Chapter 4

Exercise 4A

1. (a) $$\begin{aligned}\mathbf{r}_\mathrm{A} + t\mathbf{v}_\mathrm{A} &= (5\mathbf{i} + 4\mathbf{j}) + t(10\mathbf{i} - \mathbf{j})\\ &= (5 + 10t)\mathbf{i} + (4 - t)\mathbf{j}\end{aligned}$$

   (b) $$\begin{aligned}\mathbf{r}_\mathrm{B} + t\mathbf{v}_\mathrm{B} &= (6\mathbf{i} - 8\mathbf{j}) + t(2\mathbf{i} + 8\mathbf{j})\\ &= (6 + 2t)\mathbf{i} + (-8 + 8t)\mathbf{j}\end{aligned}$$

   (c) $$\begin{aligned}\mathbf{r}_\mathrm{C} + t\mathbf{v}_\mathrm{C} &= (2\mathbf{i} + 3\mathbf{j}) + t(-4\mathbf{i} + 3\mathbf{j})\\ &= (2 - 4t)\mathbf{i} + (3 + 3t)\mathbf{j}\end{aligned}$$

   (d) If the number of hours after 8am is t then the number of hours after 7am is $t + 1$.

   $$\begin{aligned}&\mathbf{r}_\mathrm{D}+(t + 1)\mathbf{v}_\mathrm{D}\\ &= (9\mathbf{i} - 10\mathbf{j}) + (t + 1)(10\mathbf{i} + 6\mathbf{j})\\ &= (9\mathbf{i} - 10\mathbf{j}) + t(10\mathbf{i} + 6\mathbf{j}) + (10\mathbf{i} + 6\mathbf{j})\\ &= (19\mathbf{i} - 4\mathbf{j}) + t(10\mathbf{i} + 6\mathbf{j})\\ &= (19 + 10t)\mathbf{i} + (-4 + 6t)\mathbf{j}\end{aligned}$$

   (e) If the number of hours after 8am is t then the number of hours after 9am is $t - 1$.

   $$\begin{aligned}&\mathbf{r}_\mathrm{E}+(t - 1)\mathbf{v}_\mathrm{E}\\ &= (16\mathbf{i} + 7\mathbf{j}) + (t - 1)(-4\mathbf{i} + 3\mathbf{j})\\ &= (16\mathbf{i} + 7\mathbf{j}) + t(-4\mathbf{i} + 3\mathbf{j}) - (-4\mathbf{i} + 3\mathbf{j})\\ &= (20\mathbf{i} + 4\mathbf{j}) + t(-4\mathbf{i} + 3\mathbf{j})\\ &= (20 - 4t)\mathbf{i} + (4 + 3t)\mathbf{j}\end{aligned}$$

   (f) If the number of hours after 8am is t then the number of hours after 8:30am is $t - 0.5$.

   $$\begin{aligned}&\mathbf{r}_\mathrm{F}+(t - 0.5)\mathbf{v}_\mathrm{F}\\ &= (2\mathbf{i} + 3\mathbf{j}) + (t - 0.5)(12\mathbf{i} - 8\mathbf{j})\\ &= (2\mathbf{i} + 3\mathbf{j}) + t(12\mathbf{i} - 8\mathbf{j}) - (6\mathbf{i} - 4\mathbf{j})\\ &= (-4\mathbf{i} + 7\mathbf{j}) + t(12\mathbf{i} - 8\mathbf{j})\\ &= (-4 + 12t)\mathbf{i} + (7 - 8t)\mathbf{j}\end{aligned}$$

2. (a) $$\begin{aligned}&(7\mathbf{i} + 10\mathbf{j})+(6 - 5)(3\mathbf{i} + 4\mathbf{j})\\ &= (7\mathbf{i} + 10\mathbf{j}) + (3\mathbf{i} + 4\mathbf{j})\\ &= (10\mathbf{i} + 14\mathbf{j})\text{km}\end{aligned}$$

   (b) $$\begin{aligned}&(7\mathbf{i} + 10\mathbf{j})+(7 - 5)(3\mathbf{i} + 4\mathbf{j})\\ &= (7\mathbf{i} + 10\mathbf{j}) + 2(3\mathbf{i} + 4\mathbf{j})\\ &= (7\mathbf{i} + 10\mathbf{j}) + (6\mathbf{i} + 8\mathbf{j})\\ &= (13\mathbf{i} + 18\mathbf{j})\text{km}\end{aligned}$$

   (Alternatively, add $(3\mathbf{i} + 4\mathbf{i})$ to the previous answer.)

   (c) $$\begin{aligned}&(7\mathbf{i} + 10\mathbf{j})+(9 - 5)(3\mathbf{i} + 4\mathbf{j})\\ &= (7\mathbf{i} + 10\mathbf{j}) + 4(3\mathbf{i} + 4\mathbf{j})\\ &= (7\mathbf{i} + 10\mathbf{j}) + (12\mathbf{i} + 16\mathbf{j})\\ &= (19\mathbf{i} + 26\mathbf{j})\text{km}\end{aligned}$$

   (d) Speed $= |3\mathbf{i} + 4\mathbf{j}| = \sqrt{3^2 + 4^2} = 5\text{km/h}$

   (e) Ships position at 8am:

   $$\begin{aligned}&(7\mathbf{i} + 10\mathbf{j})+(8 - 5)(3\mathbf{i} + 4\mathbf{j})\\ &= (7\mathbf{i} + 10\mathbf{j}) + 3(3\mathbf{i} + 4\mathbf{j})\\ &= (7\mathbf{i} + 10\mathbf{j}) + (9\mathbf{i} + 12\mathbf{j})\\ &= (16\mathbf{i} + 22\mathbf{j})\text{km}\end{aligned}$$

   Distance from lighthouse:

   $$\begin{aligned}(16\mathbf{i} + 22\mathbf{j}) - (21\mathbf{i} + 20\mathbf{j}) &= -5\mathbf{j} + 2\mathbf{j}\\ |-5\mathbf{j} + 2\mathbf{j}| &= \sqrt{(-5)^2 + 2^2}\\ &= \sqrt{29}\text{km}\end{aligned}$$

3. (a) Position (in km) at 9am is

   $$(9\mathbf{i} + 36\mathbf{j}) - (2\mathbf{i} + 12\mathbf{j}) = 7\mathbf{i} + 24\mathbf{j}$$

   This represents distance from the origin of

   $$\sqrt{7^2 + 24^2} = 25\text{km}$$

   (b) Position (in km) at 8am is

   $$(7\mathbf{i} + 24\mathbf{j}) - (2\mathbf{i} + 12\mathbf{j}) = 5\mathbf{i} + 12\mathbf{j}$$

   This represents distance from the origin of

   $$\sqrt{5^2 + 12^2} = 13\text{km}$$

4. (a) At 3pm the respective positions are

   $$\mathbf{r}_\mathrm{A} = 21\mathbf{i} + 7\mathbf{j} \qquad \mathbf{r}_\mathrm{B} = 25\mathbf{i} - 6\mathbf{j}$$

   The distance between them is

   $$\begin{aligned}d &= |\mathbf{r}_\mathrm{B} - \mathbf{r}_\mathrm{A}|\\ &= |(25\mathbf{i} - 6\mathbf{j}) - (21\mathbf{i} + 7\mathbf{j})|\\ &= |4\mathbf{i} - 13\mathbf{j}|\\ &= \sqrt{4^2 + (-13)^2}\\ &= \sqrt{185}\text{km}\end{aligned}$$

   (b) At 4pm the respective positions are

   $$\begin{aligned}\mathbf{r}_\mathrm{A} &= (21\mathbf{i} + 7\mathbf{j}) + (10\mathbf{i} + 5\mathbf{j})\\ &= 31\mathbf{i} + 12\mathbf{j}\\ \mathbf{r}_\mathrm{B} &= (25\mathbf{i} - 6\mathbf{j}) + (7\mathbf{i} + 10\mathbf{j})\\ &= 32\mathbf{i} + 4\mathbf{j}\end{aligned}$$

   The distance between them is

   $$\begin{aligned}d &= |\mathbf{r}_\mathrm{B} - \mathbf{r}_\mathrm{A}|\\ &= |(32\mathbf{i} + 4\mathbf{j}) - (31\mathbf{i} + 12\mathbf{j})|\\ &= |\mathbf{i} - 8\mathbf{j}|\\ &= \sqrt{1^2 + 8^2}\\ &= \sqrt{65}\text{km}\end{aligned}$$

(c) At 5pm the respective positions are

$$\begin{aligned}\mathbf{r}_\text{A} &= (21\mathbf{i} + 7\mathbf{j}) + 2(10\mathbf{i} + 5\mathbf{j})\\ &= 41\mathbf{i} + 17\mathbf{j}\\ \mathbf{r}_\text{B} &= (25\mathbf{i} - 6\mathbf{j}) + 2(7\mathbf{i} + 10\mathbf{j})\\ &= 39\mathbf{i} + 14\mathbf{j}\end{aligned}$$

The distance between them is

$$\begin{aligned}d &= |\mathbf{r}_\text{B} - \mathbf{r}_\text{A}|\\ &= |(39\mathbf{i} + 14\mathbf{j}) - (41\mathbf{i} + 17\mathbf{j})|\\ &= |-2\mathbf{i} - 3\mathbf{j}|\\ &= \sqrt{(-2)^2 + (-3)^2}\\ &= \sqrt{13}\text{km}\end{aligned}$$

5. (a) At 9am the respective positions are

$$\begin{aligned}\mathbf{r}_\text{A} &= (-5\mathbf{i} + 13\mathbf{j}) + (7\mathbf{i} - 2\mathbf{j})\\ &= 2\mathbf{i} + 11\mathbf{j}\\ \mathbf{r}_\text{B} &= (-3\mathbf{j}) + (-3\mathbf{i} + 2\mathbf{j})\\ &= -3\mathbf{i} - \mathbf{j}\end{aligned}$$

The distance between them is

$$\begin{aligned}d &= |\mathbf{r}_\text{B} - \mathbf{r}_\text{A}|\\ &= |(-3\mathbf{i} - \mathbf{j}) - (2\mathbf{i} + 11\mathbf{j})|\\ &= |-5\mathbf{i} - 12\mathbf{j}|\\ &= \sqrt{(-5)^2 + (-12)^2}\\ &= \sqrt{169}\\ &= 13\text{km}\end{aligned}$$

(b) At 10am the respective positions are

$$\begin{aligned}\mathbf{r}_\text{A} &= (-5\mathbf{i} + 13\mathbf{j}) + 2(7\mathbf{i} - 2\mathbf{j})\\ &= 9\mathbf{i} + 9\mathbf{j}\\ \mathbf{r}_\text{B} &= (-3\mathbf{j}) + 2(-3\mathbf{i} + 2\mathbf{j})\\ &= -6\mathbf{i} + \mathbf{j}\end{aligned}$$

The distance between them is

$$\begin{aligned}d &= |\mathbf{r}_\text{B} - \mathbf{r}_\text{A}|\\ &= |(-6\mathbf{i} + \mathbf{j}) - (9\mathbf{i} + 9\mathbf{j})|\\ &= |-15\mathbf{i} - 8\mathbf{j}|\\ &= \sqrt{(-15)^2 + (-8)^2}\\ &= \sqrt{289}\\ &= 17\text{km}\end{aligned}$$

6. (a) At a time t hours after 8am:

$$\begin{aligned}\mathbf{r}_\text{A}(t) &= (28\mathbf{i} - 5\mathbf{j}) + t(-8\mathbf{i} + 4\mathbf{j})\\ &= (28 - 8t)\mathbf{i} + (-5 + 4t)\mathbf{j}\\ \mathbf{r}_\text{B}(t) &= (24\mathbf{j}) + t(6\mathbf{i} + 2\mathbf{j})\\ &= (6t)\mathbf{i} + (24 + 2t)\mathbf{j}\end{aligned}$$

