}\\ -3c\mathbf{i}-3d\mathbf{j} &= 21\mathbf{i}-3\mathbf{j}\\ c &= -7\\ d &= 1\end{aligned}$$

$$\begin{aligned}e\mathbf{i}+f\mathbf{j} &= k(2\mathbf{i}-2\mathbf{j})\\ f &= -e\end{aligned}$$

$$\begin{aligned}\sqrt{e^2+f^2} &= \sqrt{c^2+d^2}\\ 2e^2 &= c^2+d^2\\ &= (-7)^2+1^2\\ &= 50\\ e^2 &= 5\\ e &= 5\\ f &= -5\end{aligned}$$
(rejecting $e=-5, f=5$ because $\mathbf{s}$ must be in the same direction as $\mathbf{q}$ which has a positive $\mathbf{i}$ component.)

6. (a) $\text{OP} = \sqrt{7^2+(-24)^2}$
$= 25\,\text{m}$

(b) $3|(-5\mathbf{i}+12\mathbf{j})| = 3\sqrt{(-5)^2+12^2}$
$= 39\,\text{m}$

(c) $7\mathbf{i}-24\mathbf{j}+3(-5\mathbf{i}+12\mathbf{j}) = (-8\mathbf{i}+12\mathbf{j})$ m

(d) $|(-8\mathbf{i}+12\mathbf{j})| = \sqrt{(-8)^2+12^2}$
$= 4\sqrt{2^2+3^2}$
$= 4\sqrt{13}\,\text{m}$

7.
$$\begin{aligned}\overrightarrow{\text{OP}} &= \overrightarrow{\text{OA}}+\frac{2}{5}\overrightarrow{\text{AB}}\\ &= \overrightarrow{\text{OA}}+\frac{2}{5}(\overrightarrow{\text{OB}}-\overrightarrow{\text{OA}})\\ &= \frac{3}{5}\overrightarrow{\text{OA}}+\frac{2}{5}\overrightarrow{\text{OB}}\\ &= \frac{3}{5}(-7\mathbf{i}+7\mathbf{j})+\frac{2}{5}(8\mathbf{i}+2\mathbf{j})\\ &= \frac{-21+16}{5}\mathbf{i}+\frac{21+4}{5}\mathbf{j}\\ &= -\mathbf{i}+5\mathbf{j}\end{aligned}$$

8.
$$\begin{aligned}2z-\bar{z} &= -5+6\text{i}\\ 2(a+b\text{i})-(a-b\text{i}) &= -5+6\text{i}\\ a+3b\text{i} &= -5+6\text{i}\\ a &= -5\\ b &= 2\end{aligned}$$

9.
$$\begin{aligned}\sin 75^\circ &= \sin(45^\circ+30^\circ)\\ &= \sin 45^\circ\cos 30^\circ+\cos 45^\circ\sin 30^\circ\\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\frac{1}{2}\\ &= \frac{\sqrt{2}\left(\sqrt{3}+1\right)}{4}\end{aligned}$$

10.
$$\begin{aligned}(a+b\text{i})(-1+2\text{i}) &= 1+2\text{i}\\ -a+2a\text{i}-b\text{i}-2b &= 1+2\text{i}\\ (-a-2b)+(2a-b)\text{i} &= 1+2\text{i}\end{aligned}$$
This gives the following pair of equations by equating real and imaginary components:

$$\begin{aligned}-a-2b &= 1\\ 2a-b &= 2\end{aligned}$$

Solving simultaneously gives

$$\begin{aligned}a &= \frac{3}{5}\\ b &= -\frac{4}{5}\\ \therefore\qquad z &= \frac{3}{5}-\frac{4}{5}\text{i}\end{aligned}$$

11.
$$\begin{aligned}(a+b\text{i})^2 &= -5+12\text{i}\\ a^2-b^2+2ab\text{i} &= -5+12\text{i}\\ a^2-b^2 &= -5\\ 2ab &= 12\\ b &= \frac{6}{a}\\ a^2-\left(\frac{6}{a}\right)^2 &= -5\\ a^2-\frac{36}{a^2} &= -5\\ a^4+5a^2-36 &= 0\\ (a^2-4)(a^2+9) &= 0\\ a &= \pm 2\\ b &= \pm 3\\ z &= \pm(2+3\text{i})\end{aligned}$$
(Note that we disregard $(a^2+9)=0$ as leading to any solutions because a must be real.)

