Chapter 1

Exercise 1A

1. $\sqrt{-25} = \sqrt{25 \times -1}$
$= \sqrt{25} \times \sqrt{-1}$
$= 5\text{i}$

2. $\sqrt{-144} = \sqrt{144 \times -1}$
$= \sqrt{144} \times \sqrt{-1}$
$= 12\text{i}$

3. $\sqrt{-9} = \sqrt{9 \times -1}$
$= \sqrt{9} \times \sqrt{-1}$
$= 3\text{i}$

4. $\sqrt{-49} = \sqrt{49 \times -1}$
$= \sqrt{49} \times \sqrt{-1}$
$= 7\text{i}$

5. $\sqrt{-400} = \sqrt{400 \times -1}$
$= \sqrt{400} \times \sqrt{-1}$
$= 20\text{i}$

6. $\sqrt{-5} = \sqrt{5 \times -1}$
$= \sqrt{5} \times \sqrt{-1}$
$= \sqrt{5}\text{i}$

7. $\sqrt{-8} = \sqrt{8 \times -1}$
$= \sqrt{8} \times \sqrt{-1}$
$= 2\sqrt{2}\text{i}$

8. $\sqrt{-45} = \sqrt{45 \times -1}$
$= \sqrt{9 \times 5} \times \sqrt{-1}$
$= 3\sqrt{5}\text{i}$

No working required for questions 9, 10 and 11.

12. $$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 5}}{2 \times 1} = \frac{-2 \pm \sqrt{4 - 20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4\text{i}}{2}$$
$x = -1 + 2\text{i}$ or $x = -1 - 2\text{i}$

13. $$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 3}}{2 \times 1} = \frac{-2 \pm \sqrt{4 - 12}}{2} = \frac{-2 \pm \sqrt{-8}}{2} = \frac{-2 \pm 2\sqrt{2}\text{i}}{2}$$
$x = -1 + \sqrt{2}\text{i}$ or $x = -1 - \sqrt{2}\text{i}$

14. $$x = \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 6}}{2 \times 1} = \frac{-4 \pm \sqrt{16 - 24}}{2} = \frac{-4 \pm \sqrt{-8}}{2} = \frac{-4 \pm 2\sqrt{2}\text{i}}{2}$$
$x = -2 + \sqrt{2}\text{i}$ or $x = -2 - \sqrt{2}\text{i}$

15. $$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 10}}{2 \times 1} = \frac{-2 \pm \sqrt{4 - 40}}{2} = \frac{-2 \pm \sqrt{-36}}{2} = \frac{-2 \pm 6\text{i}}{2}$$
$x = -1 + 3\text{i}$ or $x = -1 - 3\text{i}$

16. $$x = \frac{4 \pm \sqrt{4^2 - 4 \times 1 \times 6}}{2 \times 1} = \frac{4 \pm \sqrt{16 - 24}}{2} = \frac{4 \pm \sqrt{-8}}{2} = \frac{4 \pm 2\sqrt{2}\text{i}}{2}$$
$x = 2 + \sqrt{2}\text{i}$ or $x = 2 - \sqrt{2}\text{i}$

17. $$x = \frac{1 \pm \sqrt{(-1)^2 - 4 \times 2 \times 1}}{2 \times 2} = \frac{1 \pm \sqrt{1 - 8}}{4} = \frac{1 \pm \sqrt{-7}}{4} = \frac{1 \pm \sqrt{7}\text{i}}{4}$$
$x = \frac{1}{4} + \frac{\sqrt{7}}{4}\text{i}$ or $x = \frac{1}{4} - \frac{\sqrt{7}}{4}\text{i}$

18. $$x = \frac{-1 \pm \sqrt{1^2 - 4 \times 2 \times 1}}{2 \times 2} = \frac{-1 \pm \sqrt{1 - 8}}{4} = \frac{-1 \pm \sqrt{-7}}{4} = \frac{-1 \pm \sqrt{7}\text{i}}{4}$$
$x = -\frac{1}{4} + \frac{\sqrt{7}}{4}\text{i}$ or $x = -\frac{1}{4} - \frac{\sqrt{7}}{4}\text{i}$

19. $x = \dfrac{-6 \pm \sqrt{6^2 - 4 \times 2 \times 5}}{2 \times 2}$
$= \dfrac{-6 \pm \sqrt{36 - 40}}{4}$
$= \dfrac{-6 \pm \sqrt{-4}}{4}$
$= \dfrac{-6 \pm 2\mathrm{i}}{4}$
$x = -\dfrac{3}{2} + \dfrac{1}{2}\mathrm{i} \quad \text{or} \quad x = -\dfrac{3}{2} - \dfrac{1}{2}\mathrm{i}$

