Chapter 6

Exercise 6A

1. Vertical asymptote when the denominator is zero: $x = 0$.
2. Vertical asymptote when the denominator is zero: $x = 1$.
3. Vertical asymptote when the denominator is zero: $x = 3$ or $x = \frac{1}{2}$.
4. Vertical asymptote when the denominator is zero: $x = 3$.
5. x^2 is always non-negative (for x real) so $y \not< 0$.
6. The square root is non-negative. The function is defined for all x such that $x - 3$ is non-negative; this has no upper limit so there is no upper limit on y: $y \not< 0$.
7. $y \neq 0$
8. $\frac{3}{x} \neq 0$ so $y = 2 + \frac{3}{x} \neq 2$
9. $y \neq 0$
10. As x increases without bound, or decreases without bound, y approaches 1, but since the numerator and denominator can never be equal, $y \neq 1$.
11. As $x \to +\infty$ the x^2 term dominates, so $y \to +\infty$.
    As $x \to -\infty$ the x^2 term dominates, so $y \to +\infty$.
12. As $x \to +\infty$ the $-2x^3$ term dominates, so $y \to -\infty$.
    As $x \to -\infty$ the $-2x^3$ term dominates, so $y \to +\infty$.
13. As $x \to +\infty$ the x term in the denominator dominates, so $y \to 0$.
    As $x \to -\infty$ the x term in the denominator dominates, so $y \to 0$.
14. As $x \to +\infty$ the x term in the numerator and denominator dominates, so $y \to 1$.
    As $x \to -\infty$ the x term in the numerator and denominator dominates, so $y \to 1$.
15. As $x \to +\infty$ the $5x^2$ term in the numerator and the x^2 term in the denominator dominates, so $y \to 5$.
    As $x \to -\infty$ the $5x^2$ term in the numerator and the x^2 term in the denominator dominates, so $y \to 5$.
16. As $x \to +\infty$ the $3x^2$ term in the numerator and the x^2 term in the denominator dominates, so $y \to 3$.
    As $x \to -\infty$ the $3x^2$ term in the numerator and the x^2 term in the denominator dominates, so $y \to 3$.
17. For $x > 0$, $\frac{1}{x} > 0$ so as $x \to 0^+$ then $y \to +\infty$.
    For $x < 0$, $\frac{1}{x} < 0$ so as $x \to 0^-$ then $y \to -\infty$.
18. For $x > 3$, $x - 3 > 0$ so as $x \to 3^+$ then $y \to +\infty$.
    For $x < 3$, $x - 3 < 0$ so as $x \to 3^-$ then $y \to -\infty$.
19. For $x > 1$, $1 - x < 0$ so as $x \to 1^+$ then $y \to -\infty$.
    For $x < 1$, $1 - x > 0$ so as $x \to 1^-$ then $y \to +\infty$.
20. As $x \to 0$ the $\frac{1}{x^2}$ term dominates.
    For $x > 0$, $\frac{1}{x^2} > 0$ so as $x \to 0^+$ then $y \to +\infty$.
    For $x < 0$, $\frac{1}{x^2} > 0$ so as $x \to 0^-$ then $y \to +\infty$.
21. (a) Key features: $y \to +\infty$ as $x \to 3$ from above or from below. We expect a denominator that goes to 0 only when $x \to 3$ but that is positive either side of $x = 3$ so the denominator must be raised to an even power. The only equation that matches is $y = \frac{1}{(x-3)^2}$.
    (b) Key features: y goes to infinity as x approaches 3 or -3. For $x > 3$, $y > 0$. The equation that matches is $y = \frac{1}{(x+3)(x-3)}$.
    (c) Key features: $y \to +\infty$ as $x \to 3^+$ and $y \to -\infty$ as $x \to 3^-$. We expect a denominator that goes to 0 only when $x \to 3$ and the expression is positive for $x > 3$. The only equation that matches is $y = \frac{1}{(x-3)}$.

Exercise 6B

1. (a) Third piece: $f(5) = 3 \times 5 = 15$
   (b) First piece: $f(-2) = -2$
   (c) Second piece: $f(3) = 3^2 = 9$
   (d) First piece: $f(-4) = -4$
   (e) Second piece: $f(2.5) = (2.5)^2 = 6.25$
2. (a) Third piece: $f(3) = 2 \times 3 = 6$
   (b) First piece: $f(0) = 0 + 1 = 1$
   (c) Second piece: $f(1) = 3$
   (d) First piece: $f(-4) = -4 + 1 = -3$
   (e) Third piece: $f(3.5) = 2(3.5) = 7$

3. (a) Not a function for domain $\Re$ because f(0) has two values ($2(0) + 3 = 3$ and 1) where the two pieces overlap, so it fails the vertical line test at $x = 0$.
   (b) Not a function for domain $\Re$ because f(x) is not defined for $x < -3$.
   (c) Not a function for domain $\Re$ because f(x) is not defined for $7 < x < 8$. (The first and second pieces overlap at $x = 4$, but this is not a problem because both pieces give f(4) = 8.)
   (d) Not a function because f(5) = ±2: it fails the vertical line test at $x = 5$.

4. The key point for $|x - 1|$ is at $x = 1$. This gives us

$$f(x) = \begin{cases} -(x-1) & \text{for } x < 1 \\ x - 1 & \text{for } x \geq 1 \end{cases}$$

that simplifies to

$$f(x) = \begin{cases} 1 - x & \text{for } x < 1 \\ x - 1 & \text{for } x \geq 1 \end{cases}$$

5. The key point for $|2x - 5|$ is at $x = 2.5$. This gives us

$$f(x) = \begin{cases} -(2x-5) & \text{for } x < 2.5 \\ 2x - 5 & \text{for } x \geq 2.5 \end{cases}$$

that simplifies to

$$f(x) = \begin{cases} 5 - 2x & \text{for } x < 2.5 \\ 2x - 5 & \text{for } x \geq 2.5 \end{cases}$$

6. Working left to right, the first part (for $x < -2$) has equation $y = x + 4$. The second part (for $-2 \leq x < 2$) has equation $y = x^2 - 2$. The third part (for $x \geq 2$) has equation $y = -x + 4$. This gives

$$f(x) = \begin{cases} x + 4 & \text{for } x < -2 \\ x^2 - 2 & \text{for } -2 \leq x < 2 \\ -x + 4 & \text{for } x \geq 2 \end{cases}$$

7. Working left to right, the first part (for $x < -2$) has equation $y = 4$. The second part (for $-2 \leq x < 1$) has equation $y = x^2$. The third part (for $x \geq 1$) has equation $y = -x + 2$. This gives

$$f(x) = \begin{cases} 4 & \text{for } x < -2 \\ x^2 & \text{for } -2 \leq x < 1 \\ -x + 2 & \text{for } x \geq 1 \end{cases}$$