(b) The distance between ships t hours after 8am is

$$\begin{aligned}d &= |\mathbf{r}_\text{B}(t) - \mathbf{r}_\text{A}(t)|\\ &= |\,((6t)\mathbf{i} + (24 + 2t)\mathbf{j}) -\\ &\qquad ((28 - 8t)\mathbf{i} + (-5 + 4t)\mathbf{j})\,|\\ &= |(-28 + 14t)\mathbf{i} + (29 - 2t)\mathbf{j}|\\ d^2 &= (-28 + 14t)^2 + (29 - 2t)^2\end{aligned}$$

solving this for $d = 25$:

$$\begin{aligned}25^2 &= (-28 + 14t)^2 + (29 - 2t)^2\\ t &= 2 \text{ or } t = 2.5\end{aligned}$$

The ships will be 25 km apart at 10am and again at 10:30am.
You should be able to solve the quadratic in the second-last line of working above manually, but provided you're confident that you can it's acceptable to use your ClassPad here. If you're unsure of the manual solution,

$$\begin{aligned}(-28 + 14t)^2 + (29 - 2t)^2 &= 25^2\\ 784 - 784t + 196t^2 + 841 - 116t + 4t^2 &= 625\\ 200t^2 - 900t + 1625 &= 625\\ 200t^2 - 900t + 1000 &= 0\\ 2t^2 - 9t + 10 &= 0\\ (2t - 5)(t - 2) &= 0\\ t &= \frac{5}{2} \text{ or } t = 2\end{aligned}$$

7. The position of both ships t hours after 8am is

$$\begin{aligned}\mathbf{r}_\text{A}(t) &= (12\mathbf{i} + 61\mathbf{j}) + t(7\mathbf{i} - 8\mathbf{j})\\ &= (12 + 7t)\mathbf{i} + (61 - 8t)\mathbf{j}\\ \mathbf{r}_\text{B}(t) &= (57\mathbf{i} - 29\mathbf{j}) + t(-2\mathbf{i} + 10\mathbf{j})\\ &= (57 - 2t)\mathbf{i} + (-29 + 10t)\mathbf{j}\end{aligned}$$

Ships will collide if for some value of t:

$$\begin{aligned}\mathbf{r}_\text{A}(t) &= \mathbf{r}_\text{B}(t)\\ (12 + 7t)\mathbf{i} + (61 - 8t)\mathbf{j} &= (57 - 2t)\mathbf{i} + (-29 + 10t)\mathbf{j}\end{aligned}$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components separately:

$$\begin{aligned}12 + 7t &= 57 - 2t & 61 - 8t &= -29 + 10t\\ 9t &= 45 & 90 &= 18t\\ t &= 5 & t &= 5\end{aligned}$$

Thus when $t = 5$ the position vectors have the same $\mathbf{i}$ components and the same $\mathbf{j}$ components: the ships collide at 1pm. The position vector of the collision is

$$(12 + 7 \times 5)\mathbf{i} + (61 - 8 \times 5)\mathbf{j} = (47\mathbf{i} + 21\mathbf{j})\text{km}$$

8. The position of both ships t hours after 8am is

$$\begin{aligned}\mathbf{r}_\text{A}(t) &= (-11\mathbf{i} - 8\mathbf{j}) + t(7\mathbf{i} - \mathbf{j}) \\ &= (-11 + 7t)\mathbf{i} + (-8 - t)\mathbf{j} \\ \mathbf{r}_\text{B}(t) &= (-2\mathbf{i} - 4\mathbf{j}) + t(4\mathbf{i} + 5\mathbf{j}) \\ &= (-2 + 4t)\mathbf{i} + (-4 + 5t)\mathbf{j}\end{aligned}$$

Ships will collide if for some value of t:

$$\begin{aligned}\mathbf{r}_\text{A}(t) &= \mathbf{r}_\text{B}(t) \\ (-11 + 7t)\mathbf{i} + (-8 - t)\mathbf{j} &= (-2 + 4t)\mathbf{i} + (-4 + 5t)\mathbf{j}\end{aligned}$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components separately:

$$\begin{aligned}-11 + 7t &= -2 + 4t & -8 - t &= -4 + 5t \\ 3t &= 9 & -4 &= 6t \\ t &= 3 & t &= -\frac{2}{3}\end{aligned}$$

The position vectors have the same $\mathbf{i}$ components and the same $\mathbf{j}$ components at different times: the ships do not collide.

9. The position of both ships t hours after 8am is

$$\begin{aligned}\mathbf{r}_\text{A}(t) &= (24\mathbf{i} - 25\mathbf{j}) + t(-3\mathbf{i} + 4\mathbf{j}) \\ &= (24 - 3t)\mathbf{i} + (-25 + 4t)\mathbf{j} \\ \mathbf{r}_\text{B}(t) &= (-9\mathbf{i} + 33\mathbf{j}) + (t - 1)(2\mathbf{i} - 5\mathbf{j}) \\ &= (-9\mathbf{i} + 33\mathbf{j}) + t(2\mathbf{i} - 5\mathbf{j}) - (2\mathbf{i} - 5\mathbf{j}) \\ &= (-11 + 2t)\mathbf{i} + (38 - 5t)\mathbf{j}\end{aligned}$$

Ships will collide if for some value of t:

$$\begin{aligned}\mathbf{r}_\text{A}(t) &= \mathbf{r}_\text{B}(t) \\ (24 - 3t)\mathbf{i} + (-25 + 4t)\mathbf{j} &= (-11 + 2t)\mathbf{i} + (38 - 5t)\mathbf{j}\end{aligned}$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components separately:

$$\begin{aligned}24 - 3t &= -11 + 2t & -25 + 4t &= 38 - 5t \\ 35 &= 5t & 9t &= 63 \\ t &= 7 & t &= 7\end{aligned}$$

Thus when $t = 7$ the position vectors have the same $\mathbf{i}$ components and the same $\mathbf{j}$ components: the ships collide at 3pm. The position vector of the collision is

$$(24 - 3 \times 7)\mathbf{i} + (-25 + 4 \times 7)\mathbf{j} = (3\mathbf{i} + 3\mathbf{j})\text{km}$$

10. The position of both ships t hours after 9am is

$$\begin{aligned}\mathbf{r}_\text{A}(t) &= (-6\mathbf{i} + 44\mathbf{j}) + (t - 0.5)(4\mathbf{i} - 6\mathbf{j}) \\ &= (-6\mathbf{i} + 44\mathbf{j}) + t(4\mathbf{i} - 6\mathbf{j}) - (2\mathbf{i} - 3\mathbf{j}) \\ &= (-8 + 4t)\mathbf{i} + (47 - 6t)\mathbf{j} \\ \mathbf{r}_\text{B}(t) &= (2\mathbf{i} - 18\mathbf{j}) + t(2\mathbf{i} + 7\mathbf{j}) \\ &= (2 + 2t)\mathbf{i} + (-18 + 7t)\mathbf{j}\end{aligned}$$

Ships will collide if for some value of t:

$$\begin{aligned}\mathbf{r}_\text{A}(t) &= \mathbf{r}_\text{B}(t) \\ (-8 + 4t)\mathbf{i} + (47 - 6t)\mathbf{j} &= (2 + 2t)\mathbf{i} + (-18 + 7t)\mathbf{j}\end{aligned}$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components separately:

$$\begin{aligned}-8 + 4t &= 2 + 2t & 47 - 6t &= -18 + 7t \\ 2t &= 10 & 65 &= 13t \\ t &= 5 & t &= 5\end{aligned}$$

Thus when $t = 5$ the position vectors have the same $\mathbf{i}$ components and the same $\mathbf{j}$ components: the ships collide at 2pm. The position vector of the collision is

$$(-8 + 4 \times 5)\mathbf{i} + (47 - 6 \times 5)\mathbf{j} = (12\mathbf{i} + 17\mathbf{j})\text{km}$$

11. The position of both ships t hours after noon is

$$\begin{aligned}\mathbf{r}_\text{A}(t) &= (-11\mathbf{i} + 4\mathbf{j}) + t(10\mathbf{i} - 4\mathbf{j}) \\ &= (-11 + 10t)\mathbf{i} + (4 - 4t)\mathbf{j} \\ \mathbf{r}_\text{B}(t) &= (3\mathbf{i} - 5\mathbf{j}) + (t - 0.5)(7\mathbf{i} + 5\mathbf{j}) \\ &= (3\mathbf{i} - 5\mathbf{j}) + t(7\mathbf{i} + 5\mathbf{j}) - (3.5\mathbf{i} + 2.5\mathbf{j}) \\ &= (-0.5 + 7t)\mathbf{i} + (-7.5 + 5t)\mathbf{j}\end{aligned}$$

Ships will collide if for some value of t:

$$\begin{aligned}\mathbf{r}_\text{A}(t) &= \mathbf{r}_\text{B}(t) \\ (-11 + 10t)\mathbf{i} + (4 - 4t)\mathbf{j} &= (-0.5 + 7t)\mathbf{i} + (-7.5 + 5t)\mathbf{j}\end{aligned}$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components separately:

$$\begin{aligned}-11 + 10t &= -0.5 + 7t & 4 - 4t &= -7.5 + 5t \\ 3t &= 10.5 & 11.5 &= 9t \\ t &= 3.5 & t &= 1.25\end{aligned}$$

The position vectors have the same $\mathbf{i}$ components and the same $\mathbf{j}$ components at different times: the ships do not collide.

12. (a) The position of the three ships t hours after 8am is

$$\begin{aligned}\mathbf{r}_\text{P}(t) &= (-23\mathbf{i} + 3\mathbf{j}) + t(18\mathbf{i} + 4\mathbf{j}) \\ &= (-23 + 18t)\mathbf{i} + (3 + 4t)\mathbf{j} \\ \mathbf{r}_\text{Q}(t) &= (7\mathbf{i} + 30\mathbf{j}) + t(12\mathbf{i} - 10\mathbf{j}) \\ &= (7 + 12t)\mathbf{i} + (30 - 10t)\mathbf{j} \\ \mathbf{r}_\text{R}(t) &= (32\mathbf{i} - 30\mathbf{j}) + t(2\mathbf{i} + 14\mathbf{j}) \\ &= (32 + 2t)\mathbf{i} + (-30 + 14t)\mathbf{j}\end{aligned}$$

Ships P and Q will collide if for some value of t:

$$\begin{aligned}\mathbf{r}_\text{P}(t) &= \mathbf{r}_\text{Q}(t) \\ (-23 + 18t)\mathbf{i} + (3 + 4t)\mathbf{j} &= (7 + 12t)\mathbf{i} + (30 - 10t)\mathbf{j}\end{aligned}$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components separately:

$$\begin{aligned}-23 + 18t &= 7 + 12t & 3 + 4t &= 30 - 10t \\ 6t &= 30 & 14t &= 27 \\ t &= 5 & t &= \frac{27}{14}\end{aligned}$$

The position vectors have the same **i** components and the same **j** components at different times: ships P and Q do not collide.
Ships P and R will collide if for some value of t:

$$\mathbf{r}_{\mathrm{P}}(t) = \mathbf{r}_{\mathrm{R}}(t)$$
$$(-23 + 18t)\mathbf{i} + (3 + 4t)\mathbf{j} = (32 + 2t)\mathbf{i} + (-30 + 14t)\mathbf{j}$$

Equating **i** and **j** components separately:

$$\begin{aligned} -23 + 18t &= 32 + 2t & 3 + 4t &= -30 + 14t \\ 16t &= 55 & 33 &= 10t \\ t &= \frac{55}{16} & t &= \frac{33}{10} \end{aligned}$$

The position vectors have the same **i** components and the same **j** components at different times: ships P and R do not collide.
Ships Q and R will collide if for some value of t:

$$\mathbf{r}_{\mathrm{Q}}(t) = \mathbf{r}_{\mathrm{R}}(t)$$
$$(7 + 12t)\mathbf{i} + (30 - 10t)\mathbf{j} = (32 + 2t)\mathbf{i} + (-30 + 14t)\mathbf{j}$$