20. $x = \dfrac{2 \pm \sqrt{(-2)^2 - 4 \times 2 \times 25}}{2 \times 2}$
$= \dfrac{2 \pm \sqrt{4 - 200}}{4}$
$= \dfrac{2 \pm \sqrt{-196}}{4}$
$= \dfrac{2 \pm 14\mathrm{i}}{4}$
$x = \dfrac{1}{2} + \dfrac{7}{2}\mathrm{i} \quad \text{or} \quad x = \dfrac{1}{2} - \dfrac{7}{2}\mathrm{i}$

21. $x = \dfrac{2 \pm \sqrt{(-2)^2 - 4 \times 5 \times 13}}{2 \times 5}$
$= \dfrac{2 \pm \sqrt{4 - 260}}{10}$
$= \dfrac{2 \pm \sqrt{-256}}{10}$
$= \dfrac{2 \pm 16\mathrm{i}}{10}$
$x = 0.2 + 1.6\mathrm{i} \quad \text{or} \quad x = 0.2 - 1.6\mathrm{i}$

22. $x = \dfrac{1 \pm \sqrt{(-1)^2 - 4 \times 1 \times 1}}{2 \times 1}$
$= \dfrac{1 \pm \sqrt{1 - 4}}{2}$
$= \dfrac{1 \pm \sqrt{-3}}{2}$
$= \dfrac{1 \pm \sqrt{3}\mathrm{i}}{2}$
$x = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\mathrm{i} \quad \text{or} \quad x = \dfrac{1}{2} - \dfrac{\sqrt{3}}{2}\mathrm{i}$

23. $x = \dfrac{3 \pm \sqrt{(-3)^2 - 4 \times 5 \times 1}}{2 \times 5}$
$= \dfrac{3 \pm \sqrt{9 - 20}}{10}$
$= \dfrac{3 \pm \sqrt{-11}}{10}$
$= \dfrac{3 \pm \sqrt{11}\mathrm{i}}{10}$
$x = 0.3 + \dfrac{\sqrt{11}}{10}\mathrm{i} \quad \text{or} \quad x = 0.3 - \dfrac{\sqrt{11}}{10}\mathrm{i}$

Exercise 1B

1. $(2 + 5) + (3 - 1)\mathrm{i} = 7 + 2\mathrm{i}$
2. $(5 - 2) + (-6 - 4)\mathrm{i} = 3 - 10\mathrm{i}$
3. $(2 - 5) + (3 - -1)\mathrm{i} = -3 + 4\mathrm{i}$
4. $(5 + 2) + (-6 + 4)\mathrm{i} = 7 - 2\mathrm{i}$
5. $(2 - 5) + (3 - 1)\mathrm{i} = -3 + 2\mathrm{i}$
6. $(5 + 2) + (-6 + 4)\mathrm{i} = 7 - 2\mathrm{i}$
7. $(3 + 4 + 6) + (1 - 2 + 5)\mathrm{i} = 13 + 4\mathrm{i}$
8. $(6 + 4\mathrm{i}) + (6 + 3\mathrm{i}) = (6 + 6) + (4 + 3)\mathrm{i} = 12 + 7\mathrm{i}$
9. $(10 + 5\mathrm{i}) + (3 - 3\mathrm{i}) = (10 + 3) + (5 - 3)\mathrm{i} = 13 + 2\mathrm{i}$
10. $(10 + 5\mathrm{i}) - (3 - 3\mathrm{i}) = (10 - 3) + (5 - -3)\mathrm{i} = 7 + 8\mathrm{i}$
11. $(3 - 15\mathrm{i}) + 7\mathrm{i} = 3 + (-15 + 7)\mathrm{i} = 3 - 8\mathrm{i}$
12. $(3 - 15\mathrm{i}) + 7 = (3 + 7) - 15\mathrm{i} = 10 - 15\mathrm{i}$
13. $2 + 5 = 7$
14. $4 + 1 = 5$
15. $6 + 15\mathrm{i} + 4\mathrm{i} + 10\mathrm{i}^2 = 6 + 19\mathrm{i} - 10 = -4 + 19\mathrm{i}$
16. $3 + 2\mathrm{i} + 9\mathrm{i} + 6\mathrm{i}^2 = 3 + 11\mathrm{i} - 6 = -3 + 11\mathrm{i}$
17. $2 - 2\mathrm{i} + \mathrm{i} - \mathrm{i}^2 = 2 - \mathrm{i} - -1 = 3 - \mathrm{i}$
18. $-10 - 2\mathrm{i} + 15\mathrm{i} + 3\mathrm{i}^2 = -10 + 13\mathrm{i} - 3 = -13 + 13\mathrm{i}$

19. $$\frac{3+2i}{1+5i} = \frac{3+2i}{1+5i} \times \frac{1-5i}{1-5i} = \frac{3-15i+2i-10i^2}{1^2-(5i)^2} = \frac{3-13i--10}{1--25} = \frac{13-13i}{26} = 0.5+0.5i$$