8. Working left to right, the first part (for $x \leq -2$) has equation $y = 2$. The second part (for $-2 < x < 2$) has equation $y = x$. The third part (for $x = 2$) has equation $y = 1$. The fourth part (for $x > 2$) has equation $y = 0.5x - 1$. This gives

$$f(x) = \begin{cases} 2 & \text{for } x \leq -2 \\ x & \text{for } -2 < x < 2 \\ 1 & \text{for } x = 2 \\ 0.5x - 1 & \text{for } x > 2 \end{cases}$$

9. Refer to solutions in Sadler.

10. Refer to solutions in Sadler.

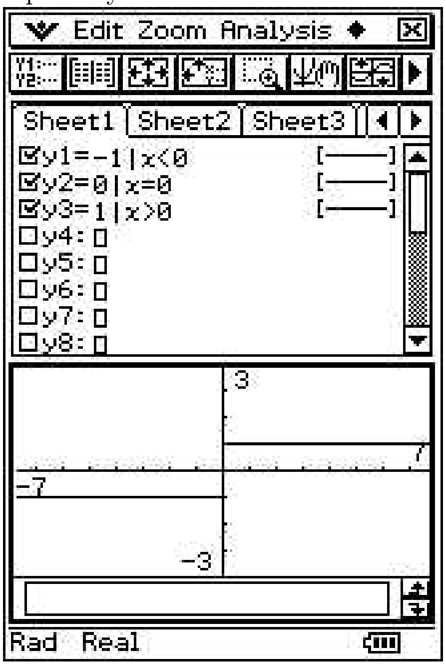
11. (a)

$$\text{sgn}(x) = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$$

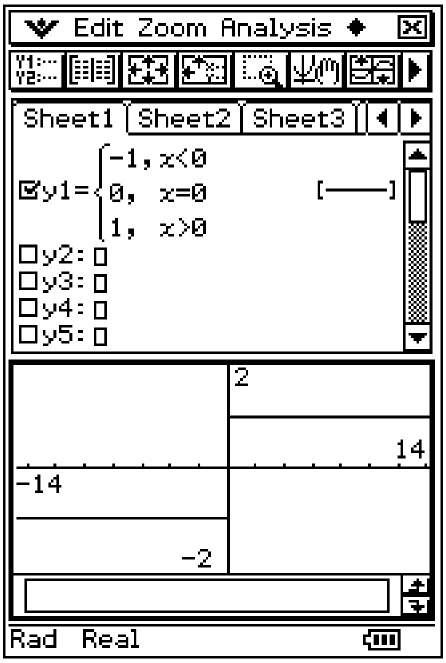
   (b) Refer to solutions in Sadler.

   (c) This can be done several ways. The classpad has a "signum" function that is the same as sgn(x). It may we worth showing here how to graph this in a way that reflects the piecewise definition.

   Graphing a piecewise-defined function on the ClassPad involves entering the pieces separately as illustrated below.



   If you have at least version 3.0.4 of the ClassPad operating system, you can also enter the piecewise-defined function directly:



12. (a) See the solution in Sadler.

    (b) The ClassPad has an int function.

    (c)

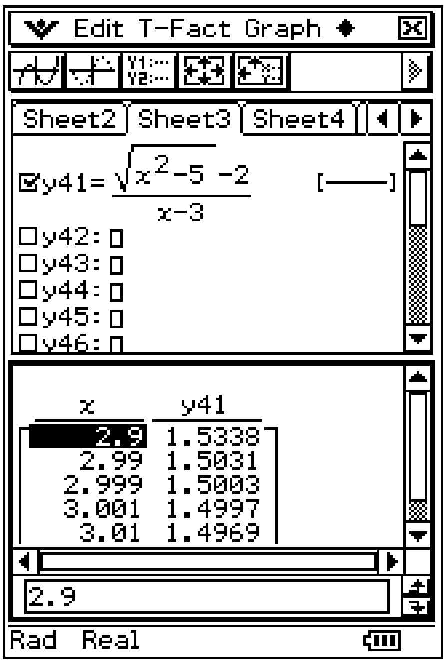
$$\text{int}(x) = \begin{cases} -1 & \text{for } -1 \le x < 0 \\ 0 & \text{for } 0 \le x < 1 \\ 1 & \text{for } 1 \le x < 2 \\ 2 & \text{for } x = 2 \end{cases}$$

In fact with a bit of ingenuity we can also come up with a recursive definition that works for a domain of $x \in \Re$:

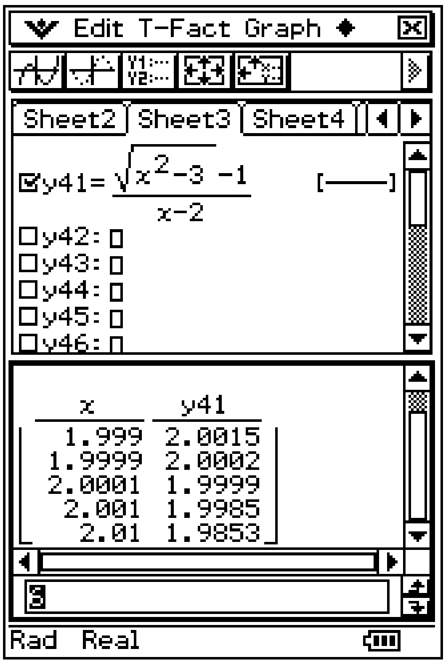
$$\text{int}(x) = \begin{cases} \text{int}(x+1) - 1 & \text{for } x < 0 \\ 0 & \text{for } 0 \le x < 1 \\ \text{int}(x-1) + 1 & \text{for } x \ge 0 \end{cases}$$

Try to understand how this works by using the definition to find, for instance, int(2.3).

Exercise 6C

1. 

The limit appears to be 1.5.

2. 

The limit appears to be 2.

3. $f(x)$ is continuous at a so
$$\lim_{x\to a} f(x) = f(a) = 10$$

4. $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = 10$ so
$$\lim_{x\to a} f(x) = 10$$
(The actual value of $f(a)$ is of no relevance.)

5. $\lim_{x\to a^-} f(x) = 10$; $\lim_{x\to a^+} f(x) = 5$ The limits from above and below are not equal so $\lim_{x\to a} f(x)$ does not exist.

6. $\lim_{x\to a^-} f(x) = 10$; $\lim_{x\to a^+} f(x) = 0$. The limits from above and below are not equal so $\lim_{x\to a} f(x)$ does not exist. (The actual value of $f(a)$ is of no relevance.)

7. $f(x)$ is continuous at a so
$$\lim_{x\to a} f(x) = f(a) = 10$$

8. $f(x)$ is continuous at a so
$$\lim_{x\to a} f(x) = f(a) = 5$$

9. $\lim_{x\to a^-} f(x) = 5$; $\lim_{x\to a^+} f(x) = 10$. The limits from above and below are not equal so $\lim_{x\to a} f(x)$ does not exist.

10. $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = 10$ so
$$\lim_{x\to a} f(x) = 10$$
(It makes no difference that $f(x)$ is not defined at $x = a$.)

