Chapter 8

Exercise 8A

1. Cubic, $y = x^3$ (by observation)

2. Quadratic, $y = x^2 - 1$ (by observation)

3. Note the common product. Relationship is reciprocal $xy = -24$ so $y = -\frac{24}{x}$

4. There is a common ratio of $\frac{1}{2}$ so the relationship is exponential. $y = 128(0.5)^x$

5. There is a common first difference of -5, so the relationship is linear. $y = -5x + 13$

6. First differences are -5, -3, -1, 1, 3,5, 7, 9. Common second difference of 2, so the relationship is quadratic: $y = x^2 + 2x - 3$

7. Looks like it might be a common ratio. Checking gives a common ratio of 1.5; the relationship is exponential. $y = 1280(1.5)^x$

8. Common first difference of 2.5 $\implies$ linear. $y = 2.5x + 1.5$

9. First differences are -12,-10,-8,-6,-4,-2,0,2. Common second difference of 2 so it's a quadratic with $a = 1$. When $x = 0$, $y = 1$ so $c = 1$. $a + b = -4$ so $b = -5$ giving $y = x^2 - 5x + 1$

10. Looks like it could be a cubic. Construct a table of differences.

x	-4	-3	-2	-1	0	1	2	3	4
y	-69	-32	-13	-6	-5	-4	3	22	59
	37	19	7	1	1	7	19	37	
	-18	-12	-6	0	6	12	18		
	6	6	6	6	6	6			

The common third difference confirms that this is a cubic. From here you could use your calculator to find the cubic of best fit, or proceed as follows.

For the general cubic $y = ax^3 + bx^2 + cx + d$. Constructing a difference table for this gives:

x	0	1	2	3
y	d	$a+b+c+d$	$8a+4b+2c+d$	$27a+9b+3c+d$
	$a+b+c$	$7a+3b+c$	$19a+5b+c$	
	$6a+2b$	$12a+2b$		
	$6a$			

Compare this with the corresponding part of the difference table above and the following results appear:

$$\begin{aligned} 6a &= 6 & a &= 1 \\ 6a + 2b &= 6 & b &= 0 \\ a + b + c &= 1 & c &= 0 \\ & & d &= -5 \end{aligned}$$

giving $y = x^3 - 5$.
You might have spotted this by observation without going through all this effort, but this is a generic approach that will work for any cubic.

11. This is the absolute value function $y = |x|$ and is none of linear, quadratic, cubic, exponential nor reciprocal.

12. This is is none of linear, quadratic, cubic, exponential nor reciprocal. (It is actually a log function, $y = \log_2 x$)

13. It's a linear pattern, increasing by 6 each time. The n^{th} term is $6n - 1$

14. First differences are 4, 6, 8, 10,12,14. It's a quadratic pattern where the n^{th} term is $n(n+1)$ or $n^2 + n$.

15. It's an exponential pattern with each term 3 times the previous term. The n^{th} term is 2×3^n.

16. It's an exponential pattern with each term twice the previous term. The n^{th} term is $\frac{3}{2} \times 2^n$.

Exercise 8B

1. (a) One-to-one . . . function
   (b) One-to-many . . . not a function
   (c) Many-to-one . . . function
   (d) Many-to-many . . . not a function
   (e) Many-to-one . . . function
   (f) Many-to-many . . . not a function

2. (a) x maps to unique y . . . function
   (b) x maps to unique y . . . function
   (c) x maps to either zero, one or two values of y . . . not a function
   (d) x maps to either zero, one or two values of y . . . not a function
   (e) x maps to unique y . . . function
   (f) x maps to either one, two or three values of y . . . not a function

3. (a) Range=$\{1\times 2+3,\ 2\times 2+3,\ 3\times 2+3,\ 4\times 2+3\}=\{5,\ 7,\ 9,\ 11\}$
   (b) Range=$\{(1+3)\times 2,\ (2+3)\times 2,\ (3+3)\times 2,\ (4+3)\times 2\}=\{8,\ 10,\ 12,\ 14\}$
   (c) Range=$\{1\div 1,\ 2\div 2,\ 3\div 3,\ 4\div 4\}=\{1\}$
   (d) The function machine maps $x\to x^2$. The range is the set of non-negative real numbers.

4. (a) $\mathrm{f}(4)=5\times 4-2=18$
   (b) $\mathrm{f}(-1)=5\times -1-2=-7$
   (c) $\mathrm{f}(3)=5\times 3-2=13$
   (d) $\mathrm{f}(1.2)=5\times 1.2-2=4$
   (e) $\mathrm{f}(3)+\mathrm{f}(2)=(5\times 3-2)+(5\times 2-2)=21$
   (f) $\mathrm{f}(5)=5\times 5-2=23$
   (g) $\mathrm{f}(-5)=5\times -5-2=-27$
   (h) $\mathrm{f}(a)=5\times a-2=5a-2$
   (i) $\mathrm{f}(2a)=5\times 2a-2=10a-2$
   (j) $\mathrm{f}(a^2)=5\times a^2-2=5a^2-2$
   (k) $3\mathrm{f}(2)=3(5\times 2-2)=3\times 8=24$
   (l) $\mathrm{f}(a+b)=5\times(a+b)-2=5a+5b-2$
   (m)
$$\begin{aligned}\mathrm{f}(p)&=33\\5p-2&=33\\5p&=35\\p&=7\end{aligned}$$
   (n)
$$\begin{aligned}\mathrm{f}(q)&=-12\\5q-2&=-12\\5q&=-10\\q&=-2\end{aligned}$$

5. (a) $\mathrm{f}(4)=4(4)-7=9$
   (b) $\mathrm{f}(0)=4(0)-7=-7$
   (c) $\mathrm{g}(3)=3^2-12=-3$
   (d) $\mathrm{g}(-3)=(-3)^2-12=-3$
   (e) $\mathrm{h}(-5)=(-5)^2-3(-5)+3=43$
   (f) $\mathrm{h}(5)=5^2-3(5)+3=13$
   (g) $\mathrm{h}(-2)=(-2)^2-3(-2)+3=13$
   (h) $3\mathrm{f}(a)=3(4a-7)=12a-21$
   (i) $\mathrm{f}(3a)=4(3a)-7=12a-7$
   (j) $3\mathrm{g}(a)=3(a^2-12)=3a^2-36$
   (k) $\mathrm{g}(3a)=(3a)^2-12=9a^2-12$
   (l)
$$\begin{aligned}\mathrm{g}(p)&=24\\p^2-12&=24\\p^2&=36\\p&=\pm 6\end{aligned}$$
   (m)
$$\begin{aligned}\mathrm{g}(q)&=\mathrm{h}(q)\\q^2-12&=q^2-3q+3\\-12&=-3q+3\\-15&=-3q\\q&=5\end{aligned}$$
   (n)
$$\begin{aligned}\mathrm{h}(r)&=\mathrm{f}(r)+28\\r^2-3r+3&=(4r-7)+28\\r^2-3r+3&=4r-7+28\\r^2-3r+3&=4r+21\\r^2-7r+3&=21\\r^2-7r-18&=0\\(r-9)(r+2)&=0\\r&=9\\\text{or } r&=-2\end{aligned}$$