20. $$\frac{3+i}{1-2i} = \frac{3+i}{1-2i} \times \frac{1+2i}{1+2i} = \frac{3+6i+i+2i^2}{1^2-(2i)^2} = \frac{3+7i+-2}{1--4} = \frac{1+7i}{5} = 0.2+1.4i$$

21. $$\frac{4}{1+3i} = \frac{4}{1+3i} \times \frac{1-3i}{1-3i} = \frac{4-12i}{1^2-(3i)^2} = \frac{4-12i}{1--9} = \frac{4-12i}{10} = 0.4+1.2i$$

22. $$\frac{2i}{1+4i} = \frac{2i}{1+4i} \times \frac{1-4i}{1-4i} = \frac{2i-8i^2}{1^2-(4i)^2} = \frac{2i--8}{1--16} = \frac{8+2i}{17} = \frac{8}{17}+\frac{2}{17}i$$

23. $$\frac{-3+2i}{2+3i} = \frac{-3+2i}{2+3i} \times \frac{2-3i}{2-3i} = \frac{-6+9i+4i-6i^2}{2^2-(3i)^2} = \frac{-6+13i--6}{4--9} = \frac{13i}{13} = i$$

24. $$\frac{5+i}{2i+3} = \frac{5+i}{2i+3} \times \frac{-2i+3}{-2i+3} = \frac{-10i+15-2i^2+3i}{-(2i)^2+3^2} = \frac{15-7i--2}{--4+9} = \frac{17-7i}{13} = \frac{17}{13}-\frac{7}{13}i$$

25. (a) $w+z=(5+4)+(-2+3)i=9+i$

(b) $w-z-(5-4)+(-2-3)i=1-5i$

(c) $3w-2z=(15-6i)-(8+6i)=(15-8)+(-6-6)i=7-12i$

(d) $wz=(5-2i)(4+3i)=20+15i-8i-6i^2=26+7i$

(e) $z^2=(4+3i)^2=16+24i+9i^2=7+24i$

(f) $$\frac{w}{z} = \frac{5-2i}{4+3i} \times \frac{4-3i}{4-3i} = \frac{20-15i-8i+6i^2}{4^2-(3i)^2} = \frac{20-23i+-6}{16--9} = \frac{14-23i}{25} = 0.56-0.92i$$

26. (a) $Z_1+Z_2=(3+1)+(5-5)i=4$

(b) $Z_2-Z_1=(1-3)+(-5-5)i=-2-10i$

(c) $Z_1+3Z_2=(3+5i)+(3-15i)=(3+3)+(5-15)i=6-10i$

(d) $Z_1Z_2=(3+5i)(1-5i)=3-15i+5i-25i^2=28-10i$

(e) $Z_1^2=(3+5i)^2=9+30i+25i^2=-16+30i$

(f) $$\frac{Z_1}{Z_2} = \frac{3+5i}{1-5i} \times \frac{1+5i}{1+5i} = \frac{3+15i+5i+25i^2}{1^2-(5i)^2} = \frac{3+20i+-25}{1--25} = \frac{-22+20i}{26} = -\frac{11}{13}+\frac{10}{13}i$$

27. (a) $\bar{z}=24+7i$

(b) $z+\bar{z}=24-7i+24+7i=48$

(c) $z\bar{z}=(24-7i)(24+7i)=24^2-(7i)^2=576--49=625$

(d) $$\begin{aligned}\frac{z}{\bar{z}} &= \frac{z}{\bar{z}} \times \frac{z}{z}\\ &= \frac{(24-7\mathrm{i})^2}{625}\\ &= \frac{576-336\mathrm{i}+49\mathrm{i}^2}{625}\\ &= \frac{527-336\mathrm{i}}{625}\\ &= \frac{527}{625} - \frac{336}{625}\mathrm{i}\end{aligned}$$

28. (a) $\bar{z} = 4 - 9\mathrm{i}$

(b) $z - \bar{z} = (4+9\mathrm{i}) - (4-9\mathrm{i}) = 18\mathrm{i}$

(c) $2z + 3\bar{z} = (8+18\mathrm{i}) + (12-27\mathrm{i}) = 20 - 9\mathrm{i}$

(d) $2z - 3\bar{z} = (8+18\mathrm{i}) - (12-27\mathrm{i}) = -4 + 45\mathrm{i}$

(e) $z\bar{z} = (4+9\mathrm{i})(4-9\mathrm{i}) = 4^2 - (9\mathrm{i})^2 = 16 - -81 = 97$