11. $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = 10$ so
$$\lim_{x\to a} f(x) = 10$$
(The discontinuity does not matter here because we only care about what happens in the vicinity of $x = a$.)

12. $\lim_{x\to a^-} f(x) = -\infty$; $\lim_{x\to a^+} f(x) = +\infty$. The limits from above and below do not exist (differently) so $\lim_{x\to a} f(x)$ does not exist.

13. $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = 2$ so
$$\lim_{x\to a} f(x) = 2$$
(The discontinuity does not matter here because we only care about what happens in the vicinity of $x = a$.)

14. $\lim_{x\to a^-} f(x) = +\infty$; $\lim_{x\to a^+} f(x) = +\infty$. The limits from above and below do not exist so $\lim_{x\to a} f(x)$ does not exist.

15. $3x + 5$ is continuous everywhere, so
$$\lim_{x\to 1}(3x+5) = 3\times 1 + 5 = 8$$

16. $2x^2 + x + 3$ is continuous everywhere, so
$$\lim_{x\to 1}(2x^2 + x + 1) = 2(1)^2 + 1 + 3 = 6$$

17. $\frac{5}{x-2}$ has a discontinuity at $x = 2$ but is continuous at $x = 4$ so
$$\lim_{x\to 4} \frac{5}{x-2} = \frac{5}{4-2} = 2.5$$

18. $\frac{x+3}{x+2}$ has a discontinuity at $x = -2$ but is continuous at $x = 2$ so
$$\lim_{x\to 4} \frac{x+3}{x+2} = \frac{2+3}{2+2} = 1.25$$

19. $\frac{4x^2-36}{x-3}$ is not defined at $x = 3$ and gives $\frac{0}{0}$ so it needs further investigation. We can factorise and simplify to obtain a function that is identical except for being defined at $x = 3$ thus:
$$\begin{aligned} \frac{4x^2-36}{x-3} &= \frac{4(x^2-9)}{x-3} \\ &= \frac{4(x+3)(x-3)}{x-3} \\ &= 4(x+3) \qquad (x \neq 3) \\ \therefore \quad \lim_{x\to 3} \frac{4x^2-36}{x-3} &= 4(3+3) \\ &= 24 \end{aligned}$$

20. $\frac{x^2+3x-10}{x-2}$ is not defined at $x = 2$ and gives $\frac{0}{0}$ so it needs further investigation. We can factorise and simplify to obtain a function that is identical except for being defined at $x = 2$ thus:
$$\begin{aligned} \frac{x^2+3x-10}{x-2} &= \frac{(x+5)(x-2)}{x-2} \\ &= (x+5) \qquad (x \neq 2) \\ \therefore \quad \lim_{x\to 2} \frac{x^2+3x-10}{x-2} &= 2+5 \\ &= 7 \end{aligned}$$

21. $\frac{x^2-3x}{x-3}$ is not defined at $x = 3$ and gives $\frac{0}{0}$ so it needs further investigation. We can factorise and simplify to obtain a function that is identical except for being defined at $x = 3$ thus:
$$\begin{aligned} \frac{x^2-3x}{x-3} &= \frac{x(x-3)}{x-3} \\ &= x \qquad (x \neq 3) \\ \therefore \quad \lim_{x\to 3} \frac{x^2-3x}{x-3} &= 3 \end{aligned}$$

22. $\frac{2x^2-10x}{x-5}$ is not defined at $x = 5$ and gives $\frac{0}{0}$ so it needs further investigation. We can factorise and simplify to obtain a function that is identical except for being defined at $x = 5$ thus:

$$\begin{aligned} \frac{2x^2-10x}{x-5} &= \frac{2x(x-5)}{x-5} \\ &= 2x \qquad (x \neq 5) \\ \therefore \quad \lim_{x\to 5} \frac{2x^2-10x}{x-5} &= 2 \times 5 \\ &= 10 \end{aligned}$$

23. $\frac{x^2+5x+6}{x-1}$ is continuous at $x = 2$ (its only discontinuity is at $x = 1$) so

$$\begin{aligned} \lim_{x\to 2} \frac{x^2+5x+6}{x-1} &= \frac{2^2+5\times 2+6}{2-1} \\ &= 20 \end{aligned}$$

24. $\frac{3x^2-12}{x-2}$ is not defined at $x = 2$ and gives $\frac{0}{0}$ so it needs further investigation. We can factorise and simplify to obtain a function that is identical except for being defined at $x = 2$ thus:

$$\begin{aligned} \frac{3x^2-12}{x-2} &= \frac{3(x^2-4)}{x-2} \\ &= \frac{3(x+2)(x-2)}{x-2} \\ &= 3(x+2) \qquad (x \neq 2) \\ \therefore \quad \lim_{x\to 2} \frac{3x^2-12}{x-2} &= 3(2+2) \\ &= 12 \end{aligned}$$

25. $(x+2)^3$ is a polynomial function so it is continuous everywhere, so

$$\begin{aligned} \lim_{x\to 2}(x+2)^3 &= (2+2)^3 \\ &= 64 \end{aligned}$$

26. If we simply substitute $x = 5$ we get

$$\begin{aligned} \frac{x^2-4x-5}{x^2-7x+10} &= \frac{25-20-5}{25-35+10} \\ &= \frac{0}{0} \end{aligned}$$

so we need further investigation. Factorising and simplifying gives:

$$\begin{aligned} \frac{x^2-4x-5}{x^2-7x+10} &= \frac{(x-5)(x+1)}{(x-5)(x-2)} \\ &= \frac{x+1}{x-2} \qquad (x \neq 5) \\ \therefore \quad \lim x \to 5 \frac{x^2-4x-5}{x^2-7x+10} &= \frac{5+1}{5-2} \\ &= 2 \end{aligned}$$

27. If we simply substitute $x = 1$ we get

$$\begin{aligned} \frac{x^2-1}{x^2-x} &= \frac{1-1}{1-1} \\ &= \frac{0}{0} \end{aligned}$$

so we need further investigation. Factorising and simplifying gives:

$$\begin{aligned} \frac{x^2-1}{x^2-x} &= \frac{(x+1)(x-1)}{x(x-1)} \\ &= \frac{x+1}{x} \qquad (x \neq 1) \\ \therefore \quad \lim x \to 1 \frac{x^2-1}{x^2-x} &= \frac{1+1}{1} \\ &= 2 \end{aligned}$$

28. If we simply substitute $x = 5$ we get

$$\frac{2+x}{5-x} = \frac{7}{0}$$

The limit does not exist: the denominator approaches zero as the numerator approaches 7 so the limit increases or decreases without bound, depending on whether we approach from above or below. ($\lim_{x\to 5^-} 2 + x5 - x = +\infty$ since the denominator is positive for $x < 5$ and $\lim_{x\to 5^+} 2 + x5 - x = -\infty$ since the denominator is negative for $x > 5$. The numerator is positive in either case.)