6. Add 5 to domain to get range.
   $\{y\in\mathbb{R}:5\le y\le 8\}$

7. Subtract 3 from domain to get range.
   $\{y\in\mathbb{R}:-3\le y\le 0\}$

8. Multiply domain by 3 to get range.
   $\{y\in\mathbb{R}:-6\le y\le 15\}$

9. Multiply domain by 4 to get range.
   $\{y\in\mathbb{R}:20\le y\le 40\}$

10. Multiply domain by 2 and subtract 1 to get range.
    $\{y\in\mathbb{R}:-1\le y\le 9\}$

11. Here the minimum for the domain maps to the maximum for the range and vice versa, so the range is $\{y\in\mathbb{R}:(1-5)\le y\le(1-0)\}=\{y\in\mathbb{R}:-4\le y\le 1\}$

12. Here the minimum value of the function is zero (when $x=0$) and the maximum is 3^2. The range is $\{y\in\mathbb{R}:0\le y\le 9\}$

13. Here the minimum value of the function is zero (when $x=-1$) and the maximum is $(3+1)^2$. The range is $\{y\in\mathbb{R}:0\le y\le 16\}$

14. The minimum value of x^2+1 is 1 (when $x=0$. The maximum for the given domain is 3^2+1. The range is $\{y\in\mathbb{R}:1\le y\le 10\}$

15. Here the minimum for the domain maps to the maximum for the range and vice versa, so the range is $\{y\in\mathbb{R}:\frac{1}{4}\le y\le 1\}$

16. This function has a minimum value of 1 but it has no maximum. $\{y\in\mathbb{R}:y\ge 1\}$

17. Absolute value has a minimum of 0. $\{y\in\mathbb{R}:0\le y\le 3\}$

18. Absolute value has a minimum of 0. $\{y\in\mathbb{R}:y\ge 0\}$

19. $|x|$ has a minimum of 0 so $|x|+2$ has a minimum of 2. $\{y\in\mathbb{R}:y\ge 2\}$

20. x^2 has a minimum of 0 so x^2-1 has a minimum of -1. $\{y\in\mathbb{R}:y\ge -1\}$

21. x^2 has a minimum of 0 so x^2+4 has a minimum of 4. $\{y\in\mathbb{R}:y\ge 4\}$

22. This function has no minimum or maximum, but it cannot have a value of zero. $\{y \in \mathbb{R} : y \neq 0\}$
23. It's difficult to visualise this function without graphing it. Use your Classpad to graph it. It should be clear that the output of the function can be any real number except 1. $\{y \in \mathbb{R} : y \neq 1\}$
24. one-to-one
25. one-to-one (since every positive number has its own unique square)
26. many-to-one (since a positive and negative number can map to the same square)
27. many-to-one (as for the previous question)
28. one-to-one (any number can be the square root of at most one other number)
29. The natural domain of the function is $\{x \in \mathbb{R} : x \geq 0\}$ (since the function is not defined for negative x), and it is a one-to-one function (like the previous question).
30. Both x and y can take any real value.
    $\{x \in \mathbb{R}\}, \quad \{y \in \mathbb{R}\}$
31. x can take any real value, but y can not be negative.
    $\{x \in \mathbb{R}\}, \quad \{y \in \mathbb{R} : y \geq 0\}$
32. Neither x nor y can be negative.
    $\{x \in \mathbb{R} : x \geq 0\}, \quad \{y \in \mathbb{R} : y \geq 0\}$
33. x must not be less than 3 (so that $x - 3$ is not negative) and y can not be negative.
    $\{x \in \mathbb{R} : x \geq 3\}, \quad \{y \in \mathbb{R} : y \geq 0\}$
34. x must not be less than -3 (so that $x + 3$ is not negative) and y can not be negative.
    $\{x \in \mathbb{R} : x \geq -3\}, \quad \{y \in \mathbb{R} : y \geq 0\}$
35. x must not be less than 3 (so that $x - 3$ is not negative) and y can not be less than 5.
    $\{x \in \mathbb{R} : x \geq 3\}, \quad \{y \in \mathbb{R} : y \geq 5\}$
36. Neither x nor y can be zero.
    $\{x \in \mathbb{R} : x \neq 0\}, \quad \{y \in \mathbb{R} : y \neq 0\}$
37. Neither $x - 1$ nor y can be zero.
    $\{x \in \mathbb{R} : x \neq 1\}, \quad \{y \in \mathbb{R} : y \neq 0\}$
38. Neither $x - 3$ nor y can be zero.
    $\{x \in \mathbb{R} : x \neq 3\}, \quad \{y \in \mathbb{R} : y \neq 0\}$
39. $x - 3$ must be non-negative and non-zero and y similarly can be neither zero nor a negative number.
    $\{x \in \mathbb{R} : x > 3\}, \quad \{y \in \mathbb{R} : y > 0\}$

Exercise 8C

1. (a) $x \to \mathrm{f}(x) \to \mathrm{g\,f}(x)$

    $0 \to 1 \to -1$
    $1 \to 2 \to 1$
    $2 \to 3 \to 3$
    $3 \to 4 \to 5$
    $4 \to 5 \to 7$

    Range is {-1, 1, 3, 5, 7}.

   (b) $x \to \mathrm{g}(x) \to \mathrm{f\,g}(x)$

    $0 \to -3 \to -2$
    $1 \to -1 \to 0$
    $2 \to 1 \to 2$
    $3 \to 3 \to 4$
    $4 \to 5 \to 6$

    Range is {-2, 0, 2, 4, 6}.

   (c) $x \to \mathrm{g}(x) \to \mathrm{g\,g}(x)$

    $0 \to -3 \to -9$
    $1 \to -1 \to -5$
    $2 \to 1 \to -1$
    $3 \to 3 \to 3$
    $4 \to 5 \to 7$

    Range is {-9, -5, -1, 3, 7}.

2. (a) $x \to \mathrm{f}(x) \to \mathrm{g\,f}(x)$

    $1 \to 4 \to 9$
    $2 \to 5 \to 16$
    $3 \to 6 \to 25$

    Range is {9, 16, 25}.

   (b) $x \to \mathrm{h}(x) \to \mathrm{g\,h}(x) \to \mathrm{f\,g\,h}(x)$

    $1 \to 1 \to 0 \to 3$
    $2 \to 8 \to 49 \to 52$
    $3 \to 27 \to 676 \to 679$

    Range is {3, 52, 679}.

   (c) $x \to \mathrm{f}(x) \to \mathrm{g\,f}(x) \to \mathrm{h\,g\,f}(x)$

    $1 \to 4 \to 9 \to 729$
    $2 \to 5 \to 16 \to 4096$
    $3 \to 6 \to 25 \to 15\,625$

    Range is {729, 4096, 15 625}.