Equating **i** and **j** components separately:

$$\begin{aligned} 7 + 12t &= 32 + 2t & 30 - 10t &= -30 + 14t \\ 10t &= 25 & 60 &= 24t \\ t &= \frac{5}{2} & t &= \frac{5}{2} \end{aligned}$$

Ships Q and R collide at 10:30am.
The position of ships Q and R at the collision is:

$$(7+12\times 2.5)\mathbf{i}+(30-10\times 2.5)\mathbf{j} = (37\mathbf{i}+5\mathbf{j})\text{km}$$

(b) The position of ship P at 10:30am is

$$\begin{aligned} &(-23 + 18 \times 2.5)\mathbf{i} + (3 + 4 \times 2.5)\mathbf{j} \\ &= (22\mathbf{i} + 13\mathbf{j})\text{km} \end{aligned}$$

Distance from the collision is

$$\begin{aligned} d &= \sqrt{(37 - 22)^2 + (5 - 13)^2} \\ &= \sqrt{15^2 + (-8)^2} \\ &= \sqrt{225 + 64} \\ &= \sqrt{289} \\ &= 17\text{km} \end{aligned}$$

13. At 10:05am when *Brig* starts moving her position is $(-2\mathbf{i} + 7\mathbf{j})$km. Her destination is the location of *Ajax* at 10:45am:

$$(2\mathbf{i} + 12\mathbf{j}) + \frac{45}{60}(8\mathbf{i} - 4\mathbf{j}) = (8\mathbf{j} + 9\mathbf{j})\text{km}$$

She needs to travel a total displacement of

$$(8\mathbf{i} + 9\mathbf{j}) - (-2\mathbf{i} + 7\mathbf{j}) = (10\mathbf{i} + 2\mathbf{j})\text{km}$$

To do this in the 40 minutes between 10:05 and 10:45 she needs to have velocity

$$\begin{aligned} v &= (10\mathbf{i} + 2\mathbf{j}) \div \frac{40}{60} \\ &= (10\mathbf{i} + 2\mathbf{j}) \times \frac{3}{2} \\ &= (15\mathbf{i} + 3\mathbf{j})\text{km/h} \end{aligned}$$

Exercise 4B

Note that the answers given where a vector or parametric equation of a line is requested are not uniquely correct. You may have correct answers that look different from those given here or in Sadler. For example,

$$(2 + 5\lambda)\mathbf{i} + (3 - \lambda)\mathbf{j}$$

and

$$(-3 - 5\lambda)\mathbf{i} + (4 + \lambda)\mathbf{j}$$

and

$$(7 + 10\lambda)\mathbf{i} + (2 - 2\lambda)\mathbf{j}$$

all represent the same line. In each the parameter λ means something different, so the same value of λ will give different points, but every point on one is also a point on the others, just for different values of λ.

1. $$\begin{aligned} \mathbf{r} &= \mathbf{a} + \lambda\mathbf{b} \\ &= 2\mathbf{i} + 3\mathbf{j} + \lambda(5\mathbf{i} - \mathbf{j}) \\ &= (2 + 5\lambda)\mathbf{i} + (3 - \lambda)\mathbf{j} \end{aligned}$$

2. $$\begin{aligned} \mathbf{r} &= \mathbf{a} + \lambda\mathbf{b} \\ &= 3\mathbf{i} - 2\mathbf{j} + \lambda(\mathbf{i} + \mathbf{j}) \\ &= (3 + \lambda)\mathbf{i} + (-2 + \lambda)\mathbf{j} \end{aligned}$$

3. $$\begin{aligned} \mathbf{r} &= \mathbf{a} + \lambda\mathbf{b} \\ &= 5\mathbf{i} + 3\mathbf{j} + \lambda(-2\mathbf{j}) \\ &= 5\mathbf{i} + (3 - 2\lambda)\mathbf{j} \end{aligned}$$

4. $$\begin{aligned} \mathbf{r} &= \mathbf{a} + \lambda\mathbf{b} \\ &= 5\mathbf{j} + \lambda(3\mathbf{i} - 10\mathbf{j}) \\ &= 3\lambda\mathbf{i} + (5 - 10\lambda)\mathbf{j} \end{aligned}$$

5.
$$\begin{aligned}\mathbf{r} &= \mathbf{a}+\lambda\mathbf{b}\\ &= \begin{pmatrix}2\\-3\end{pmatrix}+\lambda\begin{pmatrix}1\\4\end{pmatrix}\\ &= \begin{pmatrix}2+\lambda\\-3+4\lambda\end{pmatrix}\end{aligned}$$

6.
$$\begin{aligned}\mathbf{r} &= \mathbf{a}+\lambda\mathbf{b}\\ &= \begin{pmatrix}0\\5\end{pmatrix}+\lambda\begin{pmatrix}5\\0\end{pmatrix}\\ &= \begin{pmatrix}5\lambda\\5\end{pmatrix}\end{aligned}$$

7.
$$\begin{aligned}\mathbf{r} &= \mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})\\ &= 5\mathbf{i}+3\mathbf{j}+\lambda(2\mathbf{i}-\mathbf{j}-5\mathbf{i}-3\mathbf{j})\\ &= 5\mathbf{i}+3\mathbf{j}+\lambda(-3\mathbf{i}-4\mathbf{j})\\ &= (5-3\lambda)\mathbf{i}+(3-4\lambda)\mathbf{j}\end{aligned}$$

8.
$$\begin{aligned}\mathbf{r} &= \mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})\\ &= 6\mathbf{i}+7\mathbf{j}+\lambda(-5\mathbf{i}+2\mathbf{j}-6\mathbf{i}-7\mathbf{j})\\ &= 6\mathbf{i}+7\mathbf{j}+\lambda(-11\mathbf{i}-5\mathbf{j})\\ &= (6-11\lambda)\mathbf{i}+(7-5\lambda)\mathbf{j}\end{aligned}$$

9.
$$\begin{aligned}\mathbf{r} &= \mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})\\ &= \begin{pmatrix}-6\\3\end{pmatrix}+\lambda\left(\begin{pmatrix}2\\4\end{pmatrix}-\begin{pmatrix}-6\\3\end{pmatrix}\right)\\ &= \begin{pmatrix}-6\\3\end{pmatrix}+\lambda\begin{pmatrix}8\\1\end{pmatrix}\\ &= \begin{pmatrix}-6+8\lambda\\3+\lambda\end{pmatrix}\end{aligned}$$

10.
$$\begin{aligned}\mathbf{r} &= \mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})\\ &= \begin{pmatrix}1\\-3\end{pmatrix}+\lambda\left(\begin{pmatrix}-3\\1\end{pmatrix}-\begin{pmatrix}1\\-3\end{pmatrix}\right)\\ &= \begin{pmatrix}1\\-3\end{pmatrix}+\lambda\begin{pmatrix}-4\\4\end{pmatrix}\\ &= \begin{pmatrix}1-4\lambda\\-3+4\lambda\end{pmatrix}\end{aligned}$$

11.
$$\begin{aligned}\mathbf{r} &= \mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})\\ &= \begin{pmatrix}1\\4\end{pmatrix}+\lambda\left(\begin{pmatrix}3\\-1\end{pmatrix}-\begin{pmatrix}1\\4\end{pmatrix}\right)\\ &= \begin{pmatrix}1\\4\end{pmatrix}+\lambda\begin{pmatrix}2\\-5\end{pmatrix}\\ &= \begin{pmatrix}1+2\lambda\\4-5\lambda\end{pmatrix}\end{aligned}$$

12.
$$\begin{aligned}\mathbf{r} &= \mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})\\ &= \begin{pmatrix}5\\0\end{pmatrix}+\lambda\left(\begin{pmatrix}-1\\-4\end{pmatrix}-\begin{pmatrix}5\\0\end{pmatrix}\right)\\ &= \begin{pmatrix}5\\0\end{pmatrix}+\lambda\begin{pmatrix}-6\\-4\end{pmatrix}\\ &= \begin{pmatrix}5-6\lambda\\-4\lambda\end{pmatrix}\end{aligned}$$

13. (a)
$$\begin{aligned}\overrightarrow{\mathrm{AB}} &= (2\mathbf{i}-3\mathbf{j}+1(\mathbf{i}-4\mathbf{j}))\\ &\quad-(2\mathbf{i}-3\mathbf{j}+-1(\mathbf{i}-4\mathbf{j}))\\ &= (1--1)(\mathbf{i}-4\mathbf{j})\\ &= 2(\mathbf{i}-4\mathbf{j})\\ &= 2\mathbf{i}-8\mathbf{j}\end{aligned}$$

(b)
$$\begin{aligned}|\overrightarrow{\mathrm{BC}}| &= |(2-1)(\mathbf{i}-4\mathbf{j})|\\ &= |(\mathbf{i}-4\mathbf{j})|\\ &= \sqrt{1^2+4^2}\\ &= \sqrt{17}\end{aligned}$$

(c)
$$\begin{aligned}\overrightarrow{\mathrm{AB}}:\overrightarrow{\mathrm{BC}} &= 2(\mathbf{i}-4\mathbf{j}):1(\mathbf{i}-4\mathbf{j})\\ &= 2:1\end{aligned}$$

14. (a) Vector equation:
$$\begin{aligned}\mathbf{r} &= 5\mathbf{i}-\mathbf{j}+\lambda(7\mathbf{i}+2\mathbf{j})\\ &= (5+7\lambda)\mathbf{i}+(-1+2\lambda)\mathbf{j}\end{aligned}$$

(b) Parametric equation:
$$x\mathbf{i}+y\mathbf{j} = (5+7\lambda)\mathbf{i}+(-1+2\lambda)\mathbf{j}$$

giving parametric equations
$$\begin{aligned}x &= 5+7\lambda\\ y &= -1+2\lambda\end{aligned}$$

(c) Cartesian equation:
$$\begin{aligned}2x &= 10+14\lambda\\ -7y &= 7-14\lambda\\ 2x-7y &= 17\end{aligned}$$

15. (a) Vector equation:
$$\begin{aligned}\mathbf{r} &= \begin{pmatrix}2\\-1\end{pmatrix}+\lambda\begin{pmatrix}-3\\4\end{pmatrix}\\ &= \begin{pmatrix}2-3\lambda\\-1+4\lambda\end{pmatrix}\end{aligned}$$

(b) Parametric equation:
$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2-3\lambda\\-1+4\lambda\end{pmatrix}$$

giving parametric equations
$$\begin{aligned}x &= 2-3\lambda\\ y &= -1+4\lambda\end{aligned}$$

(c) Cartesian equation:
$$\begin{aligned}4x &= 8-12\lambda\\ 3y &= -3+12\lambda\\ 4x+3y &= 5\end{aligned}$$

16. (a) Vector equation:
$$\begin{aligned}\mathbf{r} &= \begin{pmatrix}0\\3\end{pmatrix}+\lambda\begin{pmatrix}7\\-8\end{pmatrix}\\ &= \begin{pmatrix}7\lambda\\3-8\lambda\end{pmatrix}\end{aligned}$$

(b) Parametric equation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7\lambda \\ 3-8\lambda \end{pmatrix}$$

giving parametric equations

$$\begin{aligned} x &= 7\lambda \\ y &= 3-8\lambda \end{aligned}$$

(c) Cartesian equation:

$$\begin{aligned} 8x &= 56\lambda \\ 7y &= 21-56\lambda \\ 8x+7y &= 21 \end{aligned}$$

17. (a) Vector equation:

$$\begin{aligned} \mathbf{r} &= \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} 2-3\lambda \\ -5+2\lambda \end{pmatrix} \end{aligned}$$

(b) Cartesian equation:

$$\begin{aligned} 2x &= 4-6\lambda \\ 3y &= -15+6\lambda \\ 2x+3y &= -11 \end{aligned}$$