29. If we simply substitute $x = 3$ we get

$$\frac{x-5}{x-3} = \frac{-2}{0}$$

The limit does not exist: the denominator approaches zero as the numerator approaches -2 so the limit increases or decreases without bound, depending on whether we approach from above or below. ($\lim_{x\to 3^-} x - 5x - 3 = +\infty$ since the denominator is negative for $x < 3$ and $\lim_{x\to 5^+} x - 5x - 3 = -\infty$ since the denominator is positive for $x > 3$. The numerator is negative in either case.)

30. If we simply substitute $x = 0$ we get

$$\frac{x^2+4x}{x} = \frac{0}{0}$$

so we need further investigation. Factorising and simplifying gives:

$$\begin{aligned} \frac{x^2+4x}{x} &= \frac{x(x+4)}{x} \\ &= x+4 \qquad (x \neq 0) \\ \therefore \quad \lim x \to 0 \frac{x^2+4x}{x} &= 0+4 \\ &= 4 \end{aligned}$$

31. If we simply substitute $x = 1$ we get
$$\frac{5x^2-5}{2x-2} = \frac{5-5}{2-2} = \frac{0}{0}$$
so we need further investigation. Factorising and simplifying gives:
$$\begin{aligned}\frac{5x^2-5}{2x-2} &= \frac{5(x^2-1)}{2(x-1)}\\ &= \frac{5(x+1)(x-1)}{2(x-1)}\\ &= \frac{5(x+1)}{2} \qquad (x \neq 1)\\ \therefore \quad \lim x \to 1 \frac{5x^2-5}{2x-2} &= \frac{5(1+1)}{2}\\ &= 5\end{aligned}$$

32. If we simply substitute $x = 2$ we get
$$\frac{3x-6}{x-2} = \frac{6-6}{2-2} = \frac{0}{0}$$
so we need further investigation. Factorising and simplifying gives:
$$\begin{aligned}\frac{3x-6}{x-2} &= \frac{3(x-2)}{x-2}\\ &= 3 \qquad (x \neq 1)\\ \therefore \quad \lim x \to 2 \frac{3x-6}{x-2} &= 3\end{aligned}$$

33.
$$\begin{aligned}\lim_{x\to 4^-} f(x) &= 2(4) - 3\\ &= 5\\ \lim_{x\to 4^+} f(x) &= (4) + 1 \qquad = 5\\ \therefore \quad \lim_{x\to 4} f(x) &= 5\end{aligned}$$

34.
$$\begin{aligned}\lim_{x\to 2^-} f(x) &= 3(2) - 2\\ &= 4\\ \lim_{x\to 2^+} f(x) &= 2(2) - 3\\ &= 1\\ \therefore \quad \lim_{x\to 2} f(x) &\text{ does not exist.}\end{aligned}$$

35. $f(x)$ is continuous at $x = 3$ so $\lim_{x\to 3} f(x) = f(3) = 2(3) - 4 = 2$.

36.
$$\begin{aligned}\lim_{x\to 5^-} f(x) &= 3(5) - 2\\ &= 13\\ \lim_{x\to 5^+} f(x) &= 13(5)\\ &= 65\\ \therefore \quad \lim_{x\to 5} &\text{ does not exist.}\end{aligned}$$

37.
$$\begin{aligned}\lim_{x\to 3^-} f(x) &= (3)^2\\ &= 9\\ \lim_{x\to 3^+} f(x) &= 3(3)\\ &= 9\\ \therefore \quad \lim_{x\to 3} &= 9\end{aligned}$$

38.
$$\begin{aligned}\lim_{x\to 2^-} f(x) &= (2-1)^2\\ &= 1\\ \lim_{x\to 2^+} f(x) &= 3(2) - 5\\ &= 1\\ \therefore \quad \lim_{x\to 2} &= 1\end{aligned}$$

39. (a) As $x \to 0^+$, $x > 0$ so $|x| = x$ and $\frac{|x|}{x} = 1$ so $\lim_{x\to 0^+} \frac{|x|}{x} = 1$.

(b) As $x \to 0^-$, $x < 0$ so $|x| = -x$ and $\frac{|x|}{x} = -1$ so $\lim_{x\to 0^-} \frac{|x|}{x} = -1$.

(c) As $\lim_{x\to 0^+} \frac{|x|}{x} \neq \lim_{x\to 0^-} \frac{|x|}{x}$ we conclude that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

40. (a) For $x > 0$ there is no discontinuity in $\sqrt{x}$ so $\lim_{x\to 0^+} \sqrt{x} = \sqrt{0} = 0$.

(b) For $x < 0$, $\sqrt{x}$ has no real value, so $\lim_{x\to 0^-} \sqrt{x}$ does not exist.

41. (a) $\lim_{x\to 1^+} \text{sgn}x = 1$ (since $\text{sgn}x = 1 \quad \forall x > 0$.)

(b) $\lim_{x\to 1^-} \text{sgn}x = 1$ (since $\text{sgn}x = 1 \quad \forall x > 0$.)

(c) $\lim_{x\to 1} \text{sgn}x = 1$ (since $\text{sgn}x = 1 \quad \forall x > 0$.)

(d) $\lim_{x\to -1^+} \text{sgn}x = -1$ (since $\text{sgn}x = -1 \quad \forall x < 0$.)

(e) $\lim_{x\to -1^-} \text{sgn}x = -1$ (since $\text{sgn}x = -1 \quad \forall x < 0$.)

(f) $\lim_{x\to -1} \text{sgn}x = -1$ (since $\text{sgn}x = -1 \quad \forall x < 0$.)

(g) $\lim_{x\to 0^+} \text{sgn}x = 1$ (since $\text{sgn}x = 1 \quad \forall x > 0$.)

(h) $\lim_{x\to 0^-} \text{sgn}x = -1$ (since $\text{sgn}x = -1 \quad \forall x < 0$.)

(i) $\lim_{x\to 0} \text{sgn}x$ does not exist (because $\lim_{x\to 0^+} \text{sgn}x \neq \lim_{x\to 0^-} \text{sgn}x$).

Exercise 6D

1. Considering dominant powers as $x \to \infty$,
$$\frac{3x^2+2x-1}{x} = \frac{3x^2}{x} = 3x$$
so
$$\lim_{x\to\infty} \frac{3x^2+2x-1}{x} = \infty$$
that is, $\lim_{x\to\infty} \frac{3x^2+2x-1}{x}$ does not exist.