3. (a) Domain: $\{x \in \mathbb{R}\}$; Range: $\{y \in \mathbb{R}\}$

   (b) Domain: $\{x \in \mathbb{R}\}$; Range: $\{y \in \mathbb{R}\}$

   (c) $\mathrm{f}(x) + \mathrm{g}(x) = (x + 5) + (x - 5) = 2x$
   Domain: $\{x \in \mathbb{R}\}$; Range: $\{y \in \mathbb{R}\}$

   (d) $\mathrm{f}(x) - \mathrm{g}(x) = (x + 5) - (x - 5) = 10$
   Domain: $\{x \in \mathbb{R}\}$; Range: $\{10\}$

(e) $\text{f}(x) \cdot \text{g}(x) = (x+5)(x-5)$
$= x^2 - 25$
Domain: $\{x \in \mathbb{R}\}$; Range: $\{y \in \mathbb{R} : y \geq -25\}$

(f) $\dfrac{\text{f}(x)}{\text{g}(x)} = \dfrac{x+5}{x-5}$
$= \dfrac{(x-5)+10}{x-5}$
$= 1 + \dfrac{10}{x-5}$
Domain: $\{x \in \mathbb{R} : x \neq 5\}$; Range: $\{y \in \mathbb{R} : y \neq 1\}$

4. (a) $\frac{2}{3x+2} = \frac{2}{\text{f}(x)} = \text{g}\,\text{f}(x)$
(b) $\sqrt{3x+2} = \sqrt{\text{f}(x)} = \text{h}\,\text{f}(x)$
(c) $\frac{6}{x} + 2 = 3 \times \frac{2}{x} + 2 = 3\text{g}(x) + 2 = \text{f}\,\text{g}(x)$
(d) $3\sqrt{x} + 2 = 3\text{h}(x) + 2 = \text{f}\,\text{h}(x)$
(e) $\frac{2}{\sqrt{x}} = \frac{2}{\text{h}(x)} = \text{g}\,\text{h}(x)$
(f) $\sqrt{\frac{2}{x}} = \sqrt{\text{g}(x)} = \text{h}\,\text{g}(x)$
(g) $9x+8 = (9x+6)+2 = 3(3x+2)+2 = \text{f}\,\text{f}(x)$
(h) $x^{0.25} = \sqrt{\sqrt{x}} = \text{h}\,\text{h}(x)$
(i) $27x + 26 = 3(9x+8) + 2 = \text{f}\,\text{f}\,\text{f}(x)$

5. (a) $\text{f} \circ \text{f}(x) = 2(\text{f}(x)) - 3$
$= 2(2x-3) - 3$
$= 4x - 6 - 3$
$= 4x - 9$

(b) $\text{g} \circ \text{g}(x) = 4(g(x)) + 1$
$= 4(4x+1) + 1$
$= 16x + 4 + 1$
$= 16x + 5$

(c) $\text{h} \circ \text{h}(x) = (\text{h}(x))^2 + 1$
$= (x^2+1)^2 + 1$
$= x^4 + 2x^2 + 1 + 1$
$= x^4 + 2x^2 + 2$

(d) $\text{f} \circ \text{g}(x) = 2(g(x)) - 3$
$= 2(4x+1) - 3$
$= 8x + 2 - 3$
$= 8x - 1$

(e) $\text{g} \circ \text{f}(x) = 4(\text{f}(x)) + 1$
$= 4(2x-3) + 1$
$= 8x - 12 + 1$
$= 8x - 11$

(f) $\text{f} \circ \text{h}(x) = 2(\text{h}(x)) - 3$
$= 2(x^2+1) - 3$
$= 2x^2 + 2 - 3$
$= 2x^2 - 1$

(g) $\text{h} \circ \text{f}(x) = (\text{f}(x))^2 + 1$
$= (2x-3)^2 + 1$
$= 4x^2 - 12x + 9 + 1$
$= 4x^2 - 12x + 10$

(h) $\text{g} \circ \text{h}(x) = 4(\text{h}(x)) + 1$
$= 4(x^2+1) + 1$
$= 4x^2 + 4 + 1$
$= 4x^2 + 5$

(i) $\text{h} \circ \text{g}(x) = (\text{g}(x))^2 + 1$
$= (4x+1)^2 + 1$
$= 16x^2 + 8x + 1 + 1$
$= 16x^2 + 8x + 2$

6. (a) $\text{f} \circ \text{f}(x) = 2(\text{f}(x)) + 5$
$= 2(2x+5) + 5$
$= 4x + 10 + 5$
$= 4x + 15$

(b) $\text{g} \circ \text{g}(x) = 3(g(x)) + 1$
$= 3(3x+1) + 1$
$= 9x + 3 + 1$
$= 9x + 4$

(c) $\text{h} \circ \text{h}(x) = 1 + \dfrac{2}{\text{h}(x)}$
$= 1 + \dfrac{2}{1 + \frac{2}{x}}$
$= 1 + \dfrac{2}{\frac{x+2}{x}}$
$= 1 + \dfrac{2x}{x+2}$

(d) $\text{f} \circ \text{g}(x) = 2(g(x)) + 5$
$= 2(3x+1) + 5$
$= 6x + 2 + 5$
$= 6x + 7$

(e) $\text{g} \circ \text{f}(x) = 3(\text{f}(x)) + 1$
$= 3(2x+5) + 1$
$= 6x + 15 + 1$
$= 6x + 16$

(f) $\text{f} \circ \text{h}(x) = 2(\text{h}(x)) + 5$
$= 2(1 + \dfrac{2}{x}) + 5$
$= 2 + \dfrac{4}{x} + 5$
$= \dfrac{4}{x} + 7$

(g) $\text{h} \circ \text{f}(x) = 1 + \dfrac{2}{\text{f}(x)}$
$= 1 + \dfrac{2}{2x+5}$

(h) $\text{g} \circ \text{h}(x) = 3(\text{h}(x)) + 1$
$= 3(1 + \dfrac{2}{x}) + 1$
$= 3 + \dfrac{6}{x} + 1$
$= \dfrac{6}{x} + 4$

(i) $\text{h} \circ \text{g}(x) = 1 + \dfrac{2}{\text{g}(x)}$

$= 1 + \dfrac{2}{3x+1}$

7. $\text{g}[\text{f}(x)] = \sqrt{x-4}$
$x - 4 \geq 0$
$x \geq 4$

8. $\text{g}[\text{f}(x)] = \sqrt{4-x}$
$4 - x \geq 0$
$x \leq 4$

9. $\text{g}[\text{f}(x)] = \sqrt{4-x^2}$
$4 - x^2 \geq 0$
$x^2 \leq 4$
$-2 \leq x \leq 2$

10. $\text{g}[\text{f}(x)] = \sqrt{4-|x|}$
$4 - |x| \geq 0$
$|x| \leq 4$
$-4 \leq x \leq 4$

11. $\text{g}[\text{f}(x)] = \sqrt{(x+3)-5}$
$(x+3) - 5 \geq 0$
$x - 2 \geq 0$
$x \geq 2$

12. $\text{g}[\text{f}(x)] = \sqrt{(x-6)+3}$
$(x-6) + 3 \geq 0$
$x - 3 \geq 0$
$x \geq 3$