18. (a)

$$\begin{aligned} \overrightarrow{\mathrm{EF}} &= \left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 3 \end{pmatrix}\right) \\ &\quad - \left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 3 \end{pmatrix}\right) \\ &= (3-2)\begin{pmatrix} -1 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 3 \end{pmatrix} \end{aligned}$$

(b)

$$\begin{aligned} \overrightarrow{\mathrm{ED}} &= \left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} + -1\begin{pmatrix} -1 \\ 3 \end{pmatrix}\right) \\ &\quad - \left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 3 \end{pmatrix}\right) \\ &= (-1-2)\begin{pmatrix} -1 \\ 3 \end{pmatrix} \\ &= -3\begin{pmatrix} -1 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -9 \end{pmatrix} \end{aligned}$$

(c)

$$\begin{aligned} |\overrightarrow{\mathrm{DE}}| &= |\overrightarrow{\mathrm{ED}}| \\ &= \left|\begin{pmatrix} 3 \\ -9 \end{pmatrix}\right| \\ &= \sqrt{(3)^2 + -9^2} \\ &= \sqrt{90} \\ &= 3\sqrt{10} \end{aligned}$$

(d)

$$\begin{aligned} \overrightarrow{\mathrm{DE}} : \overrightarrow{\mathrm{EF}} &= -\overrightarrow{\mathrm{ED}} : \overrightarrow{\mathrm{EF}} \\ &= -\left(-3\begin{pmatrix} -1 \\ 3 \end{pmatrix}\right) : \begin{pmatrix} -1 \\ 3 \end{pmatrix} \\ &= 3\begin{pmatrix} -1 \\ 3 \end{pmatrix} : \begin{pmatrix} -1 \\ 3 \end{pmatrix} \\ &= 3 : 1 \end{aligned}$$

(e)

$$\begin{aligned} \overrightarrow{\mathrm{DE}} : \overrightarrow{\mathrm{FE}} &= \overrightarrow{\mathrm{DE}} : -\overrightarrow{\mathrm{EF}} \\ &= 3 : -1 \end{aligned}$$

(f)

$$\begin{aligned} |\overrightarrow{\mathrm{DE}}| : |\overrightarrow{\mathrm{FE}}| &= \left|3\begin{pmatrix} -1 \\ 3 \end{pmatrix}\right| : \left|-\begin{pmatrix} -1 \\ 3 \end{pmatrix}\right| \\ &= 3 : 1 \end{aligned}$$

(Remember, the magnitude of a vector can not be negative.)

19. The vector equation is

$$\mathbf{r} = 7\mathbf{i} - 2\mathbf{j} + \lambda(-2\mathbf{i} + 6\mathbf{j})$$

or

$$\mathbf{r} = (7-2\lambda)\mathbf{i} + (-2+6\lambda)\mathbf{j}$$

Point B: solve for λ for **i** and **j** components separately.

$$\begin{aligned} 7-2\lambda &= 1 & -2+6\lambda &= 16 \\ \lambda &= 3 & \lambda &= 3 \end{aligned}$$

Point B is on the line.

Point C:

$$\begin{aligned} 7-2\lambda &= 2 & -2+6\lambda &= 13 \\ \lambda &= \frac{5}{2} & \lambda &= \frac{5}{2} \end{aligned}$$

Point C is on the line.

Point D:

$$\begin{aligned} 7-2\lambda &= 8 & -2+6\lambda &= -7 \\ \lambda &= -\frac{1}{2} & \lambda &= -\frac{5}{6} \end{aligned}$$

Point D is not on the line.

Point E:

$$\begin{aligned} 7-2\lambda &= -2 & -2+6\lambda &= 5 \\ \lambda &= \frac{9}{2} & \lambda &= \frac{7}{6} \end{aligned}$$

Point E is not on the line.

20. The vector equation is

$$\mathbf{r} = \begin{pmatrix} 4 \\ -9 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

or

$$\mathbf{r} = \begin{pmatrix} 4-\lambda \\ -9+2\lambda \end{pmatrix}$$

Point G: solve for λ for **i** and **j** components separately.

$$\begin{aligned} 4-\lambda &= 5 & -9+2\lambda &= 9 \\ \lambda &= -1 & \lambda &= 9 \end{aligned}$$

Point G is not on the line.

Point H:

$$4 - \lambda = 0 \qquad -9 + 2\lambda = -1$$
$$\lambda = 4 \qquad \lambda = 4$$

Point H is on the line.

Point I:

$$4 - \lambda = -3 \qquad -9 + 2\lambda = 5$$
$$\lambda = 7 \qquad \lambda = 7$$

Point I is on the line.

21. (a)
$$\begin{aligned} 3 + 6\lambda &= -3 \\ \lambda &= -1 \\ -1 + 8\lambda &= a \\ a &= -9 \end{aligned}$$

(b)
$$\begin{aligned} -1 + 8\lambda &= 23 \\ \lambda &= 3 \\ 3 + 6\lambda &= b \\ b &= 21 \end{aligned}$$

(c)
$$\begin{aligned} 3 + 6\lambda &= -9 \\ \lambda &= -2 \\ -1 + 8\lambda &= c \\ c &= -17 \end{aligned}$$

(d)
$$\begin{aligned} -1 + 8\lambda &= -21 \\ \lambda &= -\frac{5}{2} \\ 3 + 6\lambda &= d \\ d &= -12 \end{aligned}$$

(e)
$$\begin{aligned} 3 + 6\lambda &= 12 \\ \lambda &= \frac{3}{2} \\ -1 + 8\lambda &= e \\ e &= 11 \end{aligned}$$

(f)
$$\begin{aligned} 3 + 6\lambda &= f \\ -1 + 8\lambda &= f \\ 12 + 24\lambda &= 4f \\ 3 - 24\lambda &= -3f \\ f &= 15 \end{aligned}$$

22. Simply retain the same coefficients for λ:

$$\mathbf{r} = (5 + \lambda)\mathbf{i} + (-6 - \lambda)\mathbf{j}$$

23. Simply retain the same coefficients for λ:

$$\mathbf{r} = \begin{pmatrix} 6 + 3\lambda \\ 5 - 4\lambda \end{pmatrix}$$

24. The parametric equations are:

$$\begin{aligned} x &= 2 + 6t \\ y &= 12 - 10t \end{aligned}$$

Eliminating t:

$$\begin{aligned} 5x &= 10 + 30t \\ 3y &= 36 - 30t \\ 5x + 3y &= 46 \end{aligned}$$

25. At A, on the x-axis, the $\mathbf{j}$ component is zero:

$$\begin{aligned} 8 - 2\lambda &= 0 \\ \lambda &= 4 \\ \mathrm{A} &= 2\mathbf{i} + 8\mathbf{j} + 4(\mathbf{i} - 2\mathbf{j}) \\ &= 6\mathbf{i} \end{aligned}$$

At B, on the y-axis, the $\mathbf{i}$ component is zero:

$$\begin{aligned} 2 + \lambda &= 0 \\ \lambda &= -2 \\ \mathrm{B} &= 2\mathbf{i} + 8\mathbf{j} - 2(\mathbf{i} - 2\mathbf{j}) \\ &= 12\mathbf{j} \end{aligned}$$

26. At A, on the x-axis, the $\mathbf{j}$ component is zero:

$$\begin{aligned} -4 - \lambda &= 0 \\ \lambda &= -4 \\ \mathrm{A} &= \begin{pmatrix} 5 \\ -4 \end{pmatrix} - 4\begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \end{aligned}$$

At B,

$$\begin{aligned} 5 + 2\lambda &= 11 \\ \lambda &= 3 \\ -4 - \lambda &= cc \qquad = -7 \end{aligned}$$

27. Equate L_1 and L_2; solve for λ or μ then substitute back into the corresponding equation to find the point of intersection:

$$\begin{aligned} 14\mathbf{i} - \mathbf{j} + \lambda(5\mathbf{i} - 4\mathbf{i}) &= 9\mathbf{i} - 4\mathbf{j} + \mu(-4\mathbf{i} + 6\mathbf{j}) \\ (14 + 5\lambda - 9 + 4\mu)\mathbf{i} &= (-4 + 6\mu + 1 + 4\lambda)\mathbf{j} \\ (5 + 5\lambda + 4\mu)\mathbf{i} &= (-3 + 4\lambda + 6\mu)\mathbf{j} \\ 5 + 5\lambda + 4\mu &= 0 \\ -3 + 4\lambda + 6\mu &= 0 \\ 15 + 15\lambda + 12\mu &= 0 \\ 6 - 8\lambda - 12\mu &= 0 \\ 21 + 7\lambda &= 0 \\ \lambda &= -3 \\ \mathbf{r} &= 14\mathbf{i} - \mathbf{j} - 3(5\mathbf{i} - 4\mathbf{j}) \\ &= -\mathbf{i} + 11\mathbf{j} \end{aligned}$$

28. Equate L_1 and L_2; solve for λ or μ then substitute back into the corresponding equation to find

the point of intersection:

$$\begin{aligned}\begin{pmatrix}-3\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\end{pmatrix}&=\begin{pmatrix}-10\\2\end{pmatrix}+\mu\begin{pmatrix}-4\\1\end{pmatrix}\\(-3+\lambda+10+4\mu)\mathbf{i}&=(2+\mu-4+\lambda)\mathbf{j}\\(7+\lambda+4\mu)\mathbf{i}&=(-2+\lambda+\mu)\mathbf{j}\\7+\lambda+4\mu&=0\\-2+\lambda+\mu&=0\\9+3\mu&=0\\\mu&=-3\\\mathbf{r}&=\begin{pmatrix}-10\\2\end{pmatrix}-3\begin{pmatrix}-4\\1\end{pmatrix}\\&=\begin{pmatrix}2\\-1\end{pmatrix}\end{aligned}$$

29. Equate L_1 and L_2; solve for λ or μ then substitute back into the corresponding equation to find the point of intersection:

$$\begin{aligned}\begin{pmatrix}-1\\0\end{pmatrix}+\lambda\begin{pmatrix}4\\-10\end{pmatrix}&=\begin{pmatrix}-5\\-9\end{pmatrix}+\mu\begin{pmatrix}1\\7\end{pmatrix}\\(-1+4\lambda+5-\mu)\mathbf{i}&=(-9+7\mu+10\lambda)\mathbf{j}\\(4+4\lambda-\mu)\mathbf{i}&=(-9+10\lambda+7\mu)\mathbf{j}\\4+4\lambda-\mu&=0\\-9+10\lambda+7\mu&=0\\28+28\lambda-7\mu&=0\\19+38\lambda&=0\\\lambda&=-\frac{1}{2}\\\mathbf{r}&=\begin{pmatrix}-1\\0\end{pmatrix}-\frac{1}{2}\begin{pmatrix}4\\-10\end{pmatrix}\\&=\begin{pmatrix}-3\\5\end{pmatrix}\end{aligned}$$

30. Use points A and C to find the vector equation of the line:

$$\begin{aligned}\mathbf{r}&=2\mathbf{i}+3\mathbf{j}+\lambda\left((5\mathbf{i}-4\mathbf{j})-(2\mathbf{i}+3\mathbf{j})\right)\\&=2\mathbf{i}+3\mathbf{j}+\lambda(3\mathbf{i}-7\mathbf{j})\\&=(2+3\lambda)\mathbf{i}+(3-7\lambda)\mathbf{j}\end{aligned}$$

At point B:

$$\begin{aligned}3-7\lambda&=7\\\lambda&=-\frac{4}{7}\\2+3\lambda&=b\\b&=\frac{2}{7}\end{aligned}$$

At point D:

$$\begin{aligned}2+3\lambda&=-2\\\lambda&=-\frac{4}{3}\\3-7\lambda&=d\\d&=\frac{37}{3}\end{aligned}$$