2. Considering dominant powers as $x \to \infty$,
$$\frac{3x^2+2x-1}{x^2} = \frac{3x^2}{x^2} = 3$$
so
$$\lim_{x\to\infty} \frac{3x^2+2x-1}{x^2} = 3$$

3. Considering dominant powers as $x \to \infty$,
$$\frac{3x^2+2x-1}{x^3} = \frac{3x^2}{x^3} = \frac{3}{x}$$
so
$$\lim_{x\to\infty} \frac{3x^2+2x-1}{x^3} = 0$$

4. Considering dominant powers as $x \to \infty$,
$$\frac{2x^2-x}{5-x^2} = \frac{2x^2}{-x^2} = -2$$
so
$$\lim_{x\to\infty} \frac{2x^2-x}{5-x^2} = -2$$

5. Considering dominant powers as $x \to \infty$,
$$\frac{4x^3+2x-3}{5x^3+2x^2} = \frac{4x^3}{5x^3} = 0.8$$
so
$$\lim_{x\to\infty} \frac{4x^3+2x-3}{5x^3+2x^2} = 0.8$$

6. Considering dominant powers as $x \to -\infty$,
$$\frac{3x^4}{2x^3+x-2} = \frac{3x^4}{2x^3} = 1.5x$$
so
$$\lim_{x\to-\infty} \frac{3x^4}{2x^3+x-2} = -\infty$$
that is, $\lim_{x\to\infty} \frac{3x^4}{2x^3+x-2}$ does not exist.

7. Considering dominant powers as $x \to -\infty$,
$$\frac{3x^2}{2x^3+x-2} = \frac{3x^2}{2x^3} = \frac{1.5}{x}$$
so
$$\lim_{x\to-\infty} \frac{3x^2}{2x^3+x-2} = 0$$

8. Considering dominant powers as $x \to -\infty$,
$$\frac{3x^3}{2x^3+x-2} = \frac{3x^3}{2x^3} = 1.5$$
so
$$\lim_{x\to-\infty} \frac{3x^2}{2x^3+x-2} = 1.5$$

9. First simplify the algebraic fraction then consider dominant powers as $x \to \infty$:
$$\frac{(2x+3)(2x-5)}{(3-5x)(2x-5)} = \frac{2x+3}{3-5x} = -0.4$$
so
$$\lim_{x\to\infty} \frac{(2x+3)(2x-5)}{(3-5x)(2x-5)} = -0.4$$

10. Considering dominant powers as $x \to \infty$, the numerator tends to $7x^2$ and the denominator to $-4x^2$ so
$$\frac{7x^2+x}{(3-4x)(1+x)} = \frac{7x^2}{-4x^2} = -1.75$$
so
$$\lim_{x\to\infty} \frac{7x^2+x}{(3-4x)(1+x)} = -1.75$$

11. (a) $\lim_{x\to2^-} f(x) = \lim_{x\to2^-} x^2$
$= 4$

(b) $\lim_{x\to2^+} f(x) = \lim_{x\to2^+} \frac{3x-2}{x-1}$
$= \frac{3(2)-2}{2-1}$
$= 4$

(c) $\lim_{x\to2} f(x) = 4$

(d) $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^2$
$= \infty$
(i.e. the limit does not exist.)

(e) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x-2}{x-1}$
$= 3$

Exercise 6E

1. f(x) is discontinuous where the denominator is equal to zero, i.e.

$$\begin{aligned} x^2 - 3x - 10 &= 0 \\ (x-5)(x+2) &= 0 \\ x &= 5 \\ \text{or} \quad x &= -2 \end{aligned}$$

2. f(x) is discontinuous where the denominator is equal to zero, i.e.

$$\begin{aligned} x(2x-1)(x-1) &= 0 \\ x &= 0 \\ \text{or} \quad 2x - 1 &= 0 \\ x &= 0.5 \\ \text{or} \quad x - 1 &= 0 \\ x &= 1 \end{aligned}$$

3. Each piece is a polynomial function and hence continuous and the function is defined $\forall x \in \Re$ so the only possible discontinuities are where the pieces join.

$$\begin{aligned} \lim_{x\to-2^-} f(x) &= a(-2) + 10 \\ &= 10 - 2a \\ \lim_{x\to-2^+} f(x) &= 2 - (-2) \\ &= 4 \end{aligned}$$

for the function to be continuous,

$$\begin{aligned} \lim_{x\to-2^-} f(x) &= \lim_{x\to-2^+} f(x) \\ 10 - 2a &= 4 \\ -2a &= -6 \\ a &= 3 \\ \lim_{x\to0^-} f(x) &= 2 - (0) \\ &= 2 \\ \lim_{x\to0^+} f(x) &= 3(0) + b \\ &= b \end{aligned}$$

for the function to be continuous,

$$\begin{aligned} \lim_{x\to0^+} f(x) &= \lim_{x\to0^-} f(x) \\ b &= 2 \end{aligned}$$

4. The function must be uniquely defined $\forall x \in \Re$ so this gives us a value for c: $c = 4$. Each piece is a polynomial function and hence continuous so the only other possible discontinuities are where the pieces join.

$$\begin{aligned} \lim_{x\to0^-} f(x) &= 5(0) + a \\ &= a \\ \lim_{x\to0^+} f(x) &= b(0) - 3 \\ &= -3 \end{aligned}$$

for the function to be continuous,

$$\begin{aligned} \lim_{x\to0^-} f(x) &= \lim_{x\to0^+} f(x) \\ a &= -3 \\ \lim_{x\to4^-} f(x) &= b(4) - 3 \\ &= 4b - 3 \\ \lim_{x\to4^+} f(x) &= (4) + 1 \\ &= 5 \end{aligned}$$

for the function to be continuous,

$$\begin{aligned} \lim_{x\to0^-} f(x) &= \lim_{x\to0^+} f(x) \\ 4b - 3 &= 5 \\ b &= 2 \end{aligned}$$

5. Each individual part is a polynomial function and the function is defined $\forall x \in \Re$ so the only possible discontinuities are where the parts join.

$$\begin{aligned} \lim_{x\to1^-} f(x) &= 1^2 \\ &= 1 \\ \lim_{x\to1^+} f(x) &= 3(1) - 2 \\ &= 1 \\ f(1) &= 3(1) - 2 \\ &= 1 \end{aligned}$$

so the function is continuous at $x = 1$.

$$\begin{aligned} \lim_{x\to3^-} f(x) &= 3(3) - 2 \\ &= 7 \\ \lim_{x\to3^+} f(x) &= (3)^2 + 2 \\ &= 11 \end{aligned}$$

The function has a single discontinuity at $x = 3$ and is continuous everywhere else.

6.
   - The first piece is a polynomial function, so f(x) is continuous for $x < 4$.
   - $$\begin{aligned} \lim_{x\to4^-} f(x) &= (4) + 1 \\ &= 5 \\ \lim_{x\to4^+} f(x) &= \sqrt{(4)} + 3 \\ &= 5 \\ f(4) &= \sqrt{(4)} + 3 \\ &= 5 \end{aligned}$$

   f(x) is continuous at $x = 4$.

- The second piece is defined and continuous for all non-negative x so f(x) is continuous for $4 < x < 9$.
- $$\lim_{x\to 9^-} f(x) = \sqrt{(9)} + 3 = 6$$
  $$\lim_{x\to 9^+} f(x) = \frac{18}{12-(9)} = 6 \quad f(9) = \sqrt{(9)} + 3 = 6$$
  f(x) is continuous at $x = 9$.
- The third piece is continuous for all $x \neq 12$. The function is undefined at $x = 12$ so there is a discontinuity at $x = 12$.
- There is a discontinuity at $x = 30$ because f(30) is undefined.
- The fourth piece is a polynomial function and hence continuous everywhere, so the function is continuous for $x > 30$.