13. (a) $\text{f}(3) = (3)^2 + 3 = 12$

(b) $\text{f}(-3) = (-3)^2 + 3 = 12$

(c) $\text{g}(2) = \frac{1}{2}$

(d) $\text{fg}(1) = \text{f}(\frac{1}{1})$

$= f(1)$

$= (1)^2 + 3$

$= 4$

(e) $\text{gf}(1) = \text{g}\left((1)^2 + 3\right)$

$= g(4)$

$= \dfrac{1}{4}$

(f) $\mathbb{R} \rightarrow \boxed{\text{f}(x) = x^2 + 3} \rightarrow \{y \in \mathbb{R} : y \geq 3\}$

(g) $\{x \in \mathbb{R} : x \neq 0\} \rightarrow \boxed{\text{g}(x) = \frac{1}{x}} \rightarrow \{y \in \mathbb{R} : y \neq 0\}$

(h) It may be useful to think of these compound functions as sequential mappings, something like

$$\{x\} \xrightarrow{\text{f}(x)} \{u\} \xrightarrow{\text{g}(u)} \{y\}$$

where $\{x\}$ is the domain of the compound function and $\{y\}$ is its range. The intermediate set $\{u\}$ is the intersection of the range of f and the domain of g. Thus to determine the natural domain we work right to left, then to determine the range we work left to right.

The whole of the range of $\text{f}(x)$ lies within the domain of $\text{g}(x)$ (since the only real number excluded from the domain of $\text{g}(x)$ is 0 and this is outside the range of $\text{f}(x)$) so there is no additional restriction to the domain and the domain of $\text{gf}(x)$ is the same the domain of $\text{f}(x)$: $x \in \mathbb{R}$.

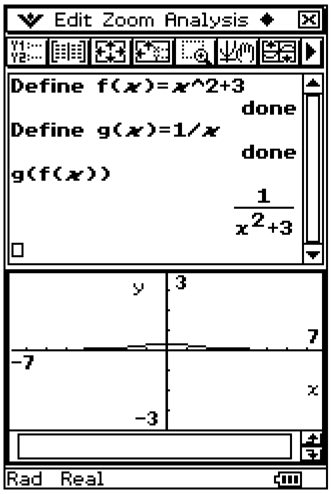
When the domain of $\text{g}(x)$ is restricted to the range of $\text{f}(x)$ (i.e. $x \in \mathbb{R} : x \geq 3$) the range is $y \in \mathbb{R} : 0 < y \leq \frac{1}{3}$.

$\mathbb{R} \rightarrow \boxed{\text{gf}(x)} \rightarrow \left\{y \in \mathbb{R} : 0 < y \leq \frac{1}{3}\right\}$

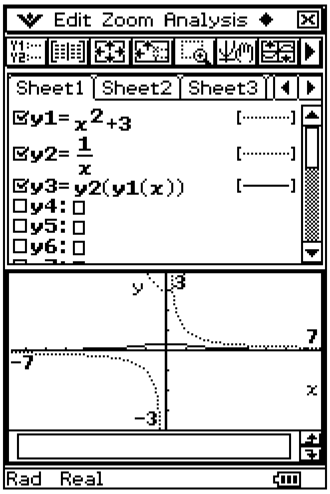
There are two main ways of graphing this on the ClassPad 330.

From the Main app
Interactive->Define
Func name: f
Variable/s: x
Expression: xˆ2+3
OK

Interactive->Define
Func name: g
Variable/s: x
Expression: 1/x
OK
g(f(x))
Tap the graph icon then highlight g(f(x)) and drag and drop it onto the graph.



From the Graph&Tab app
y1=xˆ2+3
y2=1/x
y3=y2(y1(x))
Tap the graph icon.



Note that to enter y2(y1(x)) you must use y in the abc tab, not the y button on the keyboard or under the VAR tab.

Once you have obtained the graph you can examine it in all the usual ways.

(i) The whole of the range of $g(x)$ lies within the domain of $f(x)$ (since $f(x)$ is defined for all real numbers) so there is no additional restriction to the domain and the domain of $fg(x)$ is the same the domain of $g(x)$: $\{x \in \mathbb{R} : x \neq 0\}$.

When the domain of $f(x)$ is restricted to the range of $g(x)$ (i.e. $x \in \mathbb{R} : x \neq 0$) the range is $y \in \mathbb{R} : y > 3$.

$$\{x \in \mathbb{R} : x \neq 0\} \to \boxed{fg(x)} \to \{y \in \mathbb{R} : y > 3\}$$

14. (a) $f(5) = 25 - (5)^2 = 0$

(b) $f(-5) = 25 - (-5)^2 = 0$

(c) $g(4) = \sqrt{4} = 2$

(d)
$$\begin{aligned} fg(4) &= f(\sqrt{4}) \\ &= f(2) \\ &= 25 - (2)^2 \\ &= 21 \end{aligned}$$

(e)
$$\begin{aligned} gf(4) &= g(25 - (4)^2) \\ &= g(9) \\ &= \sqrt{9} \\ &= 3 \end{aligned}$$

(f) $\mathbb{R} \to \boxed{f(x) = 25 - x^2} \to \{y \in \mathbb{R} : y \leq 25\}$

(g) $\{x \in \mathbb{R} : x \geq 0\} \to \boxed{g(x) = \sqrt{x}} \to \{y \in \mathbb{R} : y \geq 0\}$

(h) $gf(x)$:

$$\{x \in \mathbb{R} : -5 \leq x \leq 5\} \xrightarrow{f(x)=25-x^2} \{u \in \mathbb{R} : 0 \leq u \leq 25\} \xrightarrow{g(x)=\sqrt{x}} \{y \in \mathbb{R} : 0 \leq y \leq 5\}$$

Hence the domain is $\{x \in \mathbb{R} : -5 \leq x \leq 5\}$ and the range is $\{y \in \mathbb{R} : 0 \leq y \leq 5\}$.

(i) $fg(x)$:

$$\{x \in \mathbb{R} : x \geq 0\} \xrightarrow{g(x)=\sqrt{x}} \{u \in \mathbb{R} : u \geq 0\} \xrightarrow{f(x)=25-x^2} \{y \in \mathbb{R} : 0 \leq y \leq 25\}$$

Hence the domain is $\{x \in \mathbb{R} : x \geq 0\}$ and the range is $\{y \in \mathbb{R} : 0 \leq y \leq 25\}$.

15. (a) $g(x)$ is not defined for $x = 3$ so we must exclude $x = 1$ from the domain. $g \circ f(x)$:

$$\{x \in \mathbb{R} : x \neq 1\} \xrightarrow{f(x)=x+2} \{u \in \mathbb{R} : u \neq 3\} \xrightarrow{g(x)=\frac{1}{x-3}} \{y \in \mathbb{R} : y \neq 0\}$$

Hence the domain is $\{x \in \mathbb{R} : x \neq 1\}$ and the range is $\{y \in \mathbb{R} : y \neq 0\}$.