(f) $$\begin{aligned}\frac{z}{\bar{z}} &= \frac{z}{\bar{z}} \times \frac{z}{z}\\ &= \frac{(4+9\mathrm{i})^2}{97}\\ &= \frac{16+72\mathrm{i}+81\mathrm{i}^2}{97}\\ &= \frac{-65+72\mathrm{i}}{97}\\ &= -\frac{65}{97} + \frac{72}{97}\mathrm{i}\end{aligned}$$

29. $$\begin{aligned} z &= w\\ 2 + c\mathrm{i} &= d + 3\mathrm{i}\\ \mathrm{Re}(z) &= \mathrm{Re}(w)\\ 2 &= d\\ \mathrm{Im}(z) &= \mathrm{Im}(w)\\ c &= 3\end{aligned}$$

30. $$\begin{aligned} a + b\mathrm{i} &= (2-3\mathrm{i})^2\\ &= 4 - 12\mathrm{i} + 9\mathrm{i}^2\\ &= 4 - 12\mathrm{i} - 9\\ &= -5 - 12\mathrm{i}\\ a &= -5\\ b &= -12\end{aligned}$$

31. $$\begin{aligned} z &= w\\ 5 - (c+3)\mathrm{i} &= d + 1 + 7\mathrm{i}\\ \mathrm{Re}(z) &= \mathrm{Re}(w)\\ 5 &= d + 1\\ d &= 4\\ \mathrm{Im}(z) &= \mathrm{Im}(w)\\ -(c+3) &= 7\\ c + 3 &= -7\\ c &= -10\end{aligned}$$

32. $$\begin{aligned}(a+3\mathrm{i})(5-\mathrm{i}) &= p\\ 5a - a\mathrm{i} + 15\mathrm{i} - 3\mathrm{i}^2 &= p\\ 5a + 3 + (15-a)\mathrm{i} &= p\\ 15 - a &= 0\\ a &= 15\\ 5a + 3 &= p\\ 75 + 3 &= p\\ p &= 78\end{aligned}$$

33. (a) Yes, this is true. This is how the conjugate is defined: real parts equal, imaginary parts opposite.

(b) No, this is not necessarily true. For example, consider $z = 1 + 3\mathrm{i}$ and $w = 2 - 3\mathrm{i}$. Here $\mathrm{Im}(z) = -\mathrm{Im}(w)$ but $\mathrm{Re}(z) \neq \mathrm{Re}(w)$ so $w \neq \bar{z}$

34. (a) In the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

the value of the square root $\sqrt{b^2-4ac}$ either zero (for $b^2 - 4ac = 0$), real ($b^2 - 4ac > 0$) or imaginary ($b^2 - 4ac < 0$). If zero or real, the quadratic has one or two real roots. If it has complex roots we must have $\sqrt{b^2-4ac} = q\mathrm{i}$ for some real q. The two roots, then are

$$\begin{aligned} x &= \frac{-b + q\mathrm{i}}{2a} & \text{and} \quad x &= \frac{-b - q\mathrm{i}}{2a}\\ &= \frac{-b}{2a} + \frac{q}{2a}\mathrm{i} & &= \frac{-b}{2a} - \frac{q}{2a}\mathrm{i}\end{aligned}$$

from which we can see that the real parts are equal, and the imaginary parts are opposites, that is they are conjugates. □

(b) From the quadratic formula,

$$3 + 2\mathrm{i} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Substituting $a = 1$ and equating real parts we get

$$\begin{aligned} 3 &= \frac{-b}{2a}\\ &= \frac{-b}{2}\\ b &= -6\end{aligned}$$

Equating the imaginary parts,

$$\begin{aligned} 2\mathrm{i} &= \frac{\sqrt{b^2-4ac}}{2a}\\ 2\mathrm{i} &= \frac{\sqrt{(-6)^2-4c}}{2}\\ 4\mathrm{i} &= \sqrt{36-4c}\\ (4\mathrm{i})^2 &= 36 - 4c\\ -16 &= 36 - 4c\\ -52 &= -4c\\ c &= 13\end{aligned}$$

(c) From the quadratic formula,

$$5 - 3\text{i} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substituting $a = 1$ and equating real parts we get

$$\begin{aligned} 5 &= \frac{-b}{2a} \\ &= \frac{-b}{2} \\ b &= -10 \end{aligned}$$

Equating the parts,

$$\begin{aligned} -3\text{i} &= \frac{\sqrt{b^2 - 4ac}}{2a} \\ -3\text{i} &= \frac{\sqrt{(-10)^2 - 4c}}{2} \\ -6\text{i} &= \sqrt{100 - 4c} \\ (-6\text{i})^2 &= 100 - 4c \\ -36 &= 100 - 4c \\ -136 &= -4c \\ c &= 34 \end{aligned}$$