Exercise 6F

1. Not differentiable: f(a) is undefined.
2. Not differentiable: the gradient function from the left is not equal to that from the right (it is not smooth at $x = a$).
3. Not differentiable: the gradient function from the left is not equal to that from the right (it is not smooth at $x = a$).
4. Not differentiable: f(a) is undefined.
5. Not differentiable: the function is not continuous at $x = a$.
6. Not differentiable: f(a) is undefined.
7. Differentiable: f(x) is continuous and smooth at $x = a$.
8. Not differentiable: the function is not continuous at $x = a$.
9. Differentiable: f(x) is continuous and smooth at $x = a$.
10. Differentiable. (Polynomial functions are differentiable everywhere.)
11. Differentiable. (Polynomial functions are differentiable everywhere.)
12. Not differentiable: the function is not continuous at $x = 1$.
13. Differentiable: the function is smooth and continuous at $x = -1$.
14. Differentiable: the function is smooth and continuous at $x = 3$.
15. Not differentiable: the derivative from the left is -2 and the derivative from the right is 2 so the function is not smooth at $x = 2.5$.
16. f(x) is continuous (it approaches 1 from left and right as $x \to 1$) so we must consider the derivative from the left and the right.

    From the left (i.e. for $x < 1$) we obtain $f'(x) = 3x^2$ and as $x \to 1^-$, $f'(x) \to 3(1)^2 = 3$.

    From the right (i.e. for $x > 1$) we obtain $f'(x) = 1$ and as $x \to 1^+$, $f'(x) \to 1$.

    $\therefore$ f(x) is not differentiable at $x = 1$.
17. f(x) is not continuous at $x = 1$ because
    $$\lim_{x\to 1^-} = (1)^3 = 1$$
    but
    $$\lim_{x\to 1^+} = 3(1) = 3$$
    $\therefore$ f(x) is not differentiable at $x = 1$.
18. f(x) is continuous at $x = 3$ (it approaches 9 from left and right as $x \to 3$) so we must consider the derivative from the left and the right.

    From the left (i.e. for $x < 3$) we obtain $f'(x) = 6$.

    From the right (i.e. for $x > 3$) we obtain $f'(x) = 2x$ and as $x \to 3^+$, $f'(x) \to 6$.

    $\therefore$ f(x) is differentiable at $x = 3$.
19. f(x) is not defined at $x = 3$ and hence is not continuous and therefore not differentiable.
20.
    - For $x < 0$, f(x) is a polynomial function and hence both continuous and differentiable.
    - At $x = 0$ the function is continuous (it is defined and the limit approaches 0 from both directions). It is not, however, differentiable, because from the left is $f'(x) = 6$ and from the right $f'(x) = 6x$ so as $x \to 0$, $f'(x) \to 0$.

- For $0 < x < 1$, $\mathrm{f}(x)$ is a polynomial function and hence both continuous and differentiable.
- At $x = 1$ the function is not continuous as the limit from the left is $3(1)^2 = 3$ and from the right $2(1)^3 = 2$. Since it is not continuous it can not be differentiable.
- For $x > 1$, $\mathrm{f}(x)$ is a polynomial function and hence both continuous and differentiable.

21.
- For $x < 1$, $\mathrm{f}(x)$ is both continuous and differentiable.
- At $x = 1$, $\mathrm{f}(x)$ is continuous but not differentiable since the gradient from the left is -1 and the gradient from the right is 1.
- For $1 < x < 4$, $\mathrm{f}(x)$ is both continuous and differentiable.
- At $x = 4$, $\mathrm{f}(x)$ is continuous ($\lim_{x\to 4^-} \mathrm{f}(x) = |(4) - 1| = 3$ and $\lim_{x\to 4^+} \mathrm{f}(x) = (4)^2 - 7(4) + 15 = 3$). From the left the derivative is $\mathrm{f}'(x) = 1$; from the right $\mathrm{f}'(x) = 2x - 7$ so as $x \to 4^+$, $\mathrm{f}'(x) = 1$. Thus $\mathrm{f}(x)$ is differentiable at $x = 4$.
- For $4 < x < 5$, $\mathrm{f}(x)$ is both continuous and differentiable.
- At $x = 5$, $\mathrm{f}(x)$ is not continuous ($\lim_{x\to 5^-} \mathrm{f}(x) = (5)^2 - 7(5) + 15 = 5$ and $\lim_{x\to 5^+} \mathrm{f}(x) = 3(5) = 15$) and therefore not differentiable.
- For $x > 5$, $\mathrm{f}(x)$ is both continuous and differentiable.

22.
- For continuity at $x = 1$,

$$\begin{aligned}\lim_{x\to 1^-} \mathrm{f}(x) &= \lim_{x\to 1^+} \mathrm{f}(x)\\ 5(1) + 7 &= a(1)^2 + b(1) + c\\ a + b + c &= 12 \qquad (1)\end{aligned}$$

- For differentiability at $x = 1$

$$\begin{aligned}\mathrm{f}'(x) \text{ from the left} &= \mathrm{f}'(x) \text{ from the right}\\ 5 &= 2ax + b\end{aligned}$$

and as $x \to 1$ we get

$$2a + b = 5 \qquad (2)$$

- For continuity at $x = 3$,

$$\begin{aligned}\lim_{x\to 3^-} \mathrm{f}(x) &= \lim_{x\to 3^+} \mathrm{f}(x)\\ a(3)^2 + b(3) + c &= d \qquad (3)\end{aligned}$$

- For differentiability at $x = 3$

$$\begin{aligned}\mathrm{f}'(x) \text{ from the left} &= \mathrm{f}'(x) \text{ from the right}\\ 2ax + b &= 0\end{aligned}$$

and as $x \to 3$ we get

$$6a + b = 0 \qquad (4)$$

Now solving these four equations simultaneously, from (2) and (4) we get

$$\begin{aligned}4a &= -5\\ a &= -1.25\\ 6a + b &= 0\\ -7.5 + b &= 0\\ b &= 7.5\end{aligned}$$

Now substitute into (1)

$$\begin{aligned}a + b + c &= 12\\ -1.25 + 7.5 + c &= 12\\ c &= 5.75\end{aligned}$$

Finally equation (3) to find d:

$$\begin{aligned}9a + 3b + c &= d\\ d &= 9(-1.25) + 3(7.5) + 5.75\\ &= 17\end{aligned}$$

23.
- For continuity at $x = -1$,

$$\begin{aligned}\lim_{x\to -1^-} \mathrm{f}(x) &= \lim_{x\to -1^+} \mathrm{f}(x)\\ a(-1)^3 &= 6(-1) + b\\ -a &= -6 + b \qquad (1)\end{aligned}$$