(b) $f \circ g(x)$:

$$\{x \in \mathbb{R} : x \neq 3\} \xrightarrow{g(x)=\frac{1}{x-3}} \{u \in \mathbb{R} : u \neq 0\} \xrightarrow{f(x)=x+2} \{y \in \mathbb{R} : y \neq 2\}$$

Hence the domain is $\{x \in \mathbb{R} : x \neq 3\}$ and the range is $\{y \in \mathbb{R} : y \neq 2\}$.

16. (a) $g \circ f(x)$:

$$\{x \in \mathbb{R} : x \geq 0\} \xrightarrow{f(x)=\sqrt{x}} \{u \in \mathbb{R} : u \geq 0\} \xrightarrow{g(x)=2x-1} \{y \in \mathbb{R} : y \geq -1\}$$

Hence the domain is $\{x \in \mathbb{R} : x \geq 0\}$ and the range is $\{y \in \mathbb{R} : y \geq -1\}$.

(b) $\mathrm{f} \circ \mathrm{g}(x)$:

The input to f must be non-negative so for g:

$$\begin{aligned} 2x - 1 &\geq 0 \\ 2x &\geq 1 \\ x &\geq 0.5 \end{aligned}$$

$$\{x \in \mathbb{R} : x \geq 0.5\} \xrightarrow{\mathrm{g}(x)=2x-1} \{u \in \mathbb{R} : u \geq 0\} \xrightarrow{\mathrm{f}(x)=\sqrt{x}} \{y \in \mathbb{R} : y \geq 0\}$$

Hence the domain is $\{x \in \mathbb{R} : x \geq 0.5\}$ and the range is $\{y \in \mathbb{R} : y \geq 0\}$.

17. (a) $\mathrm{g} \circ \mathrm{f}(x)$:

$$\{x \in \mathbb{R} : x \neq 0\} \xrightarrow{\mathrm{f}(x)=\frac{1}{x^2}} \{u \in \mathbb{R} : u > 0\} \xrightarrow{\mathrm{g}(x)=\sqrt{x}} \{y \in \mathbb{R} : y > 0\}$$

Hence the domain is $\{x \in \mathbb{R} : x \neq 0\}$ and the range is $\{y \in \mathbb{R} : y > 0\}$.

(b) $\mathrm{f} \circ \mathrm{g}(x)$:

$$\{x \in \mathbb{R} : x > 0\} \xrightarrow{\mathrm{g}(x)=\sqrt{x}} \{u \in \mathbb{R} : u > 0\} \xrightarrow{\mathrm{f}(x)=\frac{1}{x^2}} \{y \in \mathbb{R} : y > 0\}$$

Hence the domain is $\{x \in \mathbb{R} : x > 0\}$ and the range is $\{y \in \mathbb{R} : y > 0\}$.

18. The natural domain of $\mathrm{g}(x) = \sqrt{x}$ is $\{x \in \mathbb{R} : x \geq 0\}$ and the corresponding range is $\{y \in \mathbb{R} : y \geq 0\}$. All of this range lies within the natural domain of $\mathrm{f}(x) = x+3$ so $\mathrm{f}(x)$ is defined for every element in the range of $\mathrm{g}(x)$ and hence $\mathrm{f}\,[\mathrm{g}(x)]$ is a function for all x in the natural domain of $\mathrm{g}(x)$.

The natural domain of $\mathrm{f}(x) = x + 3$ is $\mathbb{R}$ and the corresponding range is also $\mathbb{R}$. However $\mathrm{g}(x) = \sqrt{x}$ is only defined for $\{x \in \mathbb{R} : x \geq 0\}$ so $\mathrm{g}\,[\mathrm{f}(x)]$ is undefined for values of x that result in $\mathrm{f}(x) < 0$, i.e. $x < -3$, even though these values lie within the domain of $\mathrm{f}(x)$.

19. The natural domain of $\mathrm{g}(x) = \frac{1}{x-5}$ is $\{x \in \mathbb{R} : x \neq 5\}$ and the corresponding range is $\{y \in \mathbb{R} : y \neq 0\}$. All of this range lies within the natural domain of $\mathrm{f}(x) = x + 3$ so $\mathrm{f}(x)$ is defined for every element in the range of $\mathrm{g}(x)$ and hence $\mathrm{f}\,[\mathrm{g}(x)]$ is a function for all x in the natural domain of $\mathrm{g}(x)$.

The natural domain of $\mathrm{f}(x) = x + 3$ is $\mathbb{R}$ and the corresponding range is also $\mathbb{R}$. However $\mathrm{g}(x) = \frac{1}{x-5}$ is only defined for $\{x \in \mathbb{R} : x \neq 5\}$ so $\mathrm{g}\,[\mathrm{f}(x)]$ is undefined for the value of x that results in $\mathrm{f}(x) = 5$, i.e. $x = 2$, even though this value lies within the domain of $\mathrm{f}(x)$.

20. (a) $\mathrm{g} \circ \mathrm{f}(x)$:

$$\{x\} \xrightarrow{\mathrm{f}(x)=x^2-9} \{u\} \xrightarrow{\mathrm{g}(x)=\frac{1}{x}} \{y\}$$

The natural domain of $\mathrm{g}(x)$ is $\{x \in \mathbb{R} : x \neq 0\}$ so we must exclude 0 from the output of $\mathrm{f}(x)$ and hence limit the domain such that

$$\begin{aligned} x^2 - 9 &\neq 0 \\ (x+3)(x-3) &\neq 0 \\ x &\neq -3 \\ \text{and } x &\neq 3 \end{aligned}$$

The domain of $\mathrm{g} \circ \mathrm{f}(x)$ is

$$\{x \in \mathbb{R} : x \neq -3, x \neq 3\}$$

.

This, then, gives us $\{x \in \mathbb{R} : x \geq -9, x \neq 0\}$ for the domain of $\mathrm{g}(x)$. To determine the range, consider this domain in parts:

$$\{x \in \mathbb{R} : -9 \leq x < 0\} \xrightarrow{\mathrm{g}(x)} \{y \in \mathbb{R} : y \leq -\frac{1}{9}\}$$

and

$$\{x \in \mathbb{R} : x > 0\} \xrightarrow{\mathrm{g}(x)} \{y \in \mathbb{R} : y > 0\}$$

The range of $\mathrm{g} \circ \mathrm{f}(x)$ is

$$\{y \in \mathbb{R} : y \leq -\frac{1}{9}\} \cup \{y \in \mathbb{R} : y > 0\}$$

(b) $\mathrm{f} \circ \mathrm{g}(x)$:

$$\{x\} \xrightarrow{\mathrm{g}(x)=\frac{1}{x}} \{u\} \xrightarrow{\mathrm{f}(x)=x^2-9} \{y\}$$

The natural domain of $\mathrm{f}(x)$ is $\mathbb{R}$ so we need place no additional restriction on the natural domain of $\mathrm{g}(x)$

The domain of $\mathrm{f} \circ \mathrm{g}(x)$ is $\{x \in \mathbb{R} : x \neq 0\}$.