31. $\lambda\begin{pmatrix}1\\4\end{pmatrix}$ and $\mu\begin{pmatrix}2\\c\end{pmatrix}$ both represent the direction of the line, so

$$\lambda\begin{pmatrix}1\\4\end{pmatrix}=\mu\begin{pmatrix}2\\c\end{pmatrix}$$

from the **i** components:

$$\lambda=2\mu$$

so for the **j** components:

$$\begin{aligned}4\lambda&=c\mu\\4(2\mu)&=c\mu\\8\mu&=c\mu\\c&=8\end{aligned}$$

$\begin{pmatrix}9\\d\end{pmatrix}$ is a point on the line $\mathbf{r}=\begin{pmatrix}5\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\4\end{pmatrix}$ so

$$\begin{pmatrix}9\\d\end{pmatrix}=\begin{pmatrix}5\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\4\end{pmatrix}$$

i components:

$$\begin{aligned}9&=5+\lambda\\\lambda&=4\end{aligned}$$

j components:

$$\begin{aligned}d&=3+4\lambda\\&=19\end{aligned}$$

32. $e\mathbf{i}+5\mathbf{j}$ is a point on the line $\mathbf{r}=\mathbf{i}-3\mathbf{j}+\lambda(3\mathbf{i}+4\mathbf{j})$ so

$$e\mathbf{i}+5\mathbf{j}=\mathbf{i}-3\mathbf{j}+\lambda(3\mathbf{i}+4\mathbf{j})$$

j components:

$$\begin{aligned}5&=-3+4\lambda\\\lambda&=2\end{aligned}$$

i components:

$$\begin{aligned}e&=1+3\lambda\\&=7\end{aligned}$$

$\lambda(3\mathbf{i}+4\mathbf{i})$ and $\mu(\mathbf{i}+f\mathbf{j})$ both represent the direction of the line, so

$$\lambda(3\mathbf{i}+4\mathbf{i})=\mu(\mathbf{i}+f\mathbf{j})$$

from the **i** components:

$$3\lambda=\mu$$

so for the **j** components:

$$\begin{aligned}4\lambda&=f\mu\\4\lambda&=f(3\lambda)\\f&=\frac{4}{3}\end{aligned}$$

33. Convert each to a Cartesian equation.

Set ①:

$$\begin{aligned} x &= 1 + 2\lambda \\ y &= \lambda + 3 \\ -2y &= -6 - 2\lambda \\ x - 2y &= -5 \end{aligned}$$

Set ②:

$$\begin{aligned} x &= 2\lambda - 2 \\ y &= 1 + \lambda \\ -2y &= -2\lambda - 2 \\ x - 2y &= -4 \end{aligned}$$

Set ③:

$$\begin{aligned} x &= 8 + 2\lambda \\ y &= 6 + \lambda \\ -2y &= -12 - 2\lambda \\ x - 2y &= -4 \end{aligned}$$

Sets ② and ③ represent the same line. Set ① is the odd one out.

Exercise 4C

1. (a)
$$\begin{aligned} \overrightarrow{AB} &= \mathbf{r}_B - \mathbf{r}_A \\ &= 7\mathbf{i} - 15\mathbf{j} - (-9\mathbf{i} + 5\mathbf{j}) \\ &= (16\mathbf{i} - 20\mathbf{j})\text{m} \end{aligned}$$
(b)
$$\begin{aligned} {}_A\mathbf{v}_B &= \mathbf{v}_A - \mathbf{v}_B \\ &= 5\mathbf{i} - 2\mathbf{j} - (\mathbf{i} + 3\mathbf{j}) \\ &= (4\mathbf{i} - 5\mathbf{j})\text{m/s} \end{aligned}$$
(c) $\overrightarrow{AB} = 4{}_A\mathbf{v}_B$
A and B collide at $t = 4$ seconds.

2. (a)
$$\begin{aligned} \overrightarrow{AB} &= \mathbf{r}_B - \mathbf{r}_A \\ &= 11\mathbf{i} - 10\mathbf{j} - (-9\mathbf{i}) \\ &= (20\mathbf{i} - 10\mathbf{j})\text{m} \end{aligned}$$
(b)
$$\begin{aligned} {}_A\mathbf{v}_B &= \mathbf{v}_A - \mathbf{v}_B \\ &= 6\mathbf{i} + 4\mathbf{j} - (-2\mathbf{i} + 8\mathbf{j}) \\ &= (8\mathbf{i} - 4\mathbf{j})\text{m/s} \end{aligned}$$
(c) $\overrightarrow{AB} = 2.5{}_A\mathbf{v}_B$
A and B collide at $t = 2.5$ seconds.

3. (a)
$$\begin{aligned} \overrightarrow{AB} &= \mathbf{r}_B - \mathbf{r}_A \\ &= 35\mathbf{i} - 18\mathbf{j} - (-\mathbf{i} + 30\mathbf{j}) \\ &= (36\mathbf{i} - 48\mathbf{j})\text{m} \end{aligned}$$
(b)
$$\begin{aligned} {}_A\mathbf{v}_B &= \mathbf{v}_A - \mathbf{v}_B \\ &= 9\mathbf{i} + 2\mathbf{j} - (3\mathbf{i} + 10\mathbf{j}) \\ &= (6\mathbf{i} - 8\mathbf{j})\text{m/s} \end{aligned}$$
(c) $\overrightarrow{AB} = 6{}_A\mathbf{v}_B$
A and B collide at $t = 6$ seconds.

4. (a)
$$\begin{aligned} \overrightarrow{AB} &= \mathbf{r}_B - \mathbf{r}_A \\ &= 56\mathbf{i} + 4\mathbf{j} - (10\mathbf{i} + 32\mathbf{j}) \\ &= (46\mathbf{i} - 28\mathbf{j})\text{m} \end{aligned}$$
(b)
$$\begin{aligned} {}_A\mathbf{v}_B &= \mathbf{v}_A - \mathbf{v}_B \\ &= 10\mathbf{i} + 2\mathbf{j} - (-2\mathbf{i} + 12\mathbf{j}) \\ &= (12\mathbf{i} - 10\mathbf{j})\text{m/s} \end{aligned}$$
(c) $\overrightarrow{AB}$ is not a scalar multiple of ${}_A\mathbf{v}_B$ since solving $\overrightarrow{AB} = t{}_A\mathbf{v}_B$ for t gives $t = \frac{23}{6}$ for the $\mathbf{i}$ components and $t = \frac{14}{5}$ for the $\mathbf{j}$ components.
A and B do not collide.

5. (a)
$$\begin{aligned} \overrightarrow{AB} &= \mathbf{r}_B - \mathbf{r}_A \\ &= 30\mathbf{i} - 10\mathbf{j} - (-2\mathbf{i} + 22\mathbf{j}) \\ &= (32\mathbf{i} - 32\mathbf{j})\text{m} \end{aligned}$$
(b)
$$\begin{aligned} {}_A\mathbf{v}_B &= \mathbf{v}_A - \mathbf{v}_B \\ &= 4\mathbf{i} + 6\mathbf{j} - (12\mathbf{i} - 2\mathbf{j}) \\ &= (-8\mathbf{i} + 8\mathbf{j})\text{m/s} \end{aligned}$$
(c) $\overrightarrow{AB}$ is a scalar multiple of ${}_A\mathbf{v}_B$ but solving $\overrightarrow{AB} = t{}_A\mathbf{v}_B$ for t gives $t = -4$. This suggests that rather than heading for a collision A and B are diverging from a common point from which they originated 4 seconds before time $t = 0$.
A and B do not collide.

6. (a)
$$\begin{aligned} \overrightarrow{PQ} &= \mathbf{r}_Q - \mathbf{r}_P \\ &= 10\mathbf{i} + 20\mathbf{j} - (25\mathbf{i} - 22\mathbf{j}) \\ &= (-15\mathbf{i} + 42\mathbf{j})\text{m} \end{aligned}$$
(b) $\overrightarrow{PQ}$ is the displacement of Q relative to P (i.e. from P to Q).

(c)
$$\begin{aligned}{}_{\mathrm{A}}\mathbf{v}_{\mathrm{B}} &= \mathbf{v}_{\mathrm{A}} - \mathbf{v}_{\mathrm{B}}\\ &= 20\mathbf{i} + 44\mathbf{j} - (40\mathbf{i} - 12\mathbf{j})\\ &= (-20\mathbf{i} + 56\mathbf{j})\text{m/s}\\ \overrightarrow{\mathrm{PQ}} &= t_{\mathrm{A}}\mathbf{v}_{\mathrm{B}}\\ -15\mathbf{i} + 42\mathbf{j} &= t(-20\mathbf{i} + 56\mathbf{j})\end{aligned}$$

Equating **i** and **j** components separately:

$$-15 = -20t \qquad 42 = 56t$$
$$t = \frac{3}{4} \qquad t = \frac{3}{4}$$

A and B will collide at $t = 0.75$s.

7. (a) The position of A at 8:00am is

$$\begin{aligned}\mathbf{r}_{\mathrm{A}} &= 5\mathbf{i} + 28\mathbf{j} + 0.5(20\mathbf{i} + 4\mathbf{j})\\ &= (15\mathbf{i} + 30\mathbf{j})\text{km}\end{aligned}$$

$$\begin{aligned}\overrightarrow{\mathrm{BA}} &= \mathbf{r}_{\mathrm{A}} - \mathbf{r}_{\mathrm{B}}\\ &= (15\mathbf{i} + 30\mathbf{j}) - (20\mathbf{i} + 70\mathbf{j})\\ &= (-5\mathbf{i} - 40\mathbf{j})\text{km}\end{aligned}$$

(b)
$$\begin{aligned}{}_{\mathrm{B}}\mathbf{v}_{\mathrm{A}} &= \mathbf{v}_{\mathrm{B}} - \mathbf{v}_{\mathrm{A}}\\ &= (x\mathbf{i} - 12\mathbf{j}) - (20\mathbf{i} + 4\mathbf{j})\\ &= ((x - 20)\mathbf{i} - 16\mathbf{j})\text{km/h}\\ \overrightarrow{\mathrm{BA}} &= t_{\mathrm{B}}\mathbf{v}_{\mathrm{A}}\\ -5\mathbf{i} - 40\mathbf{j} &= t((x - 20)\mathbf{i} - 16\mathbf{j})\end{aligned}$$

Equating **j** components:

$$\begin{aligned}-40 &= -16t\\ t &= 2.5\,\text{hours}\end{aligned}$$

A and B will collide 2.5 hours after 8:00am, i.e. at 10:30am.
Equating **i** components:

$$\begin{aligned}-5 &= t(x - 20)\\ -5 &= 2.5(x - 20)\\ x - 20 &= -2\\ x &= 18\end{aligned}$$

8. Let **j** represent 1km due north and **i** 1km due east.

$$\begin{aligned}\overrightarrow{\mathrm{BA}} &= 6\mathbf{j}\text{km}\\ \mathbf{v}_{\mathrm{A}} &= (p\mathbf{i})\text{km/h}\\ \mathbf{v}_{\mathrm{B}} &= (8\sqrt{2}\cos 45°\mathbf{i} + 8\sqrt{2}\sin 45°\mathbf{j})\\ &= (8\mathbf{i} + 8\mathbf{j})\text{km/h}\\ {}_{\mathrm{B}}\mathbf{v}_{\mathrm{A}} &= \mathbf{v}_{\mathrm{B}} - \mathbf{v}_{\mathrm{A}}\\ &= (8\mathbf{i} + 8\mathbf{j}) - (p\mathbf{i})\\ &= ((8 - p)\mathbf{i} + 8\mathbf{j})\text{km/h}\\ \overrightarrow{\mathrm{BA}} &= t_{\mathrm{B}}\mathbf{v}_{\mathrm{A}}\\ 6\mathbf{j} &= t((8 - p)\mathbf{i} + 8\mathbf{j})\end{aligned}$$

Equating **i** and **j** components separately:

$$0 = t(8 - p) \qquad 6 = 8t$$
$$p = 8 \qquad t = 0.75$$

Tankers collide in $\frac{3}{4}$ hour, at 8:45am.