- For differentiability at $x = -1$

$$\begin{aligned}\mathrm{f}'(x) \text{ from the left} &= \mathrm{f}'(x) \text{ from the right}\\ 3ax^2 &= 6\end{aligned}$$

and as $x \to -1$ we get

$$\begin{aligned}3a(-1)^2 &= 6\\ 3a &= 6\\ a &= 2\end{aligned}$$

substituting into (1)

$$\begin{aligned}-2 &= -6 + b\\ b &= 4\end{aligned}$$

- For continuity at $x = 5$,

$$\begin{aligned}\lim_{x\to 5^-} \mathrm{f}(x) &= \lim_{x\to 5^+} \mathrm{f}(x)\\ 6(5) + b &= c(5)^2 + d(5) + 29\\ 30 + 4 &= 25c + 5d + 29\\ 25c + 5d &= 5\\ 5c + d &= 1 \qquad (2)\end{aligned}$$

- For differentiability at $x = 5$

$$\begin{aligned} f'(x) \text{ from the left} &= f'(x) \text{ from the right} \\ 6 &= 2cx + d \end{aligned}$$

and as $x \to 5$ we get

$$10c + d = 6 \qquad (3)$$

Now solving (2) and (3) simultaneously we get

$$\begin{aligned} 5c &= 5 \\ c &= 1 \\ 5c + d &= 1 \\ 5 + d &= 1 \\ d &= -4 \end{aligned}$$

Miscellaneous Exercise 6

1. (a) The amplitude is 10cm (half the peak-peak distance).

   (b) The period of the motion is 2 seconds (time taken for the cycle to repeat).

   (c) The weight passes through the equilibrium position twice per full cycle, so it passes 10 times in the first 10 seconds.

2. Left hand side:

$$\begin{aligned} \cos^4\theta - \sin^4\theta &= (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) \\ &= (1)(\cos 2\theta) \\ &= \cos 2\theta \end{aligned}$$

□

3.
$$\begin{aligned} 2\cos^2 x + \sin x &= 2\cos 2x \\ 2(1 - \sin^2 x) + \sin x &= 2(1 - 2\sin^2 x) \\ 2 - 2\sin^2 x + \sin x &= 2 - 4\sin^2 x \\ 2\sin^2 x + \sin x &= 0 \\ \sin x(2\sin x + 1) &= 0 \end{aligned}$$

Using the null factor law:

$$\begin{aligned} \sin x &= 0 \\ x &= 0 \\ \text{or} \quad x &= \pi \\ \text{or} \quad x &= 2\pi \\ \text{or} \qquad 2\sin x + 1 &= 0 \\ \sin x &= -\frac{1}{2} \\ x &= \frac{7\pi}{6} \\ \text{or} \quad x &= \frac{11\pi}{6} \end{aligned}$$

4. (a)
$$\begin{aligned} \sqrt{2^2 + 5^2} &= \sqrt{29} \\ \cos\alpha &= \frac{2}{\sqrt{29}} \\ \sin\alpha &= \frac{5}{\sqrt{29}} \\ \alpha &= \sin^{-1}\frac{5}{\sqrt{29}} \\ &= 68.2^\circ \\ 2\cos\theta + 5\sin\theta &= \sqrt{29}\left(\frac{2}{\sqrt{29}}\cos\theta + \frac{5}{\sqrt{29}}\sin\theta\right) \\ &= \sqrt{29}(\cos\theta\cos\alpha + \sin\theta\sin\alpha) \\ &= \sqrt{29}\cos(\theta - \alpha) \\ &= \sqrt{29}\cos(\theta - 68.2^\circ) \end{aligned}$$

   (b) The minimum value is $-\sqrt{29} \approx 5.4$.

$$\begin{aligned} \sqrt{29}\cos(\theta - 68.2^\circ) &= -\sqrt{29} \\ \cos(\theta - 68.2^\circ) &= -1 \\ \theta - 68.2^\circ &= 180^\circ \\ \theta &= 248.2^\circ \end{aligned}$$

5. (a) Since z has no real component, $\bar{z} = -z = 3\sqrt{5}\mathrm{i}$

   (b) $z^2 = (-3\sqrt{5}\mathrm{i})^2$
   $= 9 \times 5 \times -1$
   $= -45$

   (c) $1 - z^2 = 1 - -45 = 46$

   (d) $(1 - z)^2 = (1 - -3\sqrt{5}\mathrm{i})^2$
   $= (1 + 3\sqrt{5}\mathrm{i})^2$
   $= 1 + 6\sqrt{5}\mathrm{i} - 45$
   $= -44 + 6\sqrt{5}\mathrm{i}$

6. If $z = a + b\mathrm{i}$ and $w = c + d\mathrm{i}$ then the product is

$$\begin{aligned} zw &= (a + b\mathrm{i})(c + d\mathrm{i}) \\ &= (ac - bd) + (ad + bc)\mathrm{i} \end{aligned}$$

The conjugate of the product is

$$\overline{zw} = (ac - bd) - (ad + bc)\text{i}$$

The product of the conjugates is

$$\begin{aligned}\bar{z}\bar{w} &= (a - b\text{i})(c - d\text{i}) \\ &= (ac - bd) + (-ad - bc)\text{i} \\ &= (ac - bd) - (ad + bc)\text{i}\end{aligned}$$

$\therefore$ The conjugate of the product is equal to the product of the conjugates.

7. The centre is at $(-3, 5)$ and the radius is 7.

$$\begin{aligned}x^2 + 6x + y^2 - 10y &= 15 \\ (x+3)^2 - 9 + (x-5)^2 - 25 &= 15 \\ (x+3)^2 + (x-5)^2 &= 49\end{aligned}$$

8. (a) Expand the right hand side:

$$\begin{aligned}&(ax - b)(x^2 + cx + 4) \\ &= ax^3 + acx^2 + 4ax - bx^2 - bcx - 4b \\ &= ax^3 + (ac - b)x^2 + (4a - bc)x - 4b\end{aligned}$$

Now equate terms with corresponding powers of x gives

$$\begin{aligned}a &= 7 \\ -4b &= -12 \\ b &= 3 \\ ac - b &= 4 \\ 7c - 3 &= 4 \\ 7c &= 7 \\ c &= 1\end{aligned}$$

(b) The first factor gives

$$\begin{aligned}7x^3 + 4x^2 + 25x - 12 &= 0 \\ (7x - 3)(x^2 + x + 4) &= 0\end{aligned}$$

$$\begin{aligned}7x - 3 &= 0 \\ x &= \frac{3}{7}\end{aligned}$$

The second factor gives

$$\begin{aligned}x^2 + x + 4 &= 0 \\ x &= \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times 4}}{2 \times 1} \\ &= \frac{-1 \pm \sqrt{-15}}{2} \\ x &= -\frac{1}{2} + \frac{\sqrt{15}}{2}\text{i} \\ \text{or} \quad x &= -\frac{1}{2} - \frac{\sqrt{15}}{2}\text{i}\end{aligned}$$