12.
$$\begin{aligned}(a+b\mathrm{i})^2 &= 5+12\mathrm{i}\\ a^2-b^2+2ab\mathrm{i} &= 5+12\mathrm{i}\\ a^2-b^2 &= 5\\ 2ab &= 12\\ b &= \frac{6}{a}\\ a^2-\left(\frac{6}{a}\right)^2 &= 5\\ a^2-\frac{36}{a^2} &= 5\\ a^4-5a^2-36 &= 0\\ (a^2+4)(a^2-9) &= 0\\ a &= \pm 3\\ b &= \pm 2\\ \therefore\quad \sqrt{5+12\mathrm{i}} &= \pm(3+2\mathrm{i})\end{aligned}$$

(Note that the square root of a real number is defined uniquely to mean the positive square root. No such unique definition exists for the square root of complex numbers.)

13.
$$\begin{aligned}2\sin\left(x+\frac{\pi}{6}\right) &= \sqrt{2}\\ \sin\left(x+\frac{\pi}{6}\right) &= \frac{\sqrt{2}}{2}\\ x+\frac{\pi}{6} &= \frac{\pi}{4},\frac{3\pi}{4},\frac{9\pi}{4},\frac{11\pi}{4}\\ x &= \frac{\pi}{12},\frac{7\pi}{12},\frac{25\pi}{12},\frac{31\pi}{12}\end{aligned}$$

14.
$$y' = 2x-3$$

when $x=-1$

$$\begin{aligned}y' &= 2(-1)-3\\ &= -5\end{aligned}$$

The equation of the tangent line is

$$\begin{aligned}y-y_0 &= m(x-x_0)\\ y-9 &= -5(x--1)\\ y &= -5x-5+9\\ y &= -5x+4\end{aligned}$$

15.
$$\begin{aligned}y' &= 2(3x^2-x+1)+(6x-1)(2x-3)\\ &= 6x^2-2x+2+12x^2-18x-2x+3\\ &= 18x^2-22x+5\text{when } x=2\\ y' &= 18(2)^2-22(2)+5\\ &= 33\end{aligned}$$

The equation of the tangent line is

$$\begin{aligned}y-y_0 &= m(x-x_0)\\ y-11 &= 33(x-2)\\ y &= 33x-66+11\\ y &= 33x-55\end{aligned}$$

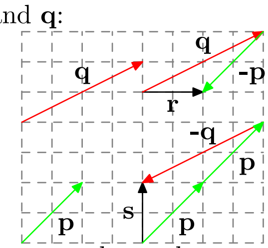
16.
$$\begin{aligned}4\theta &= 3\pi\\ \theta &= \frac{3\pi}{4}\end{aligned}$$

Polar coordinates: $\left(3\pi,\angle\frac{3\pi}{4}\right)$. (Assuming that we restrict r and θ to positive values as usual.)

The Cartesian coordinates of this point are

$$\left(3\pi\cos\frac{3\pi}{4},3\pi\sin\frac{3\pi}{4}\right)=\left(-\frac{3\sqrt{2}\pi}{2},\frac{3\sqrt{2}\pi}{2}\right)$$

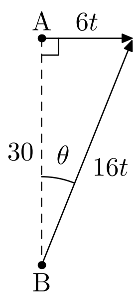
17. It may be simplest to first express **r** and **s** in terms of **p** and **q**:



then use these to work out the rest of the vectors:

$$\begin{aligned}\mathbf{t} &= \mathbf{r}+2\mathbf{s}\\ &= -\mathbf{p}+\mathbf{q}+2(2\mathbf{p}-\mathbf{q})\\ &= 3\mathbf{p}-\mathbf{q}\\ \mathbf{u} &= -\mathbf{t}\\ &= -3\mathbf{p}+\mathbf{q}\\ \mathbf{v} &= 2.5\mathbf{r}+2\mathbf{s}\\ &= 2.5(-\mathbf{p}+\mathbf{q})+2(2\mathbf{p}-\mathbf{q})\\ &= 1.5\mathbf{p}+.5\mathbf{q}\\ \mathbf{w} &= 3.5\mathbf{r}+\mathbf{s}\\ &= 3.5(-\mathbf{p}+\mathbf{q})+2\mathbf{p}-\mathbf{q}\\ &= -1.5\mathbf{p}+2.5\mathbf{q}\\ \mathbf{x} &= 2\mathbf{r}+0.5\mathbf{s}\\ &= 2(-\mathbf{p}+\mathbf{q})+0.5(2\mathbf{p}-\mathbf{q})\\ &= -\mathbf{p}+1.5\mathbf{q}\\ \mathbf{y} &= \mathbf{r}-\mathbf{s}\\ &= -\mathbf{p}+\mathbf{q}-(2\mathbf{p}-\mathbf{q})\\ &= -3\mathbf{p}+2\mathbf{q}\end{aligned}$$

18.



$$\begin{aligned}\sin\theta &= \frac{6t}{16t}\\ \theta &= \sin^{-1}\frac{3}{8}\\ &= 22°\end{aligned}$$