35. (a)
$$\begin{aligned} \frac{c + d\text{i}}{-c - d\text{i}} &= c + d\text{i}{-c} - d\text{i} \times -c + d\text{i}{-c} + d\text{i} \\ &= \frac{-c^2 + (d\text{i})^2}{(-c)^2 - (d\text{i})^2} \\ &= \frac{-c^2 - d^2}{c^2 + d^2} \\ &= -1 \end{aligned}$$

(b)
$$\begin{aligned} \frac{c + d\text{i}}{d - c\text{i}} &= c + d\text{i}d - c\text{i} \times d + c\text{i}d + c\text{i} \\ &= \frac{cd + c^2\text{i} + d^2\text{i} + cd\text{i}^2}{(d)^2 - (c\text{i})^2} \\ &= \frac{(c^2 + d^2)\text{i}}{c^2 + d^2} \\ &= \text{i} \end{aligned}$$

(c)
$$\begin{aligned} \frac{c - d\text{i}}{-d - c\text{i}} &= c - d\text{i}{-d} - c\text{i} \times -d + c\text{i}{-d} + c\text{i} \\ &= \frac{-cd + c^2\text{i} + d^2\text{i} - cd\text{i}^2}{(-d)^2 - (c\text{i})^2} \\ &= \frac{(c^2 + d^2)\text{i}}{c^2 + d^2} \\ &= \text{i} \end{aligned}$$

This should be expected, since we are just substituting $-d$ for d in the previous question. If the outcome is true for all d then it must also be true for all $-d$.

36.
$$\begin{aligned} \frac{3 + 5\text{i}}{1 + p\text{i}} &= q + 4\text{i} \\ \frac{3 + 5\text{i}}{1 + p\text{i}} \times \frac{1 - p\text{i}}{1 - p\text{i}} &= q + 4\text{i} \\ \frac{3 - 3p\text{i} + 5\text{i} - 5p\text{i}^2}{1^2 - (p\text{i})^2} &= q + 4\text{i} \\ \frac{(3 + 5p) + (5 - 3p)\text{i}}{1 + p^2} &= q + 4\text{i} \\ (3 + 5p) + (5 - 3p)\text{i} &= (1 + p^2)(q + 4\text{i}) \end{aligned}$$

Equating imaginary components:

$$\begin{aligned} 5 - 3p &= 4(1 + p^2) \\ 4p^2 + 3p - 1 &= 0 \\ (4p - 1)(p + 1) &= 0 \\ p &= \frac{1}{4} \qquad \text{or} \qquad p = -1 \end{aligned}$$

Now real components:

$$\begin{aligned} 3 + 5p &= q(1 + p^2) \\ q &= \frac{3 + 5p}{1 + p^2} \\ q &= \frac{3 + 5(\frac{1}{4})}{1 + (\frac{1}{4})^2} \qquad \text{or} \qquad q = \frac{3 + 5(-1)}{1 + (-1)^2} \\ &= \frac{\frac{17}{4}}{\frac{17}{16}} \qquad\qquad = \frac{-2}{2} \\ &= \frac{17}{4} \times \frac{16}{17} \qquad\qquad = -1 \\ &= 4 \end{aligned}$$

Solution: $p = \frac{1}{4}$, $q = 4$ or $p = -1$, $q = -1$.

37. (a)
$$\begin{aligned} (x - z)(x - w) &= ax^2 + bx + c \\ x^2 + (-w - z)x + wz &= ax^2 + bx + c \\ a &= 1 \\ b &= (-w - z) \\ c &= wz \end{aligned}$$

(b) From the above, $b = -w - z$ so $w + z = -b$ which is real.
$wz = c$ which is real. □

(c) Since $w + z$ is real, $\text{Im}(w + z) = 0$:

$$\begin{aligned} \text{Im}(p + q\text{i} + r + s\text{i}) &= 0 \\ q + s &= 0 \\ s &= -q \end{aligned}$$

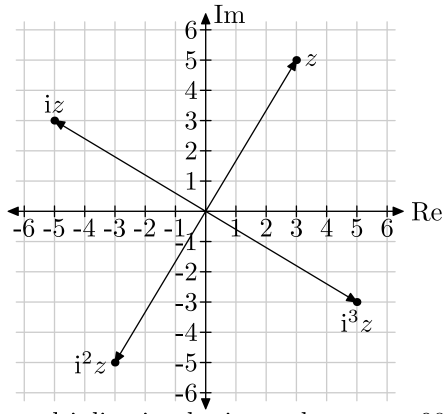
Similarly $\text{Im}(wz) = 0$:

$$\begin{aligned} \text{Im}\left((p + q\text{i})(r + s\text{i})\right) &= 0 \\ \text{Im}(pr + ps\text{i} + qr\text{i} - qs) &= 0 \\ ps + qr &= 0 \\ p(-q) + qr &= 0 \\ -p + r &= 0 \\ r &= p \end{aligned}$$