This gives us $\{x \in \mathbb{R} : x \neq 0\}$ for the domain of $\mathrm{f}(x)$. The range then is the natural range of $\mathrm{f}(x)$ excluding $f(0) = -9$, i.e. $\{y \in \mathbb{R} : y \geq -9, y \neq -9\}$. This simplifies to give the range of $\mathrm{f} \circ \mathrm{g}(x)$ as $\{y \in \mathbb{R} : y > -9\}$

Exercise 8D

1. (a) Yes. All linear functions have an inverse on their natural domains except those in the form $f(x) = k$.
   (b) Yes (linear)
   (c) Yes (linear)
   (d) No. Quadratic functions do not have an inverse on their natural domains.
   (e) No (quadratic)
   (f) No (quadratic)
   (g) Yes. All reciprocal functions have an inverse on their natural domains.
   (h) Yes (reciprocal)
   (i) No. Does not pass the horizontal line test because $f(-x) = f(x)$ (so it's not a one-to-one function).
2. $f^{-1}(x) = x + 2$; domain $\mathbb{R}$, range $\mathbb{R}$
3. $f^{-1}(x) = \frac{x+5}{2}$; domain $\mathbb{R}$, range $\mathbb{R}$
4. $f^{-1}(x) = \frac{x-2}{5}$; domain $\mathbb{R}$, range $\mathbb{R}$
5. $$\begin{aligned} y &= \frac{1}{x-4} \\ x-4 &= \frac{1}{y} \\ x &= \frac{1}{y} + 4 \\ f^{-1}(x) &= \frac{1}{x} + 4 \end{aligned}$$
   Domain $\{x \in \mathbb{R} : x \neq 0\}$, range $\{y \in \mathbb{R} : y \neq 4\}$
6. $f^{-1}(x) = \frac{1}{x} - 3$ (following the pattern of the previous question); domain $\{x \in \mathbb{R} : x \neq 0\}$, range $\{y \in \mathbb{R} : y \neq -3\}$
7. $$\begin{aligned} y &= \frac{1}{2x-5} \\ 2x-5 &= \frac{1}{y} \\ 2x &= \frac{1}{y} + 5 \\ x &= \frac{\frac{1}{y} + 5}{2} \\ &= \frac{1}{2y} + \frac{5}{2} \\ f^{-1}(x) &= \frac{1}{2x} + \frac{5}{2} \end{aligned}$$
   Domain $\{x \in \mathbb{R} : x \neq 0\}$, range $\{y \in \mathbb{R} : y \neq \frac{5}{2}\}$
8. $$\begin{aligned} y &= 1 + \frac{1}{2+x} \\ y-1 &= \frac{1}{2+x} \\ 2+x &= \frac{1}{y-1} \\ x &= \frac{1}{y-1} - 2 \\ f^{-1}(x) &= \frac{1}{x-1} - 2 \end{aligned}$$
   Domain $\{x \in \mathbb{R} : x \neq 1\}$, range $\{y \in \mathbb{R} : y \neq -2\}$
9. $$\begin{aligned} y &= 3 - \frac{1}{x-1} \\ 3-y &= \frac{1}{x-1} \\ x-1 &= \frac{1}{3-y} \\ x &= \frac{1}{3-y} + 1 \\ f^{-1}(x) &= \frac{1}{3-x} + 1 \end{aligned}$$
   Domain $\{x \in \mathbb{R} : x \neq 3\}$, range $\{y \in \mathbb{R} : y \neq 1\}$
10. $$\begin{aligned} y &= 4 + \frac{2}{2x-1} \\ y-4 &= \frac{2}{2x-1} \\ 2x-1 &= \frac{2}{y-4} \\ 2x &= \frac{2}{y-4} + 1 \\ x &= \frac{1}{y-4} + \frac{1}{2} \\ f^{-1}(x) &= \frac{1}{x-4} + \frac{1}{2} \end{aligned}$$
    Domain $\{x \in \mathbb{R} : x \neq 4\}$, range $\{y \in \mathbb{R} : y \neq \frac{1}{2}\}$
11. $f^{-1}(x) = x^2$; domain $\{x \in \mathbb{R} : x \geq 0\}$ (because that's the range of $f(x)$), range $\{y \in \mathbb{R} : y \geq 0\}$
12. $$\begin{aligned} y &= \sqrt{x+1} \\ y^2 &= x+1 \\ x &= y^2 - 1 \\ f^{-1}(x) &= x^2 - 1 \end{aligned}$$
    Domain $\{x \in \mathbb{R} : x \geq 0\}$ (range of $f(x)$), range $\{y \in \mathbb{R} : y \geq -1\}$
13. $$\begin{aligned} y &= \sqrt{2x-3} \\ y^2 &= 2x-3 \\ 2x &= y^2 + 3 \\ x &= \frac{y^2+3}{2} \\ f^{-1}(x) &= \frac{x^2+3}{2} \end{aligned}$$
    Domain $\{x \in \mathbb{R} : x \geq 0\}$ (range of $f(x)$), range $\{y \in \mathbb{R} : y \geq \frac{3}{2}\}$
14. $f^{-1}(x) = \frac{x-5}{2}$
15. $g^{-1}(x) = \frac{x-1}{3}$
16. $$\begin{aligned} y &= 1 + \frac{2}{x} \\ y-1 &= \frac{2}{x} \\ x &= \frac{2}{y-1} \\ h^{-1}(x) &= \frac{2}{x-1} \end{aligned}$$

17. We expect $\mathrm{f} \circ \mathrm{f}^{-1}(x) = x$ since f and f^{-1} are inverses. Demonstrating that this is true:

$$\begin{aligned}\mathrm{f} \circ \mathrm{f}^{-1}(x) &= f(\frac{x-5}{2})\\ &= 2(\frac{x-5}{2})+5\\ &= (x-5)+5\\ &= x\end{aligned}$$

18. Similarly for $\mathrm{f}^{-1} \circ \mathrm{f}(x)$:

$$\begin{aligned}\mathrm{f}^{-1} \circ \mathrm{f}(x) &= f^{-1}(2x+5)\\ &= \frac{(2x+5)-5}{2}\\ &= \frac{2x}{2}\\ &= x\end{aligned}$$

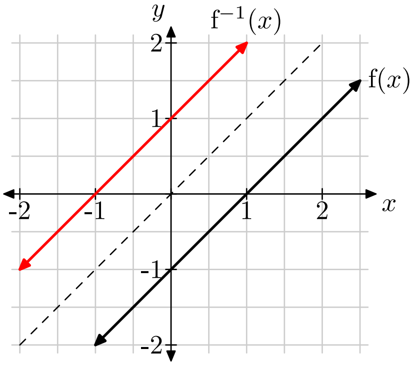
19. $$\begin{aligned}\mathrm{f} \circ \mathrm{h}^{-1}(x) &= f(\frac{2}{x-1})\\ &= 2(\frac{2}{x-1})+5\\ &= \frac{4}{x-1}+5\end{aligned}$$

20. $$\begin{aligned}\mathrm{f} \circ \mathrm{g}(x) &= f(3x+1)\\ &= 2(3x+1)+5\\ &= 6x+2+5\\ &= 6x+7\\ (\mathrm{f} \circ \mathrm{g})^{-1}(x) &= \frac{x-7}{6}\end{aligned}$$

21. $$\begin{aligned}\mathrm{g}^{-1} \circ \mathrm{f}^{-1} &= \mathrm{g}^{-1}(\frac{x-5}{2})\\ &= \frac{(\frac{x-5}{2})-1}{3}\\ &= \frac{(\frac{x-5}{2})-\frac{2}{2}}{3}\\ &= \frac{\frac{x-7}{2}}{3}\\ &= \frac{x-7}{6}\end{aligned}$$

22. $$\begin{aligned}\mathrm{f} \circ \mathrm{g}(x)^{-1} &= f(\frac{x-1}{3})\\ &= 2(\frac{x-1}{3})+5\\ &= \frac{2x-2}{3}+5\\ &= \frac{2x-2}{3}+\frac{15}{3}\\ &= \frac{2x-2+15}{3}\\ &= \frac{2x+13}{3}\end{aligned}$$

23. (a) Yes, this is a one-to-one function.