9. Let F represent the fishing vessel and C represent the coastguard boat. Let the desired velocity of the coastguard boat be $(a\mathbf{i} + b\mathbf{j})$km/h.

$$\begin{aligned}\overrightarrow{\mathrm{CF}} &= (21\mathbf{i} + 21\mathbf{j})\text{km}\\ {}_{\mathrm{C}}\mathbf{v}_{\mathrm{F}} &= \mathbf{v}_{\mathrm{C}} - \mathbf{v}_{\mathrm{F}}\\ &= (a\mathbf{i} + b\mathbf{j}) - (8\mathbf{i} - 4\mathbf{j})\\ &= ((a - 8)\mathbf{i} + (b + 4)\mathbf{j})\text{km/h}\end{aligned}$$

$$\begin{aligned}\overrightarrow{\mathrm{CF}} &= t_{\mathrm{C}}\mathbf{v}_{\mathrm{F}}\\ 21\mathbf{i} + 21\mathbf{j} &= t((a - 8)\mathbf{i} + (b + 4)\mathbf{j})\\ 21 &= t(a - 8)\\ 21 &= t(b + 4)\\ a - 8 &= b + 4\\ a &= b + 12\end{aligned}$$

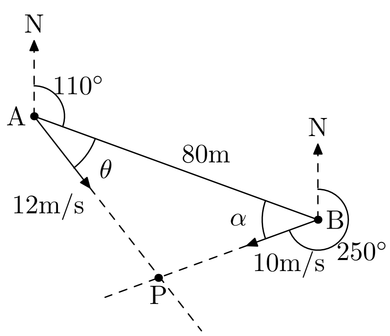
$$\begin{aligned}a^2 + b^2 &= (4\sqrt{29})^2\\ (b + 12)^2 + b^2 &= 464\\ 2b^2 + 24b - 320 &= 0\\ 2(b - 8)(b + 20) &= 0\\ b &= 8\end{aligned}$$

(disregarding the root at $b = -20$ because this would result in a negative value of time when substituted into $21 = t(b - 4)$)

$$\begin{aligned}a &= b + 12\\ &= 20\\ 21 &= t(b + 4)\\ 12t &= 21\\ t &= \frac{7}{4}\end{aligned}$$

The patrol boat should set a velocity of $(20\mathbf{i} + 8\mathbf{j})$km/h in order to intercept the fishing vessel in $1\frac{3}{4}$ hours, i.e. at 1:45pm.

10. Let P be the point where the projectiles collide.



$$\begin{aligned}\alpha &= 360 - 250 - (180 - 110)\\ &= 40°\\ \text{BP} &= 10t\\ \text{AP} &= 12t\\ \frac{\sin\theta}{10t} &= \frac{\sin\alpha}{12t}\\ \sin\theta &= \frac{10t\sin 40°}{12t}\\ \theta &= \sin^{-1}\frac{5\sin 40°}{6}\\ &= 32.4°\end{aligned}$$

The second object should be projected on a bearing of $110 + 32 = 142°$.

$$\begin{aligned}\frac{\text{AP}}{\sin\alpha} &= \frac{80}{\sin(180-\alpha-\theta)}\\ \text{AP} &= \frac{80\sin\alpha}{\sin(180-\alpha-\theta)}\\ &= \frac{80\sin 40°}{\sin(180-40-32.4)°}\\ &= 54.0\text{m}\\ t &= \frac{54}{12}\\ &= 4.5\text{s}\end{aligned}$$

You could also do this using component vectors, but it's rather more work:

Let **i** represent 1m due east and **j** represent 1m due north.

$$\begin{aligned}\overrightarrow{\text{AB}} &= 80\sin 110° + 80\cos 110°\\ &= 75.1754\mathbf{i} - 27.3616\mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{v}_\text{B} &= 10\sin 250°\mathbf{i} + 10\cos 250°\\ &= -9.3969\mathbf{i} - 3.4202\mathbf{j}\end{aligned}$$

$$\mathbf{v}_\text{A} = a\mathbf{i} + b\mathbf{j}$$

$$\begin{aligned}{}_\text{A}\mathbf{v}_\text{B} &= \mathbf{v}_\text{A} - \mathbf{v}_\text{B}\\ &= (a\mathbf{i} + b\mathbf{j}) - (-9.3969\mathbf{i} - 3.4202\mathbf{j})\\ &= (a + 9.3969)\mathbf{i} + (b + 3.4202)\mathbf{j}\end{aligned}$$

$$\begin{aligned}\overrightarrow{\text{AB}} &= t_\text{A}\mathbf{v}_\text{B}\\ 75.1754\mathbf{i} - 27.3616\mathbf{j} &= t((a + 9.3969)\mathbf{i}\\ &\quad + (b + 3.4202)\mathbf{j})\\ 75.1754 &= t(a + 9.3969)\\ t &= \frac{75.1754}{a + 9.3969}\\ -27.3616 &= t(b + 3.4202)\\ t &= \frac{-27.3616}{b + 3.4202}\\ \frac{75.1754}{a + 9.3969} &= \frac{-27.3616}{b + 3.4202}\\ 75.1754(b + 3.4202) &= -27.3616(a + 9.3969)\\ b + 3.4202 &= -0.3640(a + 9.3969)\\ &= -0.3640a - 3.4202\\ b &= -0.3640a - 6.8404\end{aligned}$$

$$\begin{aligned}a^2 + b^2 &= 12^2\\ a^2 + (-0.3640a - 6.8404)^2 &= 144\\ a &= -11.7206\\ \text{or } a &= 7.3237\end{aligned}$$

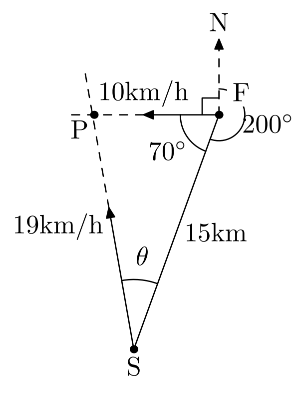
The first root will give negative t when substituted into $t = \frac{75.1754}{a+9.3969}$ so we can discard it.

$$\begin{aligned}t &= \frac{75.1754}{(7.3237) + 9.3969}\\ &= 4.4960\\ b &= -0.3640(7.3237) - 6.8404\\ &= -9.5060\\ \mathbf{v}_\text{A} &= 7.3237\mathbf{i} - 9.5060\mathbf{j}\\ \tan\theta &= \frac{7.3237}{-9.5060}\\ \theta &= 142.39°\end{aligned}$$

(Although $\tan^{-1}\frac{7.3237}{-9.5060}$ on the calculator gives $-37.61°$ that is an angle with negative sine and positive cosine; we want the angle with positive sine and negative cosine, so we must add 180°.)

The object should be projected on a bearing of 142° and collision will occur after 4.5s.

11. Let F represent the *Big Freezer* and S the *Jolly Snapper*. Let P be the point of interception.



$$\begin{aligned}
\text{FP} &= 10t\\
\text{SP} &= 19t\\
\frac{\sin\theta}{10t} &= \frac{\sin 70°}{19t}\\
\sin\theta &= \frac{10t\sin 70°}{19t}\\
\theta &= \sin^{-1}\frac{10\sin 70°}{19}\\
&= 29.6°
\end{aligned}$$

The Jolly Snapper should steam on a bearing of $(200+180)-30 = 350°$.

$$\begin{aligned}
\frac{\text{SP}}{\sin 70°} &= \frac{15}{\sin(180-70-\theta)}\\
\text{SP} &= \frac{15\sin 70°}{\sin(80.4°)}\\
&= 14.3\text{km}\\
t &= \frac{14.3}{19}\\
&= 0.752\text{hours}\\
&\approx 45\text{minutes}
\end{aligned}$$

Jolly Snapper should be travel on a bearing of 350° and will intercept *Big Freezer* after 0.752 hours, that is at about 6:45am.

Again with component vectors:

Let **i** represent 1km due east and **j** represent 1km due north.

$$\begin{aligned}
\overrightarrow{\text{FS}} &= 15\sin 200° + 15\cos 200°\\
&= -5.130\mathbf{i} - 14.095\mathbf{j}
\end{aligned}$$

$$\mathbf{v}_\text{F} = -10\mathbf{i}$$

$$\mathbf{v}_\text{S} = a\mathbf{i} + b\mathbf{j}$$

$$\begin{aligned}
{}_\text{F}\mathbf{v}_\text{S} &= \mathbf{v}_\text{F} - \mathbf{v}_\text{S}\\
&= -10\mathbf{i} - (a\mathbf{i} + b\mathbf{j})\\
&= (-10-a)\mathbf{i} - b\mathbf{j}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{\text{FS}} &= t\,{}_\text{F}\mathbf{v}_\text{S}\\
-5.130\mathbf{i} - 14.095\mathbf{j} &= t((-10-a)\mathbf{i} - b\mathbf{j})\\
-5.130 &= t(-10-a)\\
t &= \frac{-5.130}{-10-a}\\
-14.095 &= -tb\\
t &= \frac{-14.095}{-b}\\
\frac{-5.130}{-10-a} &= \frac{-14.095}{-b}\\
5.130b &= 14.095(10+a)\\
b &= 2.7475(10+a)
\end{aligned}$$

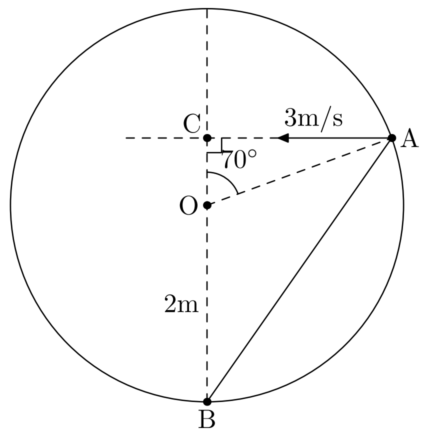
$$\begin{aligned}
a^2 + b^2 &= 19^2\\
a^2 + (2.7475(10+a))^2 &= 19^2\\
a &= -14.478\\
\text{or } a &= -3.182
\end{aligned}$$

The first root will give negative t when substituted into $t = \frac{-14.095}{-10-a}$ so we can discard it.

$$\begin{aligned}
t &= \frac{-5.130}{-10-(-3.182)}\\
&= 0.752\text{hours}\\
b &= 2.7475(10+(-3.182))\\
&= 18.732\\
\mathbf{v}_\text{S} &= -3.182\mathbf{i} + 18.732\mathbf{j}\\
\tan\theta &= \frac{-3.182}{18.732}\\
\theta &= -9.64°
\end{aligned}$$

Jolly Snapper should be travel on a bearing of 350° and will intercept *Big Freezer* after 0.752 hours, that is at about 6:45am.

12. Let C be the point where the marble from A crosses the diameter from B, thus:



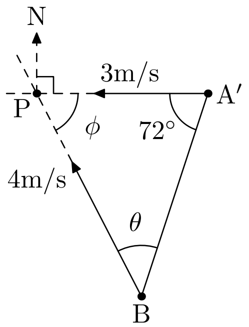
$$\begin{aligned}
\text{AC} &= 2\sin 70°\\
&= 1.88\text{m}\\
\text{BC} &= 2 + 2\cos 70°\\
&= 2.68\text{m}
\end{aligned}$$

Let A′ be the position of the first marble after it has travelled one metre.

$$\begin{aligned}
\text{A}'\text{C} &= \text{AC} - 1\\
&= 0.88\text{m}\\
\tan\angle\text{BA}'\text{C} &= \frac{\text{BC}}{\text{A}'\text{C}}\\
\angle\text{BA}'\text{C} &= \tan^{-1}\frac{2.68}{0.88}\\
&= 71.9°
\end{aligned}$$

Let P be the point where the collision occurs.