Hence $r + s\text{i} = p - q\text{i}$ □

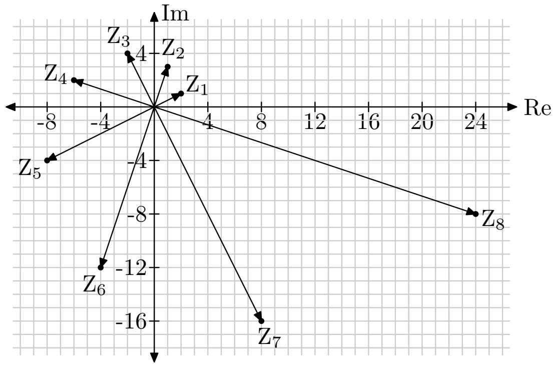
Exercise 1C

1. No working required. Refer answers in Sadler.
2. No working required. Refer answers in Sadler.
3. No working required. Refer answers in Sadler. Note that the complex conjugate produces a reflection in the Real axis on the Argand diagram (so Z_7 is a reflection of Z_1 and Z_8 of Z_3).
4. - $\frac{\text{Re}(Z_2)}{\text{Im}(Z_2)} > 1$ means Z_2 must be in either quadrant I or III on the Argand diagram and the real part larger in magnitude than the imaginary part. Only one number on the Argand diagram satisfies this: $Z_2 = -3 - 2\text{i}$
   - $\frac{\text{Re}(Z_1)}{\text{Im}(Z_1)} > 0$ means Z_1 must be in either quadrant I or III on the Argand diagram. Since we have already eliminated the number in quadrant III we can conclude $Z_1 = 1 + 2\text{i}$
   - $Z_3 = \bar{Z}_2 = -3 + 2\text{i}$
   - This leaves $Z_4 = 3 - 2\text{i}$
5. - $z = 3 + 5\text{i}$;
   - $\text{i}z = 3\text{i} + 5\text{i}^2 = -5 + 3\text{i}$;
   - $\text{i}^2 z = -z = -3 - 5\text{i}$;
   - $\text{i}^3 z = -\text{i}z = 5 - 3\text{i}$.



   Note: multiplication by i translates to a 90° rotation on the Argand diagram.

6. - $Z_1 = 2 + \text{i}$
   - $$\begin{aligned} Z_2 &= (2+\text{i})(1+\text{i}) \\ &= 2 + 2\text{i} + \text{i} + \text{i}^2 \\ &= 2 + 3\text{i} - 1 \\ &= 1 + 3\text{i} \end{aligned}$$
   - $$\begin{aligned} Z_3 &= (2+\text{i})(1+\text{i})^2 \\ &= Z_2(1+\text{i}) \\ &= (1+3\text{i})(1+\text{i}) \\ &= 1 + \text{i} + 3\text{i} + 3\text{i}^2 \\ &= 1 + 4\text{i} - 3 \\ &= -2 + 4\text{i} \end{aligned}$$
   - $$\begin{aligned} Z_4 &= (2+\text{i})(1+\text{i})^3 \\ &= Z_3(1+\text{i}) \\ &= (-2+4\text{i})(1+\text{i}) \\ &= -2 - 2\text{i} + 4\text{i} + 4\text{i}^2 \\ &= -2 + 2\text{i} - 4 \\ &= -6 + 2\text{i} \end{aligned}$$
   - $$\begin{aligned} Z_5 &= (2+\text{i})(1+\text{i})^4 \\ &= Z_4(1+\text{i}) \\ &= (-6+2\text{i})(1+\text{i}) \\ &= -6 - 6\text{i} + 2\text{i} + 2\text{i}^2 \\ &= -6 - 4\text{i} - 2 \\ &= -8 - 4\text{i} \end{aligned}$$
   - $$\begin{aligned} Z_6 &= (2+\text{i})(1+\text{i})^5 \\ &= Z_5(1+\text{i}) \\ &= (-8-4\text{i})(1+\text{i}) \\ &= -8 - 8\text{i} - 4\text{i} - 4\text{i}^2 \\ &= -8 - 12\text{i} + 4 \\ &= -4 - 12\text{i} \end{aligned}$$
   - $$\begin{aligned} Z_7 &= (2+\text{i})(1+\text{i})^6 \\ &= Z_6(1+\text{i}) \\ &= (-4-12\text{i})(1+\text{i}) \\ &= -4 - 4\text{i} - 12\text{i} - 12\text{i}^2 \\ &= -4 - 16\text{i} + 12 \\ &= 8 - 16\text{i} \end{aligned}$$
   - $$\begin{aligned} Z_8 &= (2+\text{i})(1+\text{i})^7 \\ &= Z_7(1+\text{i}) \\ &= (8-16\text{i})(1+\text{i}) \\ &= 8 + 8\text{i} - 16\text{i} - 16\text{i}^2 \\ &= 8 - 8\text{i} + 16 \\ &= 24 - 8\text{i} \end{aligned}$$