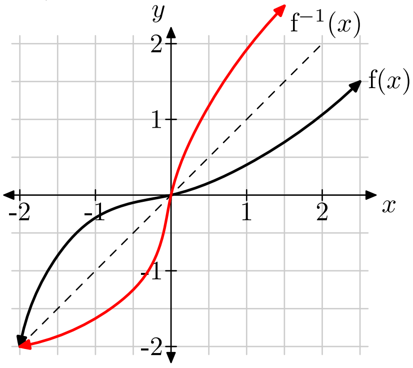
(b) Not a function.

(c) Is a function, but not one-to-one.

(d) Not a function.

(e) Not a function.

(f) Yes, this is a one-to-one function.



24. Restricted domain of $\mathrm{f}(x)$: $x \geq 0$
$\mathrm{f}^{-1}(x) = \sqrt{x-3}$
Domain and range of $\mathrm{f}^{-1}(x)$:

$$\{x \in \mathbb{R} : x \geq 3\} \quad \{y \in \mathbb{R} : y \geq 0\}$$

Alternatively: restricted domain of $\mathrm{f}(x)$: $x \leq 0$
$\mathrm{f}^{-1}(x) = -\sqrt{x-3}$
Domain and range of $\mathrm{f}^{-1}(x)$:

$$\{x \in \mathbb{R} : x \geq 3\} \quad \{y \in \mathbb{R} : y \leq 0\}$$

25. Restricted domain of $\mathrm{f}(x)$: $x \geq -3$
$\mathrm{f}^{-1}(x) = \sqrt{x}-3$
Domain and range of $\mathrm{f}^{-1}(x)$:

$$\{x \in \mathbb{R} : x \geq 0\} \quad \{y \in \mathbb{R} : y \geq -3\}$$

Alternatively: restricted domain of $\mathrm{f}(x)$: $x \leq -3$
$\mathrm{f}^{-1}(x) = -\sqrt{x}-3$
Domain and range of $\mathrm{f}^{-1}(x)$:

$$\{x \in \mathbb{R} : x \geq 0\} \quad \{y \in \mathbb{R} : y \leq -3\}$$

26. Restricted domain of $\mathrm{f}(x)$: $x \geq 3$
$\mathrm{f}^{-1}(x) = \sqrt{x-2}+3$
Domain and range of $\mathrm{f}^{-1}(x)$:

$$\{x \in \mathbb{R} : x \geq 2\} \quad \{y \in \mathbb{R} : y \geq 3\}$$

Alternatively: restricted domain of $\mathrm{f}(x)$: $x \leq 3$
$\mathrm{f}^{-1}(x) = -\sqrt{x-2}+3$
Domain and range of $\mathrm{f}^{-1}(x)$:

$$\{x \in \mathbb{R} : x \geq 2\} \quad \{y \in \mathbb{R} : y \leq 3\}$$

27. Restricted domain of $\mathrm{f}(x)$: $0 \leq x \leq 2$
$\mathrm{f}^{-1}(x) = \sqrt{4 - x^2}$
Domain and range of $\mathrm{f}^{-1}(x)$:

$$\{x \in \mathbb{R} : 0 \leq x \leq 2\} \quad \{y \in \mathbb{R} : 0 \leq y \leq 2\}$$

Alternatively:

restricted domain of $\mathrm{f}(x)$: $-2 \leq x \leq 0$
$\mathrm{f}^{-1}(x) = -\sqrt{4 - x^2}$
Domain and range of $\mathrm{f}^{-1}(x)$:

$$\{x \in \mathbb{R} : 0 \leq x \leq 2\} \quad \{y \in \mathbb{R} : -2 \leq y \leq 0\}$$

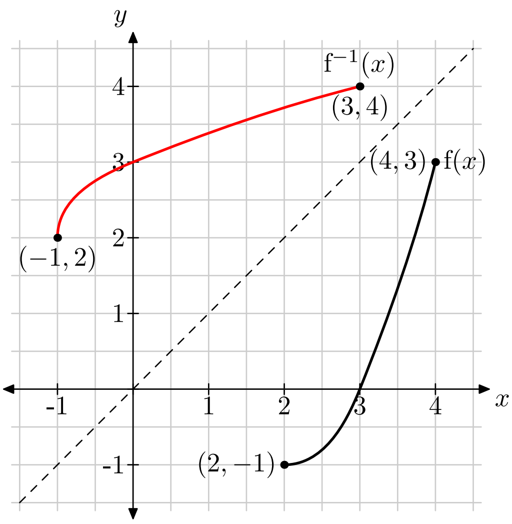
Miscellaneous Exercise 8

1. $C = 2\pi \times 6350 \cos 40° = 30\,564\text{km}$

2. (a) Domain: $\{x \in \mathbb{R} : 2 \leq x \leq 4\}$
Range: $\{y \in \mathbb{R} : -1 \leq y \leq 3\}$

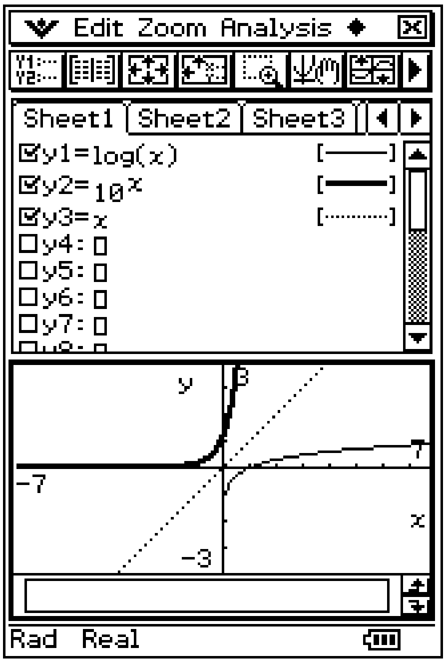
(b) Domain: $\{x \in \mathbb{R} : -1 \leq x \leq 3\}$
Range: $\{y \in \mathbb{R} : 2 \leq y \leq 4\}$

(c)