Redrawing with only the salient features gives:



$$\frac{\sin\theta}{3t} = \frac{\sin 71.9^\circ}{4t}$$
$$\sin\theta = \frac{3t\sin 71.9^\circ}{4t}$$
$$\theta = \sin^{-1}(0.75\sin 71.9^\circ) = 45.5^\circ$$
$$\phi = 180 - 71.9 - 45.5 = 62.7^\circ \approx 63^\circ$$

Thus the bearing from P to B is $90 + 63 = 153^\circ$ and the bearing from B to P is $153 + 180 = 333^\circ$.

B should roll her marble on a bearing of 333°.

Using component vectors here results in similar level of complexity:

The position of child A's marble before it is played is:

$$\mathbf{r}_A = 2\sin 70^\circ\mathbf{i} + 2\cos 70^\circ\mathbf{j} = (1.879\mathbf{i} + 0.684\mathbf{j})\text{m}$$

After it has rolled 1m this becomes

$$\mathbf{r}_A = 1.879\mathbf{i} + 0.684\mathbf{j} - 1\mathbf{i} = (0.879\mathbf{i} + 0.684\mathbf{j})\text{m}$$

Marble A has velocity

$$\mathbf{v}_A = (-3\mathbf{i})\text{m/s}$$

The position of child B's marble is initially

$$\mathbf{r}_B = -2\mathbf{j}\text{m}$$

Let $\mathbf{v}_B = (a\mathbf{i} + b\mathbf{j})$m/s.

$$\overrightarrow{BA} = \mathbf{r}_A - \mathbf{r}_B = 0.879\mathbf{i} + 0.684\mathbf{j} - (-2\mathbf{j}) = 0.879\mathbf{i} + 2.684\mathbf{j}$$

$$_B\mathbf{v}_A = \mathbf{v}_B - \mathbf{v}_A = (a\mathbf{i} + b\mathbf{j}) - (-3\mathbf{i}) = (a+3)\mathbf{i} + b\mathbf{j}$$

$$\overrightarrow{BA} = t\,_B\mathbf{v}_A$$
$$0.879\mathbf{i} + 2.684\mathbf{j} = t((a+3)\mathbf{i} + b\mathbf{j})$$
$$0.879 = t(a+3)$$
$$t = \frac{0.879}{a+3}$$
$$2.684 = tb$$
$$t = \frac{2.684}{b}$$
$$\frac{0.879}{a+3} = \frac{2.684}{b}$$
$$0.879b = 2.684(a+3)$$
$$b = 3.052(a+3)$$

$$a^2 + b^2 = 4^2$$
$$a^2 + (3.052(a+3))^2 = 4^2$$
$$a = -3.583 \quad \text{or } a = -1.836$$

The first root will give negative t when substituted into $t = \frac{0.879}{a+3}$ so we can discard it.

(We don't actually need to calculate t since the question does not ask for it, but here it is anyway.)

$$t = \frac{0.879}{(-1.836) + 3} = 0.755\text{s}$$

$$b = 3.052((-1.836) + 3) = 3.554$$
$$\mathbf{v}_B = -1.836\mathbf{i} + 3.554\mathbf{j}$$
$$\tan\theta = \frac{-1.836}{3.554}$$
$$\theta = -27.32^\circ$$

Child B should play her marble on a bearing of $360 - 27 = 333^\circ$.

Exercise 4D

1. (a) $|\mathbf{r}| = 13$
   (b) Point A:

$$\begin{aligned}|(7\mathbf{i} - 7\mathbf{j})| &= \sqrt{7^2 + 7^2}\\ &\approx 9.9\\ 9.9 &< 13\end{aligned}$$

   Point A lies inside the circle.
   Point B:

$$\begin{aligned}|(12\mathbf{i} - 5\mathbf{j})| &= \sqrt{12^2 + 5^2}\\ &= 13\end{aligned}$$

   Point B lies on the circle.

2. A: This looks like it might be a circle, but we need to check to make sure it has a positive radius:

$$\begin{aligned}x^2 + y^2 - 2x + 4y &= 6\\ (x-1)^2 - 1 + (y+2)^2 - 4 &= 6\\ (x-1)^2 + (y+2)^2 &= 11\end{aligned}$$

   This represents a circle radius $\sqrt{11}$ centred at $(1, -2)$.
   B: This is not a circle because the x^2 and y^2 terms have different coefficients. (It might be an ellipse.)
   C: This represents a circle radius $\sqrt{6}$ centred at the origin.
   D: This looks like it might be a circle. Checking:

$$\begin{aligned}x^2 + y^2 + 8x &= 10\\ (x-4)^2 - 16 + y^2 &= 10\\ (x-4)^2 + y^2 &= 26\end{aligned}$$

   This represents a circle radius $\sqrt{26}$ centred at $(4, 0)$.
   E: This is not a circle because the x^2 and y^2 terms have different coefficients. (It might be a hyperbola.)
   F: This is not a circle because it has an xy term.

3. (a) $|r| = 25$
   (b) Point A:

$$\begin{aligned}|19\mathbf{i} - 18\mathbf{j}| &= \sqrt{19^2 + 18^2}\\ &\approx 26.2\\ 26.2 &> 25\end{aligned}$$

   Point A lies outside the circle.
   Point B:

$$\begin{aligned}|-20\mathbf{i} + 15\mathbf{j}| &= \sqrt{20^2 + 15^2}\\ &= 25\end{aligned}$$

   Point B lies on the circle.
   Point C:

$$\begin{aligned}|14\mathbf{i} + 17\mathbf{j}| &= \sqrt{14^2 + 17^2}\\ &\approx 22.0\\ 22.0 &< 25\end{aligned}$$

   Point C lies inside the circle.
   Point D:

$$\begin{aligned}|-24\mathbf{i} - 7\mathbf{j}| &= \sqrt{24^2 + 7^2}\\ &= 25\end{aligned}$$

   Point D lies on the circle.

4. $x^2 + y^2 = 100$
   Point A:

$$\begin{aligned}(-6)^2 + a^2 &= 100\\ a &= \sqrt{100 - 36}\\ &= 8\end{aligned}$$

   Point B:

$$\begin{aligned}3^2 + b^2 &= 100\\ b &= \sqrt{100 - 9}\\ &= \sqrt{91}\end{aligned}$$

   Point C:

$$\begin{aligned}0^2 + c^2 &= 100\\ c &= -\sqrt{100 - 0}\\ &= -10\end{aligned}$$

   Point D:

$$\begin{aligned}d^2 + 5^2 &= 100\\ d &= -\sqrt{100 - 25}\\ &= -\sqrt{75}\\ &= -5\sqrt{3}\end{aligned}$$

5. The equation is

$$|\mathbf{r} - (-7\mathbf{i} + 4\mathbf{j})| = 4\sqrt{5}$$

$$\begin{aligned}|(\mathbf{i} + 8\mathbf{j}) - (-7\mathbf{i} + 4\mathbf{j})| &= |8\mathbf{i} + 4\mathbf{j}|\\ &= \sqrt{8^2 + 4^2}\\ &= 4\sqrt{5}\end{aligned}$$

   Point A lies on the circle.

6. (a) $(x-2)^2 + (y - -3)^2 = 5^2$
   $(x-2)^2 + (y+3)^2 = 25$

(b) $(x-3)^2+(y-2)^2=7^2$

$(x-3)^2+(y-2)^2=49$

(c) $(x--10)^2+(y-2)^2=(3\sqrt{5})^2$

$(x+10)^2+(y-2)^2=45$

(d) $(x--1)^2+(y--1)^2=6^2$

$(x+1)^2+(y+1)^2=36$

7. (a)
$$\begin{aligned}(x-3)^2+(y-5)^2&=5^2\\(x^2-6x+9)+(y^2-10y+25)^2&=25\\x^2+y^2-6x-10y+34&=25\\x^2+y^2-6x-10y&=-9\end{aligned}$$

(b)
$$\begin{aligned}(x--2)^2+(y-1)^2&=(\sqrt{7})^2\\(x+2)^2+(y-1)^2&=7\\x^2+4x+4+y^2-2y+1&=7\\x^2+y^2+4x-2y+5&=7\\x^2+y^2+4x-2y&=2\end{aligned}$$

(c)
$$\begin{aligned}(x--3)^2+(y--1)^2&=2^2\\(x+3)^2+(y+1)^2&=4\\x^2+6x+9+y^2+2y+1&=4\\x^2+y^2+6x+2y+10&=4\\x^2+y^2+6x+2y&=-6\end{aligned}$$

(d)
$$\begin{aligned}(x-3)^2+(y-8)^2&=(2\sqrt{7})^2\\x^2-6x+9+y^2-16y+64&=28\\x^2+y^2-6x-16y+73&=28\\x^2+y^2-6x-16y&=-45\end{aligned}$$

8. (a) Radius $=\sqrt{5}$; centre $=(6,3)$

(b) $|\mathbf{r}-2\mathbf{i}+3\mathbf{j}|=6$

$|\mathbf{r}-(2\mathbf{i}-3\mathbf{j})|=6$

Radius $=6$, centre $=(2,-3)$

(c) Radius $=3$, centre $=(3,-4)$

(d) Radius $=5$, centre $=(0,0)$

(e) $25x^2+25y^2=9$

$25(x^2+y^2)=9$

$x^2+y^2=\dfrac{9}{25}$

Radius $=\frac{3}{5}$, centre $=(0,0)$

(f) Radius $=5$, centre $=(3,-4)$

(g) Radius $=10$, centre $=(-7,1)$

(h) Radius $=20$, centre $=(0,0)$

(i)
$$\begin{aligned}x^2+y^2-6x+4y+4&=0\\(x-3)^2-9+(y+2)^2-4+4&=0\\(x-3)^2+(y+2)^2&=9\end{aligned}$$

Radius $=3$, centre $=(3,-2)$

(j)
$$\begin{aligned}x^2+y^2+2x-6y&=15\\(x+1)^2-1+(y-3)^2-9&=15\\(x+1)^2+(y-3)^2&=25\end{aligned}$$

Radius $=5$, centre $=(-1,3)$

9. The first circle has centre $3\mathbf{i}+7\mathbf{j}$. The second circle has centre $2\mathbf{i}+9\mathbf{j}$. The distance between these centres is

$$\begin{aligned}d&=\sqrt{(3-2)^2+(7-9)^2}\\&=\sqrt{1^2+(-2)^2}\\&=\sqrt{5}\end{aligned}$$

10.
$$\begin{aligned}\mathrm{A}&=3\mathbf{i}-4\mathbf{j}\\\mathrm{B}&=2\mathbf{i}+7\mathbf{j}\end{aligned}$$

The line through A and B is given by

$$\begin{aligned}\mathbf{r}&=\mathrm{A}+\lambda\overrightarrow{\mathrm{AB}}\\&=3\mathbf{i}-4\mathbf{j}+\lambda(2\mathbf{i}+7\mathbf{j}-(3\mathbf{i}-4\mathbf{j}))\\&=3\mathbf{i}-4\mathbf{j}+\lambda(-\mathbf{i}+11\mathbf{j})\\&=(3-\lambda)\mathbf{i}+(-4+11\lambda)\mathbf{j}\end{aligned}$$

11.
$$\begin{aligned}\mathrm{A}&=3\mathbf{i}+11\mathbf{j}\\\mathrm{B}&=12\mathbf{i}-\mathbf{j}\\|\overrightarrow{\mathrm{AB}}|&=\sqrt{(3-12)^2+(11--1)^2}\\&=\sqrt{9^2+12^2}\\&=15\end{aligned}$$

The distance between centres is 15. The circle centred at A has a radius of 12 and that centred at B has a radius of 3; 3+12=15: the circles touch at just one point (i.e. one point in common).

12.
$$\begin{aligned}\mathrm{A}&=2\mathbf{i}+3\mathbf{j}\\\mathrm{B}&=-2\mathbf{i}+5\mathbf{j}\\|\overrightarrow{\mathrm{AB}}|&=\sqrt{(2--2)^2+(3-5)^2}\\&=\sqrt{4^2+2^2}\\&=\sqrt{20}\end{aligned}$$

The distance between centres is $\sqrt{20}$. The circle centred at A has a radius of 3 and that centred at B has a radius of 1; $3+1=4=\sqrt{16}<\sqrt{20}$: the circles are further apart than the sum of their radii, so they do not intersect (i.e. no points in common).