9. For continuity at $x = 100$,

$$\begin{aligned}0.01(100)^2 - 1.2(100) + 50 &= a(100)^2 + b(100) - 250 \\ 100 - 120 + 50 &= 10\,000a + 100b - 250 \\ 280 &= 10\,000a + 100b \\ 100a + b &= 2.8 \qquad (1)\end{aligned}$$

The gradient at $x = 100$ gives:

$$\begin{aligned}0.02x - 1.2 &= 2ax + b \\ 0.02(100) - 1.2 &= 2a(100) + b \\ 2 - 1.2 &= 200a + b \\ 200a + b &= 0.8 \qquad (2)\end{aligned}$$

Solving (1) and (2) simultaneously

$$\begin{aligned}100a &= -2 \\ a &= -0.02 \\ 100a + b &= 2.8 \\ -2 + b &= 2.8 \\ b &= 4.8\end{aligned}$$

For continuity at $x = 150$

$$\begin{aligned}a(150)^2 + b(150) - 250 &= c(150)^2 + d(150) + 605 \\ -0.02(22\,500) + 4.8(150) - 250 &= 22\,500c + 150d + 605 \\ -450 + 720 - 250 &= 22\,500c + 150d + 605 \\ 22\,500c + 150d &= 20 - 605 \\ &= -585 \\ 150c + d &= -3.9 \qquad (3)\end{aligned}$$

The gradient at $x = 150$ gives:

$$\begin{aligned}2ax + b &= 2cx + d \\ -0.04(150) + 4.8 &= 2c(150) + d \\ -6 + 4.8 &= 300c + d \\ 300c + d &= -1.2 \qquad (4)\end{aligned}$$

Solving (3) and (4) simultaneously

$$\begin{aligned}150c &= 2.7 \\ c &= 0.018 \\ 150c + d &= -3.9 \\ 2.7 + d &= -3.9 \\ d &= -6.6\end{aligned}$$

At point A, $x = 0$ and $y = e$ so

$$\begin{aligned}e &= 0.01(0)^2 - 1.2(0) + 50 \\ &= 50\end{aligned}$$

At point B, $x = 100$ and $y = f$ so

$$\begin{aligned}f &= 0.01(100)^2 - 1.2(100) + 50 \\ &= 100 - 120 + 50 \\ &= 30\end{aligned}$$

At point C, $x = 150$ and $y = g$ so

$$\begin{aligned}g &= a(150)^2 + b(150) - 250 \\ &= -0.02(22\,500) + 4.8(150) - 250 \\ &= -450 + 720 - 250 \\ &= 20\end{aligned}$$

At point D, $x = h$ and $y = 0$ so

$$\begin{aligned} ch^2 + dh + 605 &= 0 \\ 0.018h^2 - 6.6h + 605 &= 0 \\ h &= \frac{550}{3} \end{aligned}$$

(using either the quadratic formula or the calculator's solver for the last step.)

10. $$\begin{aligned} \cos 3\theta &= \cos(2\theta + \theta) \\ &= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta \\ &= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta \\ &= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta \\ &= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta \\ &= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta \\ &= 4\cos^3\theta - 3\cos\theta \end{aligned}$$

    Hence $a = 4$, $b = 0$, $c = -3$, $d = 0$.

11. First piece, $x < -2$: $\mathrm{f}(x)$ is a polynomial function and hence continuous and differentiable.

    Where the first and second pieces meet, $x = -2$:

    $$\begin{aligned} \lim_{x\to -2^-} \mathrm{f}(x) &= -6 - (-2)^2 \\ &= -10 \\ \lim_{x\to -2^+} \mathrm{f}(x) &= 4(-2) - 2 \\ &= -10 \end{aligned}$$

    $\mathrm{f}(x)$ is continuous at $x = -2$.

    Derivative from the left:

    $$\begin{aligned} \mathrm{f}'(x) &= -2x \\ &= -2(-2) \\ &= 4 \end{aligned}$$

    Derivative from the right:

    $$\mathrm{f}'(x) = 4$$

    $\mathrm{f}(x)$ is differentiable at $x = -2$.

    Second piece, $-2 < x < 3$: $\mathrm{f}(x)$ is a polynomial function and hence continuous and differentiable.

    At $x = 3$,

    $$\begin{aligned} \lim_{x\to 3^-} \mathrm{f}(x) &= 4(3) - 2 \\ &= 10 \end{aligned}$$

    However, $\mathrm{f}(3) = 5$ so since $\lim_{x\to 3} \mathrm{f}(x) \neq \mathrm{f}(3)$ we must conclude that $\mathrm{f}(x)$ is not continuous at $x = 3$ and therefore not differentiable either.

    Last piece, $x > 3$: $\mathrm{f}(x)$ is a polynomial function and hence continuous and differentiable.

    Conclusion: $\mathrm{f}(x)$ is both continuous and differentiable everywhere except at $x = 3$ where it is neither.

12. (a) This is trivial; simply substitute $x = a$ to get

    $$\lim_{x\to a}(x^2 + 3x + 5) = a^2 + 3a + 5$$

    (b) Here we can't just substitute because we get $\frac{0}{0}$ so we must try another approach:

    $$\begin{aligned} \lim_{x\to a} \frac{2(x-a)}{x^2 - a^2} &= \lim_{x\to a} \frac{2(x-a)}{(x-a)(x+a)} \\ &= \lim_{x\to a} \frac{2}{(x+a)} \\ &= \frac{2}{2a} \\ &= \frac{1}{a} \end{aligned}$$

    (c) Here again we can't just substitute because we get $\frac{0}{0}$ so we must try another approach:

    $$\begin{aligned} \lim_{x\to\sqrt{a}} \frac{x - \sqrt{a}}{x^2 - a} &= \lim_{x\to\sqrt{a}} \frac{x - \sqrt{a}}{(x - \sqrt{a})(x + \sqrt{a})} \\ &= \lim_{x\to\sqrt{a}} \frac{1}{x + \sqrt{a}} \\ &= \frac{1}{2\sqrt{a}} \end{aligned}$$

13. First without a calculator:

    $$\begin{aligned} \mathrm{f}'(x) &= \frac{2(2x+a) - 2(2x+3)}{(2x+a)^2} \\ &= \frac{2a-6}{(2x+a)^2} \end{aligned}$$

    $$\begin{aligned} \mathrm{f}'(3) &= \frac{2a-6}{(6+a)^2} \\ \mathrm{f}'(3) &= -16 \\ \frac{2a-6}{(6+a)^2} &= -16 \\ 2a - 6 &= -16(6+a)^2 \\ 16(6+a)^2 + 2a - 6 &= 0 \\ 16(36 + 12a + a^2) + 2a - 6 &= 0 \\ 8(36 + 12a + a^2) + a - 3 &= 0 \\ 288 + 96a + 8a^2 + a - 3 &= 0 \\ 8a^2 + 97a + 285 &= 0 \end{aligned}$$

    Use a calculator (or the quadratic formula) to solve this quadratic: $a = -5$ or $a = -7.125$.