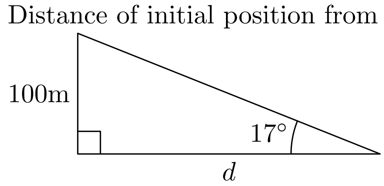


(d)
$$\begin{aligned} y &= (x-2)^2 - 1 \\ y + 1 &= (x-2)^2 \\ x - 2 &= \sqrt{y+1} \\ x &= \sqrt{y+1} + 2 \\ \mathrm{f}^{-1}(x) &= \sqrt{x+1} + 2 \end{aligned}$$

3.
$$\begin{aligned} y &= \log_{10} x \\ 10^y &= x \\ \mathrm{f}^{-1}(x) &= 10^x \end{aligned}$$

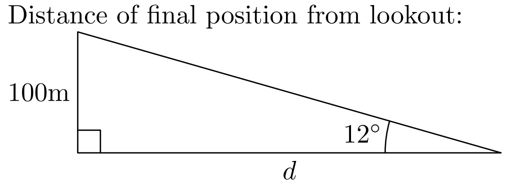


4. (a) Distance of initial position from lookout:



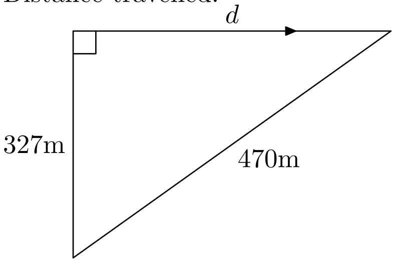
$$\begin{aligned} \tan 17° &= \frac{100}{d} \\ d &= \frac{100}{\tan 17°} \\ &\approx 327\text{m} \end{aligned}$$

Distance of final position from lookout:



$$\begin{aligned} \tan 12° &= \frac{100}{d} \\ d &= \frac{100}{\tan 12°} \\ &\approx 470\text{m} \end{aligned}$$

Distance travelled:

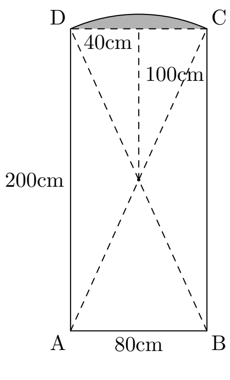


$$d = \sqrt{470^2 - 327^2} \approx 338\text{m}$$

(b) $\text{Speed} = \frac{338}{10} = 33.8\text{m/min} = 33.8 \times 60 = 2029\text{m/hr} = 2.0\text{km/h}$

5. (a) Translate up 5. Maximum t.p. $(-1, 26)$; Minimum t.p. $(3, -6)$.
   (b) Translate down 5. Maximum t.p. $(-1, 16)$; Minimum t.p. $(3, -16)$.
   (c) Reflection in y-axis. Maximum t.p. $(1, 21)$; Minimum t.p. $(-3, -11)$.
   (d) Reflection in x-axis. Minimum t.p. $(-1, -21)$; Maximum t.p. $(3, 11)$.
   (e) y-dilation ($\times 3$ enlargement). Maximum t.p. $(-1, 63)$; Minimum t.p. $(3, -33)$.
   (f) x-dilation ($\div 2$ reduction). Maximum t.p. $(-0.5, 21)$; Minimum t.p. $(1.5, -11)$.

6. Area=area of rectangle + area of segment
   Area of rectangle=$200 \times 80 = 16\,000\text{cm}^2$
   Area of segment: Radius:



$$r = \sqrt{100^2 + 40^2} = 107.7\text{cm}$$

Angle:

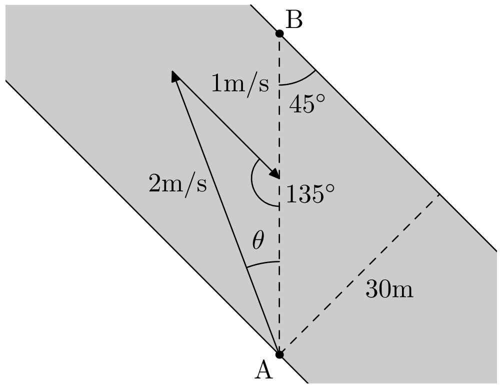
$$\tan\frac{\theta}{2} = \frac{40}{100}$$
$$\frac{\theta}{2} = \tan^{-1} 0.4$$
$$\theta = 2\tan^{-1} 0.4 = 0.761^{\text{R}}$$

$$\text{Area} = \frac{1}{2}r^2(\theta - \sin\theta) = \frac{1}{2} \times 107.7^2(0.761 - \sin 0.761) = 414\text{cm}^2$$

Total area=$16\,000 + 414 \approx 16\,410\text{cm}^2$

7. (a) $\{y \in \mathbb{R} : y \geq 0\}$
   (b) $\{y \in \mathbb{R} : y \geq 3\}$
   (c) $\{y \in \mathbb{R} : y \geq 0\}$
   (d) $\{y \in \mathbb{R} : y \geq 0\}$
   (e) $\{y \in \mathbb{R} : y \geq 3\}$
   (f) $\{y \in \mathbb{R} : y \geq 0\}$

8.



$$\frac{\sin\theta}{1} = \frac{\sin 135°}{2}$$
$$\theta = \sin^{-1}\frac{\sin 135°}{2} = 20.7°$$
$$360 - 20.7 = 339.3°$$

The person should row on a bearing of 339°.
Distance to travel = $\frac{30}{\sin 45°} = 42.43\text{m}$

Speed:

$$\frac{s}{\sin(180° - 135° - 20.7°)} = \frac{2}{\sin 135°}$$
$$s = \frac{2\sin 44.3°}{\sin 135°} = 1.16\text{m/s}$$

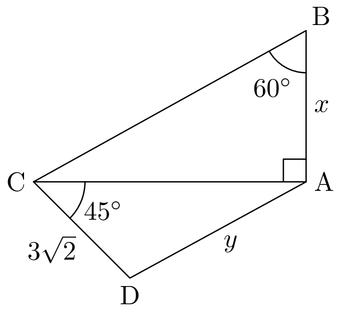
Time=$\frac{42.43}{1.16} = 36.46 \approx 36\text{s}$.

9. The critical value of 5 can only be achieved if one of the absolute values changes sign at $x = 5$, implying $a = -5$.

   For $x < 8$ and $x < 5$

$$-(x-8) = -(x-5) + b$$
$$-x + 8 = -x + 5 + b$$
$$8 = 5 + b$$
$$b = 3$$

10.



$$\tan 60^\circ = \frac{\text{AC}}{x}$$
$$\text{AC} = x\tan 60^\circ$$
$$= \sqrt{3}x$$

Using the cosine rule in triangle ACD

$$y^2 = \text{AC}^2 + \text{CD}^2 - 2\times\text{AC}\times\text{CD}\cos 45^\circ$$
$$= (\sqrt{3}x)^2 + (3\sqrt{2})^2 - 2(\sqrt{3}x)(3\sqrt{2})\left(\frac{\sqrt{2}}{2}\right)$$
$$= 3x^2 + 18 - 6\sqrt{3}x$$
$$= 3(x^2 - 2\sqrt{3}x + 6)$$
$$y = \sqrt{3(x^2 - 2\sqrt{3}x + 6)}$$

Q.E.D.