13. Substitute the expression for $\mathbf{r}$ in the equation of the line into the equation of the circle:

$$\begin{aligned}\left|\begin{pmatrix}6\\9\end{pmatrix}+\lambda\begin{pmatrix}4\\1\end{pmatrix}-\begin{pmatrix}-5\\2\end{pmatrix}\right|&=\sqrt{34}\\\left|\begin{pmatrix}11+4\lambda\\7+\lambda\end{pmatrix}\right|&=\sqrt{34}\\(11+4\lambda)^2+(7+\lambda)^2&=34\\121+88\lambda+16\lambda^2+49+14\lambda+\lambda^2&=34\\170+102\lambda+17\lambda^2&=34\\17\lambda^2+102\lambda+136&=0\\\lambda^2+6\lambda+8&=0\\(\lambda+4)(\lambda+2)&=0\\\lambda&=-4\\\text{or }\lambda&=-2\end{aligned}$$

$$\begin{aligned} r &= \begin{pmatrix}6\\9\end{pmatrix} - 4\begin{pmatrix}4\\1\end{pmatrix} \\ &= \begin{pmatrix}6\\9\end{pmatrix} - \begin{pmatrix}16\\4\end{pmatrix} \\ &= \begin{pmatrix}-10\\5\end{pmatrix} \\ \text{or } r &= \begin{pmatrix}6\\9\end{pmatrix} - 2\begin{pmatrix}4\\1\end{pmatrix} \\ &= \begin{pmatrix}6\\9\end{pmatrix} - \begin{pmatrix}8\\2\end{pmatrix} \\ &= \begin{pmatrix}-2\\7\end{pmatrix} \end{aligned}$$

14. Substitute the expression for **r** in the equation of the line into the equation of the circle:

$$\begin{aligned} |-\mathbf{i} + 8\mathbf{j} + \lambda(6\mathbf{i} + 2\mathbf{j}) - (7\mathbf{i} + 4\mathbf{j})| &= \sqrt{40} \\ |-8\mathbf{i} + 4\mathbf{j} + \lambda(6\mathbf{i} + 2\mathbf{j})| &= \sqrt{40} \\ |(-8 + 6\lambda)\mathbf{i} + (4 + 2\lambda)\mathbf{j}| &= \sqrt{40} \\ (-8 + 6\lambda)^2 + (4 + 2\lambda)^2 &= 40 \\ 64 - 96\lambda + 36\lambda^2 + 16 + 16\lambda + 4\lambda^2 &= 40 \\ 40\lambda^2 - 80\lambda + 80 &= 40 \\ 40\lambda^2 - 80\lambda + 40 &= 0 \\ \lambda^2 - 2\lambda + 1 &= 0 \\ (\lambda - 1)^2 &= 0 \\ \lambda &= 1 \end{aligned}$$

$$\begin{aligned} r &= -\mathbf{i} + 8\mathbf{j} + 1(6\mathbf{i} + 2\mathbf{j}) \\ &= 5\mathbf{i} + 10\mathbf{j} \end{aligned}$$

Miscellaneous Exercise 4

1. Let $z = a + b\mathrm{i}$ where a and b are real.

$$\begin{aligned} 3z + 2\bar{z} &= 5 + 5\mathrm{i} \\ 3(a + b\mathrm{i}) + 2(a - b\mathrm{i}) &= 5 + 5\mathrm{i} \\ 3a + 3b\mathrm{i} + 2a - 2b\mathrm{i} &= 5 + 5\mathrm{i} \\ 5a + b\mathrm{i} &= 5 + 5\mathrm{i} \\ a &= 1 \\ b &= 5 \\ z &= 1 + 5\mathrm{i} \end{aligned}$$

2. Let $z = a + b\mathrm{i}$ where a and b are real.

$$\begin{aligned} z(2 - 3\mathrm{i}) &= 5 + \mathrm{i} \\ (a + b\mathrm{i})(2 - 3\mathrm{i}) &= 5 + \mathrm{i} \\ 2a - 3a\mathrm{i} + 2b\mathrm{i} + 3b &= 5 + \mathrm{i} \\ 2a + 3b + (-3a + 2b)\mathrm{i} &= 5 + \mathrm{i} \\ 2a + 3b &= 5 \\ \text{and } -3a + 2b &= 1 \\ 6a + 9b &= 15 \\ -6a + 4b &= 2 \\ 13b &= 17 \\ b &= \frac{17}{13} \\ -3a + 2\left(\frac{17}{13}\right) &= 1 \\ -39a + 34 &= 13 \\ -39a &= -21 \\ 13a &= 7 \\ a &= \frac{7}{13} \\ z &= \frac{7}{13} + \frac{17}{13}\mathrm{i} \end{aligned}$$

3. Left Hand Side:

$$\begin{aligned}2\sin^3\theta\cos\theta + 2\cos^3\theta\sin\theta &= 2\sin\theta\cos\theta(\sin^2\theta+\cos^2\theta)\\ &= 2\sin\theta\cos\theta\\ &= \sin 2\theta\\ &= \text{R.H.S.}\end{aligned}$$

□

4.
$$\begin{aligned}2\sin x\cos x &= \sqrt{3}(1-2\sin^2 x)\\ \sin 2x &= \sqrt{3}\cos 2x\\ \tan 2x &= \sqrt{3}\\ 2x &= 60^\circ\\ \text{or } 2x &= 180+60\\ &= 240^\circ\\ \text{or } 2x &= 360+60\\ &= 420^\circ\\ \text{or } 2x &= 540+60\\ &= 600^\circ\\ x &= 30^\circ\\ \text{or } x &= 120^\circ\\ \text{or } x &= 210^\circ\\ \text{or } x &= 300^\circ\end{aligned}$$

5. Let two complex numbers be $a+b\text{i}$ and $c+d\text{i}$.

   The sum of the conjugates is
$$a-b\text{i}+c-d\text{i} = (a+c)-(b+d)\text{i}$$
   The conjugate of the sum is
$$\begin{aligned}\overline{a+b\text{i}+c+d\text{i}} &= \overline{(a+c)+(b+d)\text{i}}\\ &= (a+c)-(b+d)\text{i}\end{aligned}$$
   Hence the sum of the conjugates equals the conjugate of the sum.

6. (a)
$$\begin{aligned}R &= \sqrt{7^2+10^2}\\ &= \sqrt{149}\\ 7\sin\theta-10\cos\theta &= \sqrt{149}\left(\frac{7}{\sqrt{149}}\sin\theta-\frac{10}{\sqrt{149}}\cos\theta\right)\\ &= R(\sin\theta\cos\alpha-\cos\theta\sin\alpha)\end{aligned}$$
$$\begin{aligned}\cos\alpha &= \frac{7}{\sqrt{149}}\\ \alpha &= 0.96\\ \therefore\quad 7\sin\theta-10\cos\theta &= \sqrt{149}\sin(\theta-0.96)\end{aligned}$$
   (b) The minimum value is $-\sqrt{149}$.
$$\begin{aligned}\sqrt{149}\sin(\theta-0.96) &= -\sqrt{149}\\ \sin(\theta-0.96) &= -1\\ \theta-0.96 &= \frac{3\pi}{2}\\ \theta &= \frac{3\pi}{2}+0.96\\ &= 5.67\end{aligned}$$

7. w and $\bar{w}$ must be the vectors in the 1st and 4th quadrants, so $z=-5+3\text{i}$.
$$\begin{aligned}z^2 &= (-5+3\text{i})^2\\ &= 25-30\text{i}-9\\ &= 16-30\text{i}\end{aligned}$$

8. The vector equation of the line through C and D is
$$\begin{aligned}\mathbf{r} &= \text{C}+\mu\overrightarrow{\text{CD}}\\ &= (-\mathbf{i}-2\mathbf{j})+\mu(5\mathbf{i}+\mathbf{j}-(-\mathbf{i}-2\mathbf{j}))\\ &= (-\mathbf{i}-2\mathbf{j})+\mu(6\mathbf{i}+3\mathbf{j})\end{aligned}$$
   (We use μ as the parameter here because it is independent of λ which has already been used for the other line.)

   Where the two lines intersect,
$$\begin{aligned}(6\mathbf{i}+12\mathbf{j})+\lambda(\mathbf{i}-3\mathbf{j}) &= (-\mathbf{i}-2\mathbf{j})+\mu(6\mathbf{i}+3\mathbf{j})\\ (6+\lambda)\mathbf{i}+(12-3\lambda)\mathbf{j} &= (-1+6\mu)\mathbf{i}+(-2+3\mu)\mathbf{j}\\ (6+\lambda)\mathbf{i}-(-1+6\mu)\mathbf{i} &= (-2+3\mu)\mathbf{j}-(12-3\lambda)\mathbf{j}\\ (7+\lambda-6\mu)\mathbf{i} &= (-14+3\lambda+3\mu)\mathbf{j}\end{aligned}$$
   Because $\mathbf{i}$ and $\mathbf{j}$ are not parallel, the only way this equation can be true is if left and right sides both evaluate to the zero vector. This gives us the pair of simultaneous equations that we can solve for λ or μ. (We don't need both as either will allow us to find the point of intersection by substituting it into the corresponding equation of a line, although finding both gives us a cross check that it does in fact yield the same point from both lines.)
$$\begin{aligned}7+\lambda-6\mu &= 0\\ -14+3\lambda+3\mu &= 0\\ -28+6\lambda+6\mu &= 0\\ -21+7\lambda &= 0\\ \lambda &= 3\\ \mathbf{r} &= (6\mathbf{i}+12\mathbf{j})+3(\mathbf{i}-3\mathbf{j})\\ &= 9\mathbf{i}+3\mathbf{j}\end{aligned}$$
   Check by substituting $\lambda=3$ into the second equation above:
$$\begin{aligned}-14+3(3)+3\mu &= 0\\ -5+3\mu &= 0\\ \mu &= \frac{5}{3}\\ \mathbf{r} &= (-\mathbf{i}-2\mathbf{j})+\frac{5}{3}(6\mathbf{i}+3\mathbf{j})\\ &= (-\mathbf{i}-2\mathbf{j})+(10\mathbf{i}+5\mathbf{j})\\ &= 9\mathbf{i}+3\mathbf{j}\end{aligned}$$

This gives us A= $9\mathbf{i}+3\mathbf{j}$ so the distance from the origin is

$$\begin{aligned} d &= |\mathrm{A}| \\ &= |9\mathbf{i}+3\mathbf{j}| \\ &= \sqrt{9^2+3^2} \\ &= \sqrt{90} \\ &= 3\sqrt{10} \text{ units} \end{aligned}$$

9. If one of the complex solutions is $x = -2+2\mathrm{i}$ then another must its conjugate $x = -2-2\mathrm{i}$ and one of the quadratic factors will be a multiple of the product of the complex linear factors suggested by these two roots:

$$\begin{aligned} &(x-(-2+2\mathrm{i}))(x-(-2-2\mathrm{i})) \\ &\quad = (x+2-2\mathrm{i})(x+2+2\mathrm{i}) \\ &\quad = x^2+2x+2\mathrm{i}x+2x+4+4\mathrm{i}-2\mathrm{i}x-4\mathrm{i}+4 \\ &\quad = x^2+4x+8 \end{aligned}$$

Hence $f = 1$ and $g = 4$.

$$\begin{aligned} (5x^2+&6x+5)(x^2+4x+8) \\ &= 5x^4+20x^3+40x^2 \\ &\quad +6x^3+24x^2+48x \\ &\quad +5x^2+20x+40 \\ &= 5x^4+26x^3+69x^2+68x+40 \end{aligned}$$

Hence $a = 5$, $b = 26$, $c = 69$, $d = 68$ and $e = 40$.