Miscellaneous Exercise 1

1. $\mathbf{F} + \mathbf{P} = (13\mathbf{i} - 28\mathbf{j}) + (-6\mathbf{i} + 4\mathbf{j})$
   $= 7\mathbf{i} - 24\mathbf{j}$
   Magnitude of the resultant is $\sqrt{7^2 + 24^2} = 25\text{N}$

2. (a) $\overrightarrow{\text{OB}} = \overrightarrow{\text{OA}} + \overrightarrow{\text{AB}}$
   $= \mathbf{a} + 3\mathbf{b}$

   (b) $\overrightarrow{\text{BC}} = \overrightarrow{\text{BA}} + \overrightarrow{\text{AO}} + \overrightarrow{\text{OC}}$
   $= -\overrightarrow{\text{AB}} - \overrightarrow{\text{OA}} + \overrightarrow{\text{OC}}$
   $= -3\mathbf{b} - \mathbf{a} + 7\mathbf{b}$
   $= 4\mathbf{b} - \mathbf{a}$

   (c) $\overrightarrow{\text{BC}} = \overrightarrow{\text{BA}} + \overrightarrow{\text{AO}} + \overrightarrow{\text{OC}}$
   $= -\overrightarrow{\text{AB}} - \overrightarrow{\text{OA}} + \overrightarrow{\text{OC}}$
   $= -3\mathbf{b} - \mathbf{a} + 7\mathbf{b}$
   $= 4\mathbf{b} - \mathbf{a}$

   (d) $\overrightarrow{\text{BD}} = 0.5\overrightarrow{\text{BC}}$
   $= 0.5(4\mathbf{b} - \mathbf{a})$
   $= 2\mathbf{b} - 0.5\mathbf{a}$

   (e) $\overrightarrow{\text{OD}} = \overrightarrow{\text{OB}} + \overrightarrow{\text{BD}}$
   $= \mathbf{a} + 3\mathbf{b} + 2\mathbf{b} - 0.5\mathbf{a}$
   $= 0.5\mathbf{a} + 5\mathbf{b}$

3. (a) $\log_a 2 = \log_a \frac{10}{5}$
   $= \log_a 10 - \log_a 5$
   $= p - q$

   (b) $\log_a 50 = \log_a 10 \times 5$
   $= \log_a 10 + \log_a 5$
   $= p + q$

   (c) $\log_a 100 = \log_a 10^2$
   $= 2\log_a 10$
   $= 2p$

   (d) $\log_a 125 = \log_a 5^3$
   $= 3\log_a 5$
   $= 3q$

   (e) $\log_a 0.1 = \log_a 10^{-1}$
   $= -\log_a 10$
   $= -p$

   (f) $\log_a 0.5 = \log_a \frac{5}{10}$
   $= \log_a 5 - \log_a 10$
   $= q - p$

   (g) $\log_a 20a = \log_a \frac{100}{5}a$
   $= \log_a 10^2 - \log_a 5 + \log_a a$
   $= 2\log_a 10 - \log_a 5 + \log_a a$
   $= 2p - q + 1$

   (h) $\log_5 10 = \frac{\log_a 10}{\log_a 5}$
   $= \frac{p}{q}$

   (i) $\log 5 = \frac{\log_a 5}{\log_a 10}$
   $= \frac{q}{p}$

4. (a) $(2 + 5\text{i})(2 - 5\text{i}) = 2^2 - (5\text{i})^2$
   $= 4 - (-25)$
   $= 29$

   (b) $(3 + \text{i})(3 - \text{i}) = 3^2 - (\text{i})^2$
   $= 9 - (-1)$
   $= 10$

   (c) $(6 + 2\text{i})(6 - 2\text{i}) = 6^2 - (2\text{i})^2$
   $= 36 - (-4)$
   $= 40$

   (d) $(3 + 4\text{i})^2 = 9 + 24\text{i} + 16\text{i}^2$
   $= 9 + 24\text{i} - 16$
   $= -7 + 24\text{i}$

   (e) $\frac{2 - 3\text{i}}{3 + \text{i}} = \frac{2 - 3\text{i}}{3 + \text{i}} \times \frac{3 - \text{i}}{3 - \text{i}}$
   $= \frac{6 - 2\text{i} - 9\text{i} + 3\text{i}^2}{3^2 - \text{i}^2}$
   $= \frac{6 - 11\text{i} - 3}{9 - -1}$
   $= \frac{3 - 11\text{i}}{10}$
   $= 0.3 - 1.1\text{i}$

   (f) $\frac{3 + \text{i}}{2 - 3\text{i}} = \frac{3 + \text{i}}{2 - 3\text{i}} \times \frac{2 + 3\text{i}}{2 + 3\text{i}}$
   $= \frac{6 + 9\text{i} + 2\text{i} + 3\text{i}^2}{2^2 - (3\text{i})^2}$
   $= \frac{6 + 11\text{i} - 3}{4 - -9}$
   $= \frac{3 + 11\text{i}}{13}$
   $= \frac{3}{13} + \frac{11}{13}\text{i}$

5. (a) $z + w = 2 - 3\text{i} - 3 + 5\text{i} = -1 + 2\text{i}$

   (b) $zw = (2 - 3\text{i})(-3 + 5\text{i})$
   $= -6 + 10\text{i} + 9\text{i} - 15\text{i}^2$
   $= -6 + 19\text{i} + 15$
   $= 9 + 19\text{i}$

   (c) $\bar{z} = 2 + 3\text{i}$

   (d) $\bar{z}\bar{w} = (2 + 3\text{i})(-3 - 5\text{i})$
   $= -6 - 10\text{i} - 9\text{i} - 15\text{i}^2$
   $= -6 - 19\text{i} + 15$
   $= 9 - 19\text{i}$
   Observation: $\bar{z}\bar{w} = \overline{(zw)}$

(e) $z^2 = (2-3\text{i})^2$
$= 4 - 12\text{i} + 9\text{i}^2$
$= 4 - 12\text{i} - 9$
$= -5 - 12\text{i}$

(f) $(zw)^2 = (9+19\text{i})^2$
$= 81 + 342\text{i} + 361\text{i}^2$
$= 81 + 342\text{i} - 361$
$= -280 + 342\text{i}$

(g) $p = \text{Re}(\bar{z}) + \text{Im}(\bar{w})\text{i}$
$= \text{Re}(z) - \text{Im}(w)\text{i}$
$= 2 - 5\text{i}$

6. Calculator question: no working required. Refer answers in Sadler.

7. (a) $x+3=0$ or $x-2=0$
$x=-3$ $\quad$ $x=2$

(b) $2x-5=0$ or $x+1=0$
$2x=5$ $\quad$ $x=-1$
$x=2.5$

(c) $x^2-x-20=0$
$(x-5)(x+4)=0$
$x-5=0$ or $x+4=0$
$x=5$ $\quad$ $x=-4$

(d) $x^2-10x+24=0$
$(x-4)(x-6)=0$
$x-4=0$ or $x-6=0$
$x=4$ $\quad$ $x=6$

(e) $x^2-10x-24=0$
$(x-12)(x+2)=0$
$x-12=0$ or $x+2=0$
$x=12$ $\quad$ $x=-2$

(f) $x^2+x=12$
$x^2+x-12=0$
$(x+4)(x-3)=0$
$x+4=0$ or $x-3=0$
$x=-4$ $\quad$ $x=3$

8. (a) $(\text{Re}(2+3\text{i}))\,(\text{Re}(5-4\text{i})) = 2\times 5 = 10$

(b) $\text{Re}\,((2+3\text{i})(5-4\text{i})) = \text{Re}(10-8\text{i}+15\text{i}-12\text{i}^2)$
$= \text{Re}(10+7\text{i}+12)$
$= \text{Re}(22+7\text{i})$
$= 22$

9. (a) Beginning with the right hand side:

$$\begin{aligned}\overline{z_1+z+2} &= \overline{a+b\text{i}+c+d\text{i}}\\ &= \overline{(a+c)+(b+d)\text{i}}\\ &= (a+c)-(b+d)\text{i}\\ &= a+c-b\text{i}-d\text{i}\\ &= a-b\text{i}+c-d\text{i}\\ &= \overline{a+b\text{i}}+\overline{c+d\text{i}}\\ &= \bar{z}_1+\bar{z}_2\\ &= \text{L.H.S.}\end{aligned}$$

□

(b) Beginning with the left hand side:

$$\begin{aligned}\bar{z}_1\bar{z}_2 &= (a-b\text{i})(c-d\text{i})\\ &= ac-ad\text{i}-bc\text{i}+bd\text{i}^2\\ &= ac-(ad+bc)\text{i}-bd\\ &= (ac-bd)-(ad+bc)\text{i}\\ &= \overline{(ac-bd)+(ad+bc)\text{i}}\\ &= \overline{ac+ad\text{i}-bd+bc\text{i}}\\ &= \overline{ac+ad\text{i}+bd\text{i}^2+bc\text{i}}\\ &= \overline{a(c+d\text{i})+b\text{i}(d\text{i}+c)}\\ &= \overline{(a+b\text{i})(c+d\text{i})}\\ &= \overline{z_1z_2}\\ &= \text{R.H.S.}\end{aligned}